SEMIHOLONOMIC JETS AND CONTACT ELEMENTS, HIGHER ORDER CONNECTIONS

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last personal meeting: Banach Center, Poland, 2005: The Mathematical Legacy of Charles Ehresmann (100th anniversary of his birthday)

Paulette Libermann: C.E.'s concepts in differential geometry

she underlined semiholonomic jets and their applications

I became interested in semiholonomic jets at the end of sixties

Eduard Čech: Projective differential geometry

we did concrete geometric research of submanifolds of spaces with projective connection

I realised that semiholonomic jets play an important theoretical role there

1996, Arch. Math. (Brno), P.L: Introduction to the theory of semiholonomic jets

1. Nonholonomic jets

fibered manifold $Y \to M$, r-th jets of local sections form r-th prolongation $J^r Y$

One can be interested in $J^k(J^r Y \to M) =: J^{r,k} Y$

One possible motivation goes from PDE:

$\Psi \subset J^r Y$, $s : M \to \Psi$, the question is $s = j^r \sigma$, $\sigma : M \to Y$; we can construct $j^k s : M \to J^{r,k} Y$ ...

C.E: r-th nonholonomic prolongation

$\tilde{J}^r Y = J^1(\tilde{J}^{r-1} Y \to M)$, $\tilde{J}^1 Y = J^1 Y$,

we have $\tilde{J}^{r+k} Y \supset J^{r,k} Y \supset J^{r+k} Y$,

$j^{r+k}_x s \mapsto j^k_x (j^r s)$

We know $J^r(M,N) = J^r(M \times N \to M)$, C.E: nonholonomic r-jets $\tilde{J}^r(M,N) = \tilde{J}^r(M \times N \to M)$

and DEFINED THEIR COMPOSITION!

Basic idea on $J^{r,k}(M,N) = J^{r,k}(M \times N \to M)$:

$X \in J^{r,k}_x(M,N)_y$, $X = j^k_x F$, $F : M \to J^r(M,N)$,

$f = \beta \circ F : M \to N$; $Z \in J^{r,k}_y(N,P)_z$, $Z = j^k_y G$,

$G : N \to J^r(N,P)$

$Z \circ X := j^k_x (G(f(u)) \circ F(u)) \in J^{r,k}_x(M,P)_z$

$u \in M$

This is extended to (technique only)

$X \in \tilde{J}^r_x(M,N)_y,\ Z \in \tilde{J}^r_y(N,P)_z : Z \circ X \in \tilde{J}^r_x(M,P)_z$

2. Semiholonomic jets

$r=2$: $\bar{J}^2Y \subset \tilde{J}^2Y$, $\beta_Y : J^1Y \to Y$, $J^1\beta_Y : \tilde{J}^2Y \to$
$\to J^1Y$; $\beta_{J^1Y} : \tilde{J}^2Y \to J^1Y$

$X \in \bar{J}^2Y$ means $\beta_{J^1Y}(X) = J^1\beta_Y(X)$

$J^2Y \subset \bar{J}^2Y$; $\bar{J}^2(M,N) := \bar{J}^2(M \times N \to M)$

IMPORTANT FOR APPLICATION: $X \in \bar{J}^2_x(M,N)_y$ induces $\Delta X \in T_yN \otimes \Lambda^2 T^*_xM$ (J. Pradines: dissymétrie, I.K: difference tensor)

$\Delta X = 0$ iff X is holonomic

generally: $\bar{J}^rY \subset J^1(\bar{J}^{r-1}Y)$,
$\beta_{\bar{J}^{r-1}Y}(X) = J^1\beta_{\bar{J}^{r-2}Y}(X)$

appears in certain approaches to the repeated absolute differentiation

Instructive description: V, W vector spaces

$$J^r(V,W) = V \times \bigoplus_{k=0}^{r} W \otimes S^kV^*,$$

$$\bar{J}^r(V,W) = V \times \bigoplus_{k=0}^{r} W \otimes \bigotimes^k V^*$$

P.L.'s original concept: sesquiholonomic r-jet

$$\check{J}^r Y = \bar{J}^r Y \cap J^1 J^{r-1} Y \cap \bar{J}^2 J^{r-2} Y$$

So Δ can be applied to $\check{J}^r(M,N)$ – this is closely related with Spencer's δ.

We write $\bar{P}^r M = \operatorname{inv} \bar{J}^r_0(\mathbb{R}^m, M)$, $m = \dim M$

$\bar{P}^r M(M, \bar{G}^r_m)$, $\bar{G}^r_m = \operatorname{inv} \bar{J}^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$

NOTATION from I.K., P.W. Michor, J. Slovák

Natural Operations in Diff. geom., 1993

$P^r M = \operatorname{inv} J^r_0(\mathbb{R}^m, M)$, $G^r_m = \operatorname{inv} J^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$

3. First applications (classical objects)

(i) general connection $\Gamma : Y \to J^1 Y$

$J^1\Gamma : J^1 Y \to \tilde{J}^2 Y$, $\Gamma' = J^1\Gamma \circ \Gamma : Y \to \bar{J}^2 Y$

(Ehresmann prolongation) The curvature

$C\Gamma := \Delta \circ \Gamma' : Y \to VY \otimes \Lambda^2 T^* M$

(ii) P.L. deduced $J^1 P^1 M \simeq \bar{P}^2 M$

classical connection $\Lambda : P^1 M \to J^1 P^1 M \simeq \bar{P}^2 M$

Torsion $\tau\Lambda = \Delta \circ \Lambda \simeq P^1 M \to \mathbb{R}^m \otimes \Lambda^2 \mathbb{R}^{m*}$

S. Kobayashi $P^2 M \subset J^1 P^1 M$ and proved

Λ is torsion-free iff $\Lambda(P^1 M) \subset P^2 M$

(iii) first order G-structure $Q \subset P^1M$, $J^1Q \subset J^1P^1M \approx \bar{P}^2M$, $u \in Q$

$\sigma(u) = \{\Delta(v); v \in J^1_u Q\}$, $\sigma : Q \to H^{02}(\mathfrak{g})$

the structure function of Q

This clarifies immediately $\sigma = 0$ iff $J^1Q \cap P^2M$ is not empty

(iv) P.L. deduced $J^1T^*M \approx \bar{J}^2(M,\mathbb{R})_0$

For $\omega : M \to T^*M$, $\Delta(j^1\omega) : M \to \Lambda^2 T^*M \equiv d\omega$

4. Higher order connections

The seminal text is a very short note C.E.: Sur les connexions d'ordre supérieur, 1955. Very deep and very condensed, based on nonholonomic jets and Lie groupoids

- concept of element of connection, on P: $Q^1P = (J^1P/G) \to M$, Γ is a section $M \to Q^1P$ (bundle of first order elements of conn.)
- on groupoids - absolutely intrinsic

$PP^{-1} = P \times P/G$; $E = P[F]$, $u \in P_x$, $\tilde{u} : F \to E_x$, $v \in P_y$, $\tilde{v} \circ \tilde{u}^{-1} : E_x \to E_y$ $\quad Q^1(PP^{-1}) \leftrightarrow$ "infinitesimal displacement" by E. Cartan

- general idea of (nonholonomic) higher order absolute differentiation

C.E. – inspiration for various research about higher order connections

(i) P.L: Sur la géométrie des prolongements des espaces fibrés vectoriels, Fourier 1964

$E \to M$ vector bundle

$0 \to E \otimes \overset{r}{\otimes} T^*M \hookrightarrow \bar{J}^r E \overset{\Gamma}{\rightleftarrows} \bar{J}^{r-1} E \to 0$

also $J^r E$. P.L. deduced several deep results.

Extended to arbitrary fibered manifolds

$\Gamma : \bar{J}^{r-1} Y \to J^r Y$ (M. Modugno, M. Eastwood)

(ii) I.K: the nonholonomic r-th order connections on PP^{-1} are in bijection with right-invariant sections $\Gamma : P \to \tilde{J}^r P$.

P.L. used "surconnexion" (overconnection)

generally $\Gamma : Y \to \tilde{J}^r Y$

A general idea by J. Virsik (J. Pradines)

$\Gamma_1 : Y \to \tilde{J}^r Y$, $\Gamma_2 : Y \to \tilde{J}^k Y$,

$\tilde{J}^k \Gamma_1 : \tilde{J}^k Y \to \tilde{J}^{r+k} Y$,

$\Gamma_1 * \Gamma_2 := \tilde{J}^k \Gamma_1 \circ \Gamma_2 : Y \to \tilde{J}^{r+k} Y$

For $r = k = 1$ we reobtain the Ehresmann prolongation $\Gamma' = \Gamma * \Gamma$

5. Prolongation of G-structures

P.L. deduced $\bar{P}^{r+1}M \approx \bar{J}^r P^1 M$ (DEEP RESULT – NO HOLONOMIC ANALOGY)

Beside sesquiholonomic frames $\check{P}^{r+1}M$ she introduced almost holonomic frames $\hat{P}^{r+1}M \approx J^r P^1 M$

This opened a systematic way for studying higher order prolongation of G-structures

Here P.L. used heavily the higher order connections in the sense of P.L.– Fourier

She deduced several deep results:
P.L: Connexions d'ordre supérieur et tenseur de structure, Bologna 1967

Using these techniques, she essentially contributed to the theory of higher order G-structures (both holonomic $Q \subset P^r M$ and semiholonomic $Q \subset \bar{P}^r M$). Also P.C. Yuen

Remark. Modifying C.E, P.L. and I.K. studied principal prolongation $W^1 P = P^1 M \times_M J^1 P$ of arbitrary principal bundle P. We have $P^r M \subset W^1 P^{r-1} M$

I.K. introduced generalized G-structure as a reduction $Q \subset W^1 P$. This theory reflects several properties of h.o. G-structures

6. Semiholonomic contact elements

n-submanifold $N \subset M$, $f: \mathbb{R}^n \to M$ parametrization
$j^r_0 f \in \text{reg}\, J^r_0(\mathbb{R}^n, M)_x$, changes of parametrization
$\leftrightarrow G^r_n$, the contact element $K^r_x N := (j^r_0 f) \circ G^r_n$
$\in K^r_n M$ - bundle of contact (n, r)-elements
One writes $K^r_n M = \text{reg}\, J^r_0(\mathbb{R}^n, M) / G^r_n$
more generally $k: \text{reg}\, J^r(N, M) \to K^r_n M$
Canonical section $k^r_N : N \to K^r_n M | N$

Recently R. Vitolo, I.K. have studied systematically nonholonomic contact elements
$\tilde{K}^r_n M = K^1_n(\tilde{K}^{r-1}_n M)$, $K^r_n M \subset \tilde{K}^r_n M$
$\bar{K}^r_n M \subset \tilde{K}^r_n M$ by the same idea as for jets
Proposition $\bar{K}^r_n M = \text{reg}\, \bar{J}^r_0(\mathbb{R}^n, M) / \bar{G}^r_n$

Our approach via iterations is developped in
I. K., R. Vitolo, Absolute contact differentiation on submanifolds of Cartan spaces to appear in Diff. Geom. and its Applications, already available on web

The pure situation for this new operation:

$P(N,G)$, principal connection Γ, left G-space
F, $\dim N < \dim F$, $E = P[F]$, $s: N \to E$

In jet form, we have

$$\nabla_\Gamma s : N \to \bigcup_{x\in N} J^1_x(N, E_x)$$

ASSUME ALL $(\nabla_\Gamma s)(x)$ ARE REGULAR

We apply the viewpoint of submanifolds:

$$k(\nabla_\Gamma s(x)) \in K^1_m(E_x)$$

The first absolute contact differential

$$k\nabla_\Gamma s : N \to \bigcup_{x\in N} K^1_m(E_x) = P[K^1_m F]$$

So we can iterate

$$\nabla_\Gamma(k\nabla_\Gamma s) : N \to \bigcup_{x\in N} J^1_x(N, K^1_m E_x)$$

$$(k\nabla^2_\Gamma s)(x) := k(\nabla_\Gamma(k\nabla_\Gamma s)(x)) \in \bar{K}^2_m E_x$$

Hence $(k\nabla^2_\Gamma) s : N \to P[\bar{K}^2_m F]$

after r-1 steps: r-th absolute contact differential

$$k\nabla^r_\Gamma s : N \to P[\bar{K}^r_m F]$$

If Γ is curvature-free,

$$k\nabla^r_\Gamma s : N \to P[K^r_m F]$$

7. Submanifolds in Cartan geometries

Consider a Klein space $S = G/H$, $c = \{H\} \in S$

R.W. Sharpe: Cartan geometry $\mathcal{S}$ of type S is a principal bundle $Q(M,H)$ with $\omega : TQ \to \mathfrak{g}$ satisfying ...

S is the flat case: $Q = G(S,H)$ and ω is the Maurer-Cartan form $\varphi : TG \to \mathfrak{g}$

But we have to start from another approach (I.K: Space with Cartan connection, $\leftrightarrow$ C.E.)

$P(M,G)$, $\dim M = \dim S$, $\Gamma \leftrightarrow \omega_\Gamma$, $E = P[S]$,

$s : M \to E$ such that $\nabla_\Gamma s(x)$ is regular, $x \in M$.

COMPARISON: $u \in P_x$, $\tilde{u} : S \to E_x$

$Q = \{u \in P, \tilde{u}(c) = s(x)\}$, $\omega = \omega_\Gamma | TQ$

So $(P, \Gamma, E, s) = \mathcal{S} = (Q, \omega)$

For $N \subset M$, we restrict over N:

$$(P_N, \Gamma_N, E_N, s_N) = \mathcal{S}_N = (Q_N, \omega_N)$$

We can apply the absolute contact differentiation. Write $\bar{S}^r_m = (\bar{K}^r_m S)_c$, $S^r_m = (K^r_m S)_c$, H-spaces

$((k\nabla^r_{\Gamma_N})\,s_N)(x)$ depends on $K^r_x N$ only

$$\Gamma^r_n : K^r_n M \to Q[\bar{S}^r_n]$$

is said formal absolute contact (n,r)-differentiation on $\mathcal{S}$

$$\Gamma^r_N = \Gamma^r_n \circ K^r_N : N \to Q_N[\bar{S}^r_n]$$

is the r-th absolute contact differential of N.

In the flat case, $G[S^r_n] = K^r_n S$ and $\Gamma^r_N \approx K^r_N$.

WHERE IS AN ALGORITHM FOR Γ^r_N?

IN: É. CARTAN's METHOD OF MOVING FRAMES.

Flat case: $N \subset S = G/H$, $d\varphi + \frac{1}{2}[\varphi, \varphi] = 0$

Take a basis (e_α, e_λ) of $\mathfrak{g}$ such that e_λ is a basis of $\mathfrak{h}$. Then $\varphi = (\varphi^\alpha, \varphi^\lambda)$, φ^α – principal forms on S

zeroth order frame bundle G_N of N: $g(c) \in N$

Then we have some $m-n$ linear relations among φ^α_N

For the sake of simplicity, assume H acts transitively on S^1_n (n-dim. Grassmannians)

First order frame bundle G_{N1} of N

$g \in G_{N1}$ $Tg(e_i) \in T_{g(c)}N$, $i = 1, \ldots, n$

In first order frames: $\varphi^p_{N1} = 0$, $p = n+1, \ldots, m$

Cartan prolongation procedure: we apply exterior differentiation, structure equations and Cartan lemma.

Then one usually continues in specialisation of frames. G.F. Laptev (~1950) pointed out that one can repeat this prolongation procedure without specializing frames.

I.K.: after $r-1$ steps we obtain the coordinate functions of K^r_N. Further, this procedure yields the equations of infinitesimal action of H on S^r_m.

In the case of $\mathcal{P}$, we have $\omega : TQ \to \mathfrak{g}$, $d\omega + \frac{1}{2}[\omega, \omega] = \Omega$. For $N \subset \mathcal{P}$, we obtain $\omega_N : TQ_N \to \mathfrak{g}$. Then a quite analogous procedure yields the coordinate functions of Γ^r_N and the equations of infinitesimal action of H on $\bar{S}^r_m$.

8. Geometric objects of submanifolds

Flat case: r-th order geometric objects of n-submanifolds of S are defined on contact (n,r)-elements (-independent of parametrization, -of universal character)

We clarified: they can be viewed as H-maps $\mu : S_n^r \to A$, A being H-space

We generalized: a semiholonomic (n,r)-object of type S is an H-map $\mu : \bar{S}_n^r \to A$

μ induces $\tilde{\mu} : Q[\bar{S}_n^r] \to Q[A]$

value of μ on $N \subset \mathcal{S}$ is

$$\mu_N = \tilde{\mu} \circ \Gamma_N^r : N \to Q_N[A]$$

In the flat case, $\Gamma_N^r \leftrightarrow k_N^r$,

$$\tilde{\mu} : G[S_n^r] = K_n^r S \to G[A]$$

$$\mu_N = \tilde{\mu} \circ k_N^r : N \to G_N[A]$$

We say $N \subset \mathcal{S}$ is r-holonomic at $x \in N$, if $\Gamma_N^r(x)$ is holonomic contact (n,r)-element

So: If N is r-holonomic at x, then all its r-th order geom. objects at x are of the same type as for n-submanifolds of the corresponding Klein space.

REMARKABLE SPECIAL CASE:

Curvature of $\mathcal{S}$: $\Omega : Q \times_M \Lambda^2 TM \to \mathfrak{g}$

Write $L = T_c S = \mathfrak{g}/\mathfrak{h}$ and $\psi : \mathfrak{g} \to L$.

Def: $\sigma = \psi \circ \Omega$ is the torsion of $\mathcal{S}$

The absolute differentiation on $\mathcal{S}$ identifies $T_x M \approx T_{p(x)} E_x$. Hence

$$\sigma : M \to Q[L \otimes \Lambda^2 L^*]$$

Modifying Δ to contact elements:

Prop. If the torsion of $\mathcal{S}$ vanishes, then the values of Γ^2_n are holonomic contact $(n,2)$-elements.

Then the second order geometric objects of $N \subset \mathcal{S}$ are of the same type as for submanifolds of S.

Riemannian case: Second order geometric objects of $N \subset (M,g)$ are of the same type as for submanifolds of the Euclidean space E_m.