Symplectic geometry—an introduction

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December 1991

It is the name of a modern (and in full expansion) part of mathematics, coming from mechanics and optics (Hamilton's equations) and connected (in a very dynamical way) to a lot of other fields of mathematics: differential topology and dynamical systems, theory of representations, global analysis, singularities of differentiable mappings and algebraic geometry, to name a few.

It is the geometry of spaces—or manifolds—endowed with a *skew*-symmetric bilinear form. The beauty—but also the the difficulty—of the subject comes mainly from the fact that, in the small, all these structures look the same: there are no local invariants. Of course, in the large, it is quite different and everybody knows that global problems may be (and are very often) more difficult than local ones. It produces a situation, analogous to that in number theory, where there are many "elementary" statements with dreadfully hard proofs.

In this introductory talk, there will be mainly definitions.

1. Calculus and (bi-)linear algebra

A symplectic form can be defined as something analogous to a Riemannian metric. That is, if we begin with linear algebra, analogous with a scalar product. Suppose we are in a finite dimensional real vector space E.

Riemann (= Euclide)	Symplectic
scalar product: a nondegenerate (actually definite positive) symmetric bilinear form $\langle X,Y\rangle = \langle Y,X\rangle$. *orthonormal basis*: $(e_1,\ldots,e_k)$ such that $\langle e_i,e_j\rangle = \delta_{i,j}$.	*symplectic form*: a nondegenerate skew-symmetric bilinear form $\omega(X,Y) = -\omega(Y,X)$, thus $\omega(X,X) = 0 \quad \forall X$. *symplectic basis*: $(e_1,\ldots,e_n,f_1,\ldots,f_n)$ such that $\omega(e_i,e_j) = \omega(f_i,f_j) = 0$ and $\omega(e_i,f_j) = \delta_{i,j}$. As a consequence, $\dim E$ is even.

*This talk was prepared with the help of C. Goldstein, who pretended to understand nothing and thus made it understandable.

useful to compute (or to define) lengths, areas.
length $(u) = \|\#1\| = \sqrt{\langle u, u\rangle}$,

$$\cos\theta = \frac{\langle u, v\rangle}{\|\#1\| \cdot \|\#1\|}.$$

computes areas (with a sign).
$n = 1$, $\dim E = 2$:

$$\omega\left((p,q),(p',q')\right) = pq' - qp' = \det(u,v).$$

More generally, $\omega(X,Y)$ is the area of the parallelogram on X and Y.

Let us now go to calculus. So, let $f : E \to \mathbf{R}$ be a smooth function.

Riemann (= Euclide?)

gradient: the vector field ∇f such that

$$\forall Y \in E, \quad \langle \nabla_x f, Y\rangle = df_x(Y).$$

In an orthonormal basis

$$\nabla_x(f) = {}^t\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_k}\right).$$

Symplectic

symplectic gradient, or *hamiltonian vector field* X_f such that

$$\forall Y \in E, \omega(X_f(x), Y) = df_x(Y).$$

In a symplectic basis, $X_f(x) =$

$${}^t\left(\frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_n}, -\frac{\partial f}{\partial p_1}, \dots, -\frac{\partial f}{\partial p_n}\right).$$

Note that symplectic gradients allow also to define something very useful and special to symplectic geometry: the *Poisson bracket*

$$\{f,g\}(x) = df_x\left(X_g(x)\right) = \omega\left(X_g(x), X_f(x)\right) = -dg_x\left(X_f(x)\right) = -\{g,f\}$$

which is a bilinear pairing on $\mathcal{C}^\infty(E,\mathbf{R})$ (in fact a Lie algebra structure).

All that calculus, even in a vector space E, is very useful in mechanics, as you probably guessed from the name “hamiltonian” vector field. As everybody knows, symplectic geometry and hamiltonians were first used in the Lagrange book “Mécanique analytique” (1788); Lagrange was so kind to even use the letter H for what we now call a “hamiltonian”, after Hamilton’s name. It is also remarkable that the “hamiltonian formalism to mechanics” was invented by Hamilton in his researches on geometrical optics. It turns out that symplectic geometry is not only the good language for mechanics, but is also very useful to describe geometrical optics—for instance caustics (namely rainbows, and so on).

The last remark I want to make on nondegenerate skew-symmetric bilinear forms ω is that, in a symplectic basis, ω, as an element of $\Lambda^2(E^\star)$, can be written

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$$

thus you can consider it as a differential 2-form, which is closed: $d\omega = 0$.

Now, let us proceed from Euclide to Riemann, and analogously to symplectic manifolds.

Think of a manifold as of something looking locally very much like a vector space. For instance a surface in $\mathbf{R}^3$: near any point x, it looks very much like its tangent plane. A manifold W is a k-dimensional generalisation, it also has a tangent space (a vector space) T_xW associated to any of its points x.

Riemannian metrics

a tool to compute lengths and areas on a manifold.

Symplectic form

a tool to compute areas of surfaces in a manifold W.
more technically: a differentiable 2-form ω, that is a skew-symmetric bilinear form ω_x on each T_xW, we ask that it is nondegenerate. In order to have the same calculus we had in the linear situation, we ask that

$$d\omega = 0.$$

It is the condition which makes things difficult/interesting. One could give several talks on that condition.

Locally we have in both cases the same calculus we had in a vector space, gradients, etc.

2. Examples of symplectic manifolds

2.1. surfaces

$\mathbf{R}^2$ with its area form. Any orientable surface

endowed with an area form ω: the condition $d\omega = 0$ is automatically fullfilled in this dimension.

2.2. the complex projective plane

Let me discuss now in some details the example of the projective plane. Not only because it is a useful example, but also because the construction itself is rather interesting, being an example of *symplectic reduction*. Consider first the *complex* vector space $\mathbf{C}^3$. I shall also think at it as being a *real* 6-dimensional space. As such, I give it a symplectic (linear) structure. Here is one way to think of it: we decompose the standard hermitian product

$$\langle\langle X, Y\rangle\rangle = \Re\langle\langle X, Y\rangle\rangle + i\Im\langle\langle X, Y\rangle\rangle$$

or

$$\sum x_j\overline{y_j} = p\cdot p' + i(q\cdot p' - p\cdot q')$$

(the complex vectors X and Y being decomposed into real and imaginary parts $X = p + iq$, $Y = p' + iq'$). The real part of the hermitian product is symmetric, and is the canonical scalar product, its imaginary part is skew-symmetric and is a symplectic[1] form.

Now, let us consider the set $\mathbf{CP}^2$ of all complex lines through zero in $\mathbf{C}^3$. As any line contains a non-zero vector and any two such differ by multiplication by a non-zero scalar:

$$\mathbf{C}^3 - 0/u \sim \lambda u \cong \mathbf{CP}^2$$

which allows to define a topology on that set. Also, in any line, there are unit vectors, and

$$S^5 = \left\{u \mid \|\#1\|^2 = 1\right\}/u \cong \lambda u = \mathbf{CP}^2.$$

Now we use the symplectic form of $\mathbf{C}^3$ to define a symplectic form on $\mathbf{CP}^2$.

Let H be the function

$$\begin{array}{ccc} \mathbf{C}^3 & \longrightarrow & \mathbf{R} \\ u & \longmapsto & \frac{1}{2}\|\#1\|^2. \end{array}$$

Its hamiltonian vector field is

$$X_H(v) = iv$$

... the tangent vector to the circle of vectors spanning the same line as v.

Of course T_vS^5 is odd-dimensional, thus $\omega|_{T_vS^5}$ cannot be nondegenerate. But we already know that, if $Y \in T_vS^5$,

$$\omega_v\left(X_H(v), Y\right) = dH_v(Y) = v\cdot Y = 0,$$

thus the kernel of ω is generated by $X_H(v) = iv$. It means that ω, restricted to the *hermitian* orthogonal of v is nondegenerate. Note that $w = \lambda v$ (pour $\lambda \neq 0$) implies $w^\perp = v^\perp$, thus, this complex subspace should be the tangent space of $\mathbf{CP}^2$ at the point $[v]$ which is represented by v (or w)... and thus we have a nondegenerate 2-form on $\mathbf{CP}^2$. It is an easy exercise to check that, being defined with help of a closed form on $\mathbf{C}^3$, it is a closed form.

[1]The real and imaginary part of the canonical complex basis of $\mathbf{C}^3$ make a symplectic basis.

3. The Darboux theorem

A very special feature of symplectic geometry is that, *locally*, all symplectic manifolds are isomorphic.

THÉORÈME 1. — *Given any point x in a symplectic manifold (W, ω), there exist local coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ centered at x such that $\omega = \sum dp_i \wedge dq_i$.*

Remarks.

1. This is very different from the Riemannian situation. Consider a sphere in euclidean space, with its intrinsic metric: the distance between x and y is the euclidean length of the shortest path from x to y on the sphere (big circle).

   Everybody knows that it is impossible, even locally, to find an isometry from part of the sphere into a plane: this is the reason why there does not exist geographical maps with correct lengths and angles: every map will distot shapes. On the other hand, there exists maps with no size (i.e. areas) distortion, thanks to Darboux theorem.

2. Of cours to be evrywhere locally isomorphic does *not* mean to be (globally) isomorphic. For instance, there *do* exist symplectic forms on $\mathbf{R}^4$ which are not isomorphic to the standard (bilinear) one.

4. Moment mappings and convexity theorems

We now look at a group (more precisely a torus) acting on a symplectic manifold, (W, ω).

Examples.

1. $T^n = \{(t_1, \ldots, t_n) \in \mathbf{C}^n \mid |\#1| = 1\}$ acting on $\mathbf{C}^n$ (always considered as $\mathbf{R}^{2n}$) by

   $$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (t_1 z_1, \ldots, t_n z_n)$$

   For instance, if $n = 1$:

iz is the symplectic gradient of $H(z) = \frac{1}{2}|\#1|^2 = \frac{1}{2}(p^2+q^2)$:

$$\frac{\partial H}{\partial p} = x, \quad \frac{\partial H}{\partial q} = q$$

thus $X_H(p,q) = (-q,p) = i(p+iq)$.

For general n, one does the same coordinate by coordinate, thus getting n functions $(H_1,\dots,H_n)$, or a map

$$\begin{array}{rcl} \mu : \mathbf{C}^n & \longrightarrow & \mathbf{R}^n \\ (z_1,\dots,z_n) & \longmapsto & \frac{1}{2}(|\#1|^2,\dots,|\#1|^2) \end{array}$$

the image of which being the first "quadrant" (all coordinates nonnegative) in $\mathbf{R}^n$.

2. (variation) T^2 acts on $\mathbf{CP}^2$ by

$$(t_1,\dots,t_2)\cdot[x_1,x_2,x_3] = [t_1x_1, t_2x_2, x_3]$$

where $[x_1,x_2,x_3]$ represents the line generated by the nonzero vector (x_1,x_2,x_3). Note that the result (righthandside) depends only of the class $[x_1,x_2,x_3]$ and not on the vector (x_1,x_2,x_3)

these 2 vectors also are hamiltonian vector fields. As above they define a map: $\mu : \mathbf{CP}^2 \longrightarrow \mathbf{R}^2$

$$\mu[x_1,x_2,x_3] = \frac{1}{2}\left(\frac{|\#1|^2}{\sum|\#1|^2}, \frac{|\#1|^2}{\sum|\#1|^2}\right)$$

whose image is the triangle

Note that we get a smooth mapping whose image is a convex polyhedron.

The perspicaceous reader should have guessed there is some relationship between the triangle and the quadrant. She could even have noticed a relation between the construction of $\mathbf{CP}^2$ using T^1 acting on $\mathbf{C}^3$, the action of T^3 on $\mathbf{C}^3$ as in the first example above... and the action of T^2 on $\mathbf{CP}^2$ in the last example. All these constructions belong to the world of *moment mappings*, *symplectic reductions*... the most spectacular result in that field being the Atiyah-Guillemin-Sternberg convexity theorem, which asserts that smooth mappings (like μ) related to torus actions on compact connected symplectic manifolds (as μ is related to the T^2 action) should have convex polyhedra as images.

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