Functoriality in the semi-classical world

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Contents

Basic facts about the cotangent bundle, 1 - the canonical one form.

If X is a differentiable manifold, then its cotangent bundle T^*X carries a **canonical one form** $\alpha = \alpha_X$ defined as follows: Let

$$\pi : T^*X \to X$$

be the projection sending any covector $p \in T_x^*X$ to its base point x. If $v \in T_p(T^*X)$ is a tangent vector to T^*X at p, then

$$d\pi_p v$$

is a tangent vector to X at x. In other words, $d\pi_p v \in T_xX$. But $p \in T_x^*X$ is a linear function on T_xX, and so we can evaluate p on $d\pi_p v$. The canonical linear differential form α is defined by

$$\langle \alpha_p, v\rangle := \langle p, d\pi_p v\rangle \quad \text{if} \quad v \in T_p(T^*X). \tag{1}$$

Basic facts about the cotangent bundle, 2 - the canonical two form.

This is defined as

$$\omega_X = -d\alpha_X. \tag{2}$$

Let $q^1, \dots, q^n$ be local coordinates on X. Then $dq^1, \dots, dq^n$ are differential forms which give a basis of T_x^*X at each x in the coordinate neighborhood U. In other words, the most general element of T_x^*X can be written as $p_1(dq^1)_x + \cdots + p_n(dq^n)_x$. Thus $q^1, \dots, q^n, p_1, \dots, p_n$ are local coordinates on

$$\pi^{-1}U \subset T^*X.$$

In terms of these coordinates the canonical one-form is given by

$$\alpha = p \cdot dq = p_1 dq^1 + \cdots + p_n dq^n$$

Hence the canonical two-form has the local expression

$$\omega = dq \wedge \cdot dp = dq^1 \wedge dp_1 + \cdots + dq^n \wedge dp_n. \tag{3}$$

A local expression for ω is

$$\omega = dq \wedge \cdot dp = dq^1 \wedge dp_1 + \cdots + dq^n \wedge dp_n.$$

The form ω is closed and is of maximal rank, i.e., ω defines an isomorphism between the tangent space and the cotangent space at every point of T^*X.

A two form which is closed and is of maximal rank is called **symplectic**. A manifold M equipped with a symplectic form is called a **symplectic manifold**.

So the cotangent bundle of any differentable manifold is an example of a symplectic manifold.

Isotropic submanifolds and Lagrangian submanifolds.

A submanifold Y of a symplectic manifold is called **isotropic** if the restriction of the symplectic form ω to Y is zero. So if

$$\iota_Y : \; Y \to M$$

denotes the injection of Y as a submanifold of M, then the condition for Y to be isotropic is

$$\iota_Y^* \omega = 0$$

where ω is the symplectic form of M.

If, in addition, $\dim Y = \frac{1}{2} \dim M$ then Y is called a **Lagrangian submanifold**.

Lagrangian submanifolds of the cotangent bundle.

To say that a submanifold $\Lambda \subset T^*X$ is Lagrangian means that Λ has the same dimension as X and that the restriction to Λ of the canonical one form α_X is closed.

Suppose that Z is a submanifold of T^*X and that the restriction of $\pi : T^*X \to X$ to Z is a diffeomorphism. This means that Z is the image of a section

$$s : \; X \to T^*X.$$

Giving such a section is the same as assigning a covector at each point of X, in other words it is a linear differential form.

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Giving such a section is the same as assigning a covector at each point of X, in other words it is a linear differential form. For the purposes of the discussion we temporarily introduce a redundant notation and call the section s by the name β_s when we want to think of it as a linear differential form. We claim that

$$s^*\alpha_X = \beta_s.$$

Indeed, if $w \in T_xX$ then $d\pi_{s(x)} \circ ds_x(w) = w$ and hence

$$s^*\alpha_X(w) = \langle (\alpha_X)_{s(x)}, ds_x(w) \rangle =$$

$$= \langle s(x), d\pi_{s(x)} ds_x(w) \rangle = \langle s(x), w \rangle = \beta_s(x)(w).$$

Thus the submanifold Z is Lagrangian if and only if $d\beta_s = 0$.

Thus the submanifold Z is Lagrangian if and only if $d\beta_s = 0$. Let us suppose that X is connected and simply connected. Then $d\beta = 0$ implies that $\beta = d\phi$ where ϕ is determined up to an additive constant.

With some slight abuse of language, let us call a Lagrangian submanifold Λ of T^*X **horizontal** if the restriction of $\pi : T^*X \to X$ to Λ is a diffeomorphism. We have proved

> *Suppose that X is connected and simply connected. Then every horizontal Lagrangian submanifold of T^*X is given by a section $\gamma_\phi : X \to T^*X$ where γ_ϕ is of the form*
>
> $$\gamma_\phi(x) = d\phi(x)$$
>
> *where ϕ is a smooth function determined up to an additive constant.*

The function ϕ is called a **generating function** for the horizontal Lagrangian submanifold Λ

Over the next few slides I want to illustrate different kinds of generating function, ones which apply when the Lagrangian submanifold of the cotangent bundle need not be horizontal.

For the sake of giving an elementary introduction, I will discuss the linear case.

The Lagrangian Grassmannian.

Let $V = (V, \omega)$ be a symplectic vector space of dimension $2n$. We let $\mathcal{L}(V)$ denote the space of all Lagrangian subspaces of V. It is called the **Lagrangian Grassmannian**.

If $M \in \mathcal{L}(V)$ is a fixed Lagrangian subspace, we let $\mathcal{L}(V, M)$ denote the subset of $\mathcal{L}(V)$ consisting of those Lagrangian subspaces which are transversal to M.

Let $L \in \mathcal{L}(V, M)$ be one such subspace. The non-degenerate pairing between L and M identifies M with the dual space L^* of L and L with the dual space M^* of M. The vector space decomposition

$$V = M \oplus L = M \oplus M^*$$

tells us that any $N \in \mathcal{L}(V, M)$ projects bijectively onto L under this decomposition. In particular, this means that N is the graph of a linear map

$$T_N : L \to M = L^*.$$

So

$$N = \{(T_N\xi, \xi), \quad \xi \in L = M^*\}.$$

Giving a map from a vector space to its dual is the same as giving a bilinear form on the original vector space. In other words, N determines, and is determined by, the bilinear form β_N on $L = M^*$ where

$$\beta_N(\xi, \xi') = \frac{1}{2}\langle T_N\xi', \xi\rangle = \frac{1}{2}\omega(T_N\xi', \xi).$$

This is true for any n-dimensional subspace transversal to M.

What is the condition on β_N for N to be Lagrangian? Well, if $w = (T_N\xi, \xi)$ and $w' = (T_N\xi', \xi)'$ are two elements of N then

$$\omega(w, w') = \omega(T_N\xi, \xi') - \omega(T_N\xi', \xi)$$

since L and M are Lagrangian. So the condition is that β_N be symmetric. We have proved:

Proposition

If $M \in \mathcal{L}(V)$ and we choose $L \in \mathcal{L}(V, M)$ then we get an identification of $\mathcal{L}(V, M)$ with $S^2(L)$, the space of symmetric bilinear forms on L.
So every choice of a a pair of transverse Lagrangian subspaces L and M gives a coordinate chart on $\mathcal{L}(V)$ which is identified with $S^2(L)$. In particular, $\mathcal{L}(V)$ is a smooth manifold and

$$\dim \mathcal{L}(V) = \frac{n(n+1)}{2}$$

where $n = \frac{1}{2}\dim V$.

We have a symplectic vector space $V = M \oplus M^* = T^*M$ and we have a Lagrangian subspace $N \subset V$ which is transversal to M. This determines a linear map $T_N : M^* \to M$ and a symmetric bilinear form β_N on M^*. Suppose that we choose a basis of M and so identify M with $\mathbb{R}^n$ and so M^* with $\mathbb{R}^{n*}$. Then $T = T_N$ becomes a symmetric matrix and if we define

$$\gamma_N(\xi) := \frac{1}{2}\beta_N(\xi, \xi) = \frac{1}{2}T\xi \cdot \xi$$

then

$$T\xi = T_N\xi = \frac{\partial \gamma_N}{\partial \xi}.$$

Consider the function $\phi = \phi_N$ on $M \oplus M^*$ given by

$$\phi(x, \xi) = x \cdot \xi - \gamma_N(\xi), \quad x \in M, \ \xi \in M^*. \tag{4}$$

$$\phi(x, \xi) := x \cdot \xi - \gamma_N(\xi), \quad x \in M, \ \xi \in M^*.$$

Then the equation

$$\frac{\partial \phi}{\partial \xi} = 0 \tag{5}$$

is equivalent to

$$x = T_N\xi.$$

Of course, we have

$$\xi = \frac{\partial \phi}{\partial x}$$

and at points where (5) holds, we have

$$\frac{\partial \phi}{\partial x} = d\phi,$$

the total derivative of ϕ in the obvious notation. So

$$\phi(x, \xi) := x \cdot \xi - \gamma_N(\xi), \quad x \in M, \ \xi \in M^*. \quad (4)$$

$$\frac{\partial \phi}{\partial \xi} = 0. \quad (5)$$

Proposition

Let M be a vector space and $V = T^*M = M \oplus M^*$ its cotangent bundle with its standard symplectic structure. Let N be a Lagrangian subspace of T^*M which is transversal to M. Then

$$N = \{(x, d\phi(x, \xi))\}$$

where ϕ is the function on $M \times M^*$ given by (4) and where (x, ξ) satisfies (5).

Notice that in contrast to the previous generating, ϕ is not a function of x alone, but depends on an auxiliary variable (in this case ξ). But this type of generating function can describe a Lagrangian subspace which is not horizontal. At the extreme, the subspace M^* is described by the case $\beta_T \equiv 0$.

One of the key theorems in the subject is that, locally, every Lagrangian submanifold of a cotangent bundle can be described by this more general type of generating function, where we allow dependence on auxiliary variables.

Let $V = (V, \omega)$ be a symplectic vector space. We let $V^- = (V, -\omega)$. In other words, V is the same vector space as V but with the symplectic form $-\omega$.
We may consider the direct sum $V^- \oplus V$ (with the symplectic form $\Omega = (-\omega, \omega)$. If $T \in Sp(V)$, then its graph $\Gamma := \text{graph } T = \{(v, Tv), \quad v \in V\}$ is a Lagrangian subspace of $V^- \oplus V$. Indeed, if $v, w \in V$ then

$$\Omega((v, Tv), (w, Tw)) = \omega(Tv, Tw) - \omega(v, w) = 0.$$

Suppose that $V = X \oplus X^*$ where X is a vector space and where V is given the usual symplectic form:

$$\omega\left(\begin{pmatrix} x \\ \xi \end{pmatrix}, \begin{pmatrix} x' \\ \xi' \end{pmatrix}\right) = \langle \xi', x \rangle - \langle \xi, x' \rangle.$$

The map $\varsigma : \; V \to V$

$$\varsigma\left(\begin{pmatrix} x \\ \xi \end{pmatrix}\right) = \begin{pmatrix} x \\ -\xi \end{pmatrix}$$

is a symplectic isomorphism of V with V^-. So $\varsigma \oplus \mathrm{id}$ gives a symplectic isomorphism of $V^- \oplus V$ with $V \oplus V$.

A generating function for $(\iota \oplus \mathrm{id})(\Gamma)$ will also (by abuse of language) be called a generating function for Γ or for T.

Let us consider the simplest case, where $X = \mathbb{R}$. Then

$$V \oplus V = \mathbb{R} \oplus \mathbb{R}^* \oplus \mathbb{R} \oplus \mathbb{R}^* = T^*(\mathbb{R} \oplus \mathbb{R}).$$

Let (x, y) be coordinates on $\mathbb{R} \oplus \mathbb{R}$ and consider a generating function (of the "horizontal" type) of the form

$$\phi(x, y) = \frac{1}{2}(ax^2 + 2bxy + cy^2),$$

where

$$b \neq 0.$$

Taking into account the transformation ς, the corresponding Lagrangian subspace of $V^- \oplus V$ is given by the equations

$$\xi = -(ax + by), \qquad \eta = bx + cy.$$

$$\phi(x,y) = \frac{1}{2}(ax^2 + 2bxy + cy^2), \qquad b \neq 0.$$

Taking into account the transformation ς, the corresponding Lagrangian subspace of $V^- \oplus V$ is given by the equations

$$\xi = -(ax + by), \qquad \eta = bx + cy.$$

Solving these equations for y, η in terms of x, ξ gives

$$y = -\frac{1}{b}(ax + \xi), \quad \eta = \left(b - \frac{ca}{b}\right)x - \frac{c}{b}\xi.$$

In other words, the matrix (of) T is given by

$$\begin{pmatrix} -\frac{a}{b} & -\frac{1}{b} \\ \\ b - \frac{ca}{b} & -\frac{c}{b} \end{pmatrix}.$$

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(Notice that by inspection the determinant of this matrix is 1, which is that condition that T be symplectic.)

Notice also that the upper right hand corner of this matrix is not zero.

The converse. Going from the matrix to the generating function.

Conversely, starting with a matrix

$$T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

of determinant one, with $\beta \neq 0$ we can solve the equation

$$\begin{pmatrix} -\frac{a}{b} & -\frac{1}{b} \\ \\ b - \frac{ca}{b} & -\frac{c}{b} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

for a, b, c in terms of $\alpha, \beta, \gamma, \delta$. So the most general two by two matrix of determinant one with the upper right hand corner $\neq 0$ is represented by a generating function of the above form.

Addition of generating functions corresponds to composition of symplectic transformation.

Suppose we have two functions

$$\phi_1(x,y) = \frac{1}{2}[ax^2 + 2bxy + cy^2], \quad \phi_2(y,z) = \frac{1}{2}[Ay^2 + 2Byz + Cz^2],$$

with $b \neq 0$ and $B \neq 0$, and consider their sum:

$$\phi(x,z,y) = \phi_1(x,y) + \phi_2(y,z).$$

$$\phi(x,z,y) = \phi_1(x,y) + \phi_2(y,z).$$

Here y is now an “auxiliary variable” in the sense introduced above, so we want to impose the constraint

$$\frac{\partial \phi}{\partial y} = 0, \tag{6}$$

and on this constrained set let

$$\xi = -\frac{\partial \phi}{\partial x}, \quad \zeta = \frac{\partial \phi}{\partial z}, \tag{7}$$

and use these equations to express $\begin{pmatrix} z \\ \zeta \end{pmatrix}$ in terms of $\begin{pmatrix} x \\ \xi \end{pmatrix}$.

$$\phi(x,z,y) = \phi_1(x,y) + \phi_2(y,z).$$

$$\frac{\partial \phi}{\partial y} = 0. \quad (6)$$

Equation (6) gives

$$(A+c)y + bx + Bz = 0. \tag{8}$$

There are now two alternatives:

Case 1: $A + c \neq 0$

Then we can solve

$$(A+c)y + bx + Bz = 0. \quad (8)$$

for y in terms of x and z. This then gives a “horizontal” generating function of the above type (i.e. quadratic in x and z). It is easy to check that the matrix obtained from this generating function is indeed the product of the corresponding matrices. This is an illustration of Hamilton's principle that the composition of two symplectic transformations is given by the sum of their generating functions.
Notice also that because $\partial^2\phi/\partial y^2 = A + c \neq 0$, the effect of

$$\frac{\partial \phi}{\partial y} = 0 \quad (6)$$

was to allow us to eliminate y. This is a general phenomenon.

Case 2: $A + c = 0.$

Then

$$(A+c)y + bx + Bz = 0. \quad (8)$$

imposes no condition on y but does give $bx + Bz = 0$, i.e

$$z = -\frac{b}{B}x$$

which means precisely that the upper right hand corner of the corresponding matrix vanishes.

Since y is now a "free variable", and $b \neq 0$ we can solve the first of equations

$$\xi = -\frac{\partial \phi}{\partial x}, \quad \zeta = \frac{\partial \phi}{\partial z} \qquad (7)$$

$$\phi(x, z, y) = \phi_1(x, y) + \phi_2(y, z).$$

for y in terms of x and ξ giving

$$y = -\frac{1}{b}(\xi + ax)$$

and substitute this into the second of the equations (7) to solve for ζ in terms of x and ξ.

We see that the corresponding matrix is

$$\begin{pmatrix} -\frac{b}{B} & 0 \\ -\frac{aB}{b} - \frac{Cb}{B} & -\frac{B}{b} \end{pmatrix}.$$

Again, this is indeed the product of the correpsonding matrices.

We will now escalate this computation into the language of category theory.

The linear symplectic category.

Over 30 years ago, we introduced the following category: The objects in this category are symplectic vector spaces.

The morphisms between objects V and W in this category are defined as follows: As above, we let V^- denote the vector space V but with the opposite symplectic structure. Then a morphism Γ between V and W is a Lagrangian subspace

$$\Gamma \subset V^- \oplus W.$$

As a *set*, Γ is a relation between V and W. In other words, it is a subset of $V \times W$. Composition in the category is defined as composition of relations. In more detail:

$\Gamma_2 \star \Gamma_1$ and $\Gamma_2 \circ \Gamma_1$.

Let V, W, Z be objects in our category and let

$$\Gamma_1 \subset V^- \oplus W, \quad \Gamma_2 \subset W^- \times Z.$$

Define

$$\Gamma_2 \star \Gamma_1 \subset V \oplus W \oplus Z$$

to consist of all (v, w, z) such that

$$(v, w) \in \Gamma_1 \quad \text{and} \quad (w, z) \in \Gamma_2.$$

Let $\mathrm{pr}_{13} : V \oplus W \oplus Z \to V \oplus Z$ denote projection onto the first and third component. Then define $\Gamma_2 \circ \Gamma_1 \subset V^- \oplus Z$ by

$$\Gamma_2 \circ \Gamma_1 := \mathrm{pr}_{13}(\Gamma_2 \star \Gamma_1).$$

In words: $\Gamma_2 \circ \Gamma_1 \subset V^- \oplus Z$ consists of all (v,z) such that there is a $w \in W$ such that $(v,w) \in \Gamma_1$ and $(w,z) \in \Gamma_2$.

It is clear that $\Gamma_2 \star \Gamma_1$ is a linear subspace of $V \oplus W \oplus Z$ and that $\Gamma_2 \circ \Gamma_1$ is a linear subspace of $V^- \oplus Z$.

It is also easy to see that the restriction of $\omega_Z - \omega_V$ to $\Gamma_2 \circ \Gamma_1$ vanishes. Indeed, if (x,z) and (x',z') are elements of $\Gamma_2 \circ \Gamma_1$, then there are elements w and w' of W such that

$$(x,w) \in \Gamma_1,\ (w,z) \in \Gamma_2,\ (x',w') \in \Gamma_1,\ (',z') \in \Gamma_2.$$

Then

$$\omega_Z(z,z') - \omega_V(x,x') = \omega_Z(z,z') - \omega_W(w,w')$$

$$+\omega_W(w,w') - \omega_V(x,x') = 0.$$

A little bit of linear algebra then shows that $\Gamma_2 \circ \Gamma_1$ has the right dimension.

It is then immediate to check that the axioms for a category are satisfied.

Replacing symplectic by split orthogonal.

The entire discussion remains unchanged if we replace "symplectic" by "split orthogonal". The objects now are even dimensional real vector spaces, each equipped with a symmetric non-degenerate bilinear form of signature (n,n). "Lagrangian" still means an isotropic subspace of dimension n. This category plays a key role in important recent work by Allexeev, Bursztyn, and Meinrenken, see for example their paper "Pure spinors on Lie groups".

But I want to go back 30 years and take the next step in the symplectic setting - the work of Alan Weinstein, where symplectic vector spaces are generalized to symplectic manifolds, and Lagrangian subspaces are replaced by Lagrangian submanifolds.

Why not?

The problem comes with composition. In fact there are two problems: If $M_1,\ M_2,\ M_3$ are symplectic manifolds, and $\Gamma_1 \subset M_1^- \times M_2,\ \Gamma_2 \subset M_2^- \times M_3$ are Lagrangian submanifolds, we would like to define $\Gamma_2 \star \Gamma_1 \subset M_1 \times M_2 \times M_2 \times M_3$ to consist of all (m_1, m_2, m_2, m_3) such that

$$(m_1, m_2) \in \Gamma_1 \quad \text{and} \quad (m_2, m_3) \in \Gamma_2.$$

This means that $\Gamma_2 \star \Gamma_1$ should be the intersection

$$\Gamma_2 \star \Gamma_1 = (\Gamma_1 \times \Gamma_2) \cap (M_1 \times \Delta_{M_2} \times M_3)$$

where $\Delta_{M_2} \subset M_2 \times M_2$ is the diagonal: $\Delta_{M_2} = \{(m_2, m_2)\}$.

But the intersection of two submanifolds need not be a manifold.

Why not? continued.

Even if $\Gamma_2 \star \Gamma_1$ were a submanifold of $M_1 \times M_2 \times M_2 \times M_3$, we are not primarily interested in $\Gamma_2 \star \Gamma_1$. We are interested in $\Gamma_2 \circ \Gamma_1$ which is the image of $\Gamma_2 \star \Gamma_1$ under the map

$$\pi_{13} : M_1 \times M_2 \times M_2 \times M_3 \to M_1 \times M_3, \quad (m_1, m_2, m_2', m_3) \mapsto (m_1, m_3).$$

And the image of a manifold under such a map need not be a submanifold.

The symplectic “category”.

But Alan was not deterred: There must be some technical conditions (transverse or clean intersection, etc.) that must be imposed in order that $\Gamma_2 \circ \Gamma_1$ be a submanifold of $M_1^- \times M_3$. But once these technical conditions are satisfied, then $\Gamma_2 \circ \Gamma_1$ is a Lagrangian submanifold of $M_1^- \times M_3$.

I will not go into the details of these technical conditions. So Alan suggested putting quotation marks about the word category to indicate that composition need not always defined. We will follow this convention.

Exact symplectic manifolds.

Let (M, ω) be a symplectic manifold. It is possible that ω is exact, that is, that $\omega = -d\alpha$ for some one form α. When this happens, we say that (M, α) is an **exact sympletic manifold**. In other words, an exact symplectic manifold is a pair consisting of a manifold M together with a one form α such that $\omega = -d\alpha$ is of maximal rank. The main examples for us, of course, are cotangent bundles with their canonical one forms. Observe that

Proposition

No positive dimensional compact symplectic manifold can be exact.

Indeed, if (M, ω) is a symplectic manifold with M compact, then

$$\int_M \omega^d > 0$$

where $2d = \dim M$ assuming that $d > 0$. But if $\omega = -d\alpha$ then

$$\omega^d = -d\left(\alpha \wedge \omega^{d-1}\right)$$

and so $\int_M \omega^d = 0$ by Stokes' theorem. ☐

Exact Lagrangian submanifolds of an exact symplectic manifold.

Let (M, α) be an exact symplectic manifold and Λ a Lagrangian submanifold of (M, ω) where $\omega = -d\alpha$. Let

$$\beta_\Lambda := \iota_\Lambda^* \alpha \tag{9}$$

where

$$\iota_\Lambda : \Lambda \to M$$

is the embedding of Λ as a submanifold of M. So

$$d\beta_\Lambda = 0.$$

Suppose that β_Λ is exact, i.e. that $\beta_\Lambda = d\psi$ for some function ψ on Λ. (This will always be the case, for example, if Λ is simply connected.) We then call Λ an **exact** Lagrangian submanifold and ψ a choice of **phase function** for Λ.

The conormal bundle.

An instance of this is the conormal bundle of a submanifold: Let X be a differentiable manifold and let $Y \subset X$ be a submanifold. Its **conormal bundle** $N^*Y \subset T^*X$ consists of all $z = (x, \xi)$ such that $x \in Y$ and ξ vanishes on T_xY. Any tangent vector $\zeta \in T_zN^*Y$ has the property that $d\pi_z(\zeta) \in T_xX$ and hence

$$\langle (\alpha_X)_z, \zeta \rangle = \langle \xi,\ d\pi(v) \rangle = 0.$$

In other words, the restriction of α_X to N^*Y is zero.

The exact symplectic "category".

So we have a sub"category" of Weinstein's symplectic "category" whose objects consist of exact symplectic manifolds and whose morphsims are exact Lagrangian submanifolds of

$$M_1^- \times M_2.$$

The canonical relation associated to a map, 1.

Let X_1 and X_2 be manifolds and $f : X_1 \to X_2$ be a smooth map. We set

$$M_1 := T^*X_1 \quad \text{and} \quad M_2 := T^*X_2$$

with their canonical symplectic structures. We have the identification

$$M_1 \times M_2 = T^*X_1 \times T^*X_2 = T^*(X_1 \times X_2).$$

The graph of f is a submanifold of $X_1 \times X_2$:

$$X_1 \times X_2 \supset \mathrm{graph}(f) = \{(x_1, f(x_1))\}.$$

So the conormal bundle of the graph of f is a Lagrangian submanifold of $M_1 \times M_2$. Explicitly,

$$N^*(\mathrm{graph}(f)) = \{(x_1, \xi_1, x_2, \xi_2) | x_2 = f(x_1), \quad \xi_1 = -df^*_{x_1}\xi_2\}.$$

The canonical relation associated to a map, 2.

$$N^*(\mathrm{graph}(f)) = \{(x_1, \xi_1, x_2, \xi_2) | x_2 = f(x_1), \quad \xi_1 = -df^*_{x_1}\xi_2\}.$$

Let

$$\varsigma_1 : T^*X_1 \to T^*X_1$$

be defined by

$$\varsigma_1(x, \xi) = (x, -\xi).$$

Then $\varsigma_1^*(\alpha_{X_1}) = -\alpha_{X_1}$ and hence

$$\varsigma_1^*(\omega_{X_1}) = -\omega_{X_1}.$$

$$\varsigma_1^*(\omega_{X_1}) = -\omega_{X_1}.$$

We can think of this as saying that ς_1 is a symplectomorphism of M_1 with M_1^- and hence

$$\varsigma_1 \times \mathrm{id}$$

is a symplectomorphism of $M_1 \times M_2$ with $M_1^- \times M_2$. Let

$$\Gamma_f := (\varsigma_1 \times \mathrm{id})(N^*(\mathrm{graph}(f)).$$

Then Γ_f is a Lagrangian submanifold of $M_1^- \times M_2$. In other words,

$$\Gamma_f \in \mathrm{Morph}(M_1, M_2).$$

Explicitly,

$$\Gamma_f = \{(x_1, \xi_1, x_2, \xi_2) | x_2 = f(x_1), \quad \xi_1 = df^*_{x_1}\xi_2\}.$$

$$\Gamma_f = \{(x_1,\xi_1,x_2,\xi_2)|x_2 = f(x_1),\quad \xi_1 = df^*_{x_1}\xi_2\}.$$

Suppose that $g : X_2 \to X_3$ is a smooth map so that $\Gamma_g \in \mathrm{Morph}(M_2, M_3)$. So

$$\Gamma_g = \{(x_2,\xi_2,x_3,\xi_3)|x_3 = g(x_2), \xi_2 = dg^*_{x_2}\xi_3.\}.$$

It is easy to check that Γ_g and Γ_f satisfy the technical conditions for composibility.

Their composite $\Gamma_g \circ \Gamma_f$ consists of all (x_1,ξ_1,x_3,ξ_3) such that there exists an x_2 such that $x_2 = f(x_1)$ and $x_3 = g(x_2)$ and a ξ_2 such that $\xi_1 = df^*_{x_1}\xi_2$ and $\xi_2 = dg^*_{x_2}\xi_3$. But this is precisely the condition that $(x_1,\xi_1,x_3,\xi_3) \in \Gamma_{g\circ f}$! So

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Theorem

The assignments

$$X \mapsto T^*X$$

and

$$f \mapsto \Gamma_f$$

define a covariant functor from the category $\mathcal{C}^\infty$ of manifolds and smooth maps to the (exact) symplectic "category" $\mathcal{S}$.

Lagrangian submanifolds as "points".

Another nice idea that Alan introduced is that in the symplectic category, Lagrangian submanifolds should be regarded as "points". This works as follows:

Let us pick a distinguished one element set and call it "pt.". Giving a *map* from pt. to any set X is the same as picking a point of X. So in the category **Set** whose objects are sets and whose morphisms are maps, the points of X are the same as the morphisms from our distinguished object pt. to X.

In a more general category, where the objects are not necessarily sets, we can not talk about the points of an object X. However if we have a distinguished object pt., then we can *define* a **"point"** of any object X to be an element of $\mathrm{Morph}(\mathrm{pt.}, X)$.

The Heisenberg uncertainty principle in symplectic geometry.

For example, in our symplectic "category", let us fix a "zero dimensional symplectic vector space", and call it "$\mathrm{pt.}$". A morphism from $\mathrm{pt.}$ to a symplectic manifold M is a Lagrangian submanifold of $\mathrm{pt.}^- \times M$, which, of course, can be identified with a Lagrangian submanifold of $\Lambda \subset M$. As Alan points out, this can be considered as a manifestation of the Heisenberg uncertainty principle in symplectic geometry.

The pushforward of a Lagrangian submanifold.

For us this has the following consequence: Let M_1 and M_2 be symplectic manifolds, let $\Gamma \in \mathrm{Morph}(M_1, M_2)$ and let Λ be a Lagrangian submanifold of M_1. If we regard Λ as an element of $\mathrm{Morph}(\mathrm{pt.}, M_1)$ and if the technical conditions for composibiity are satisfied, then

$$\Gamma \circ \Lambda \in \mathrm{Morph}(\mathrm{pt.}, M_2).$$

Of course, we can regard $\Gamma \circ \Lambda$ as a Lagrangian submanifold of M_2 which we will then denote as

$$\Gamma(\Lambda)$$

and call it the "pushforward of Λ by Γ".

In particular, let Z and X be differentiable manifolds, let $f : Z \to X$ be a smooth map, and $\Lambda_Z \subset T^*Z$ be a Lagrangian submanifold. If the technical conditions of composibility are satisfied, we can form

$$\Gamma_f(\Lambda_Z)$$

which will be a Lagrangian submanifold of T^*X.

Since Γ_f consists of all $(z, \xi, x, \eta) \in T^*Z \times T^*X$ such that $x = f(z)$ and $\xi = df_x^*(\eta)$, we see that $\Gamma_f(\Lambda_Z)$ consists of all (x, η) such that there is a $(z, \xi) \in \Lambda_Z$ such that $x = f(z)$ and $\xi = df_z^*(\eta)$.

Pushforward under a fibration.

We now specialize to the case where f is a fibration. So we have a fibration $\pi : Z \to X$.
$\Gamma_\pi \in \mathrm{Morph}(T^*Z, T^*X)$ thus consists of all $(z, \xi, x, \eta) \in T^*Z \times T^*X$ such that

$$x = \pi(z) \quad \text{and} \quad \xi = (d\pi_z)^*\eta.$$

Then

$$\mathrm{pr}_1 : \Gamma_\pi \to T^*Z, \quad (z, \xi, x, \eta) \mapsto (z, \xi)$$

maps Γ_π bijectively onto the sub-bundle of T^*Z consisting of those covectors which vanish on tangents to the fibers.

We will call this sub-bundle the **horizontal sub-bundle** and denote it by H^*Z. So at each $z \in Z$, the fiber of the horizontal sub-bundle is

$$H^*(Z)_z = \{(d\pi_z)^*\eta,\ \eta \in T^*_{\pi(z)}X\}.$$

Let Λ_Z be a Lagrangian submanifold of T^*Z which we can also think of as an element of $\mathrm{Morph}(\mathrm{pt.}, T^*Z)$. One of the technical conditions for composibility (for $\Gamma_\pi \star \Lambda_Z$ to be a manifold) is that $\iota : \Lambda_Z \to T^*Z$ and pr_1 intersect cleanly. This is the same as saying that Λ_Z and H^*Z intersect cleanly in which case the intersection

$$F := \Lambda_Z \cap H^*Z$$

is a smooth manifold and we get a smooth map $\kappa : F \to T^*X$.

A more restrictive condition is that intersection be transversal, i.e. that

$$\Lambda_Z \pitchfork H^*Z$$

in which case we always get a Lagrangian immersion

$$F \to T^*X, \quad (z, d\pi_z^*\eta) \mapsto (\pi(z), \eta).$$

The additional composibility condition is that this be an embedding.

Back to generating functions.

Let us specialize further to the case where Λ_Z is a horizontal Lagrangian submanifold of T^*Z. That is, we assume that

$$\Lambda_Z = \Lambda_\phi = \gamma_\phi(Z) = \{(z, d\phi(z))\}.$$

When is

$$\Lambda_\phi \pitchfork H^*Z \quad ?$$

When is

$$\Lambda_\phi \pitchfork H^*Z?$$

Now H^*Z is a sub-bundle of T^*Z so we have the exact sequence of vector bundles

$$0 \to H^*Z \to T^*Z \to V^*Z \to 0$$

where

$$(V^*Z)_z = T_z^*Z/(H^*Z)_z = T_z^*(\pi^{-1}(x)), \quad x = \pi(z)$$

is the cotangent space to the fiber through z.

Any section $d\phi$ of T^*Z gives a section $d_{vert}\phi$ of V^*Z by the above exact sequence, and $\Lambda_\phi \pitchfork H^*Z$ if and only if this section intersects the zero section of V^*Z transversally.

Any section $d\phi$ of T^*Z gives a section $d_{vert}\phi$ of V^*Z by the above exact sequence, and $\Lambda_\phi \pitchfork H^*Z$ if and only if this section intersects the zero section of V^*Z transversally.

If this happens,

$$C_\phi := \{z \in Z | (d_{vert}\phi)_z = 0\}$$

is a submanifold of Z whose dimension is $\dim X$. Furthermore, at any $z \in C_\phi$

$$d\phi_z = (d\pi_z)^*\eta \quad \text{for a unique} \quad \eta \in T^*_{\pi(z)}X.$$

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$$d\phi_z = (d\pi_z)^*\eta \quad \text{for a unique} \quad \eta \in T^*_{\pi(z)}X.$$

Thus if Λ_ϕ and Γ_π are transversally composible then

$$C_\phi \to T^*X, \quad z \mapsto (\pi(z), \eta)$$

is a Lagrangian embedding in which case its image is a Lagrangian submanifold

$$\Lambda = \Gamma_\pi(\Lambda_\phi) = \Gamma_\pi \circ \Lambda_\phi$$

of T^*X. When this happens we say that ϕ is a **transverse generating function of Λ with respect to the fibration** (Z, π). If Λ_ϕ and Γ_π are merely cleanly composible, we say that ϕ is a **clean** generating function with respect to π.

The generating function in local coordinates.

Suppose that X is an open subset of $\mathbb{R}^n$, that

$$Z = X \times \mathbb{R}^k$$

that π is projection onto the first factor, and that (x, s) are coordinates on Z so that $\phi = \phi(x, s)$. Then $C_\phi \subset Z$ is defined by the k equations

$$\frac{\partial \phi}{\partial s_i} = 0, \qquad i = 1, \dots, k.$$

and the transversality condition is that these equations be functionally independent. This amounts to the hypothesis that their differentials

$$d\left(\frac{\partial \phi}{\partial s_i}\right) \qquad i = 1, \dots k$$

be linearly independent.

$C_\phi \subset Z$ is defined by the k equations

$$\frac{\partial \phi}{\partial s_i} = 0, \qquad i = 1, \dots, k.$$

Then $\Lambda \subset T^*X$ is the image of the embedding

$$C_\phi \to T^*X, \quad (x, s) \mapsto \frac{\partial \phi}{\partial x} = d_X\phi(x, s).$$

We see that our pushforward construction does indeed give a generalization of the generating functions that we studied at the beginning of the lecture.

Existence.

Every Lagrangian submanifold has, at every point, a local description in terms of a generating function of the above pushforward type.

Uniqueness. Local generating functions are certainly not unique. But any two generating functions can be obtained from one another by a sequence of elementary operations known as **Hörmander moves** which I will desist from describing.

Recall that if X_1 is a differentiable manifold then we defined

$$\varsigma : T^*X_1 \to T^*X_1$$

by

$$\varsigma(x, \xi) = (x, -\xi).$$

Recall also that Γ is a morphism from T^*X_1 to T^*X_2 if and only if $(\varsigma \times \mathrm{id})(\Gamma)$ is a Lagrangian submanifold of $T^*(X_1 \times X_2)$.

We shall say that ϕ is a generating function for Γ if it is a generating function for the Lagrangian submanifold $(\varsigma \times \mathrm{id})(\Gamma)$ of $T^*(X_1 \times X_2)$.

The generating function for a morphism in local coordinates.

Let

$$\Gamma \in \mathrm{Morph}(T^*X, T^*Y)$$

be a canonical relation, let

$$\pi : Z \to X \times Y$$

be a fibration and ϕ a generating function for Γ relative to this fibration. In local coordinates this says that $Z = X \times Y \times S$, that

$$C_\phi = \{(x, y, s) | \frac{\partial \phi}{\partial s} = 0\},$$

and that Γ is the image of C_ϕ under the map

$$(x, y, s) \mapsto (-d_X\phi, d_Y\phi).$$

The functorial version of Hamilton's idea.

Let X_1, X_2 and X_3 be manifolds and

$$\Gamma_1 \in \mathrm{Morph}(T^*X_1, T^*X_2), \quad \Gamma_2 \in \mathrm{Morph}(T^*X_2, T^*X_3)$$

be transversally composible. So we are assuming in particular that the maps

$$\Gamma_1 \to T^*X_2, \quad (p_1, p_2) \mapsto p_2 \quad \text{and} \quad \Gamma_2 \to T^*X_2, \quad (q_2, q_3) \mapsto q_2$$

are transverse.

$$\Gamma_1 \to T^*X_2, \quad (p_1, p_2) \mapsto p_2 \quad \text{and} \quad \Gamma_2 \to T^*X_2, \quad (q_2, q_3) \mapsto q_2$$

are transverse.

Suppose that

$$\pi_1 : Z_1 \to X_1 \times X_2, \quad \pi_2 : Z_2 \to X_2 \times X_3$$

are fibrations and that $\phi_i \in C^\infty(Z_i), \ i = 1, 2$ are generating functions for Γ_i with respect to π_i.
From π_1 and π_2 we get a map

$$\pi_1 \times \pi_2 : Z_1 \times Z_2 \to X_1 \times X_2 \times X_2 \times X_3.$$

Let

$$\Delta_2 \subset X_2 \times X_2$$

be the diagonal and let

$$Z := (\pi_1 \times \pi_2)^{-1}(X_1 \times \Delta_2 \times X_3).$$

$$\pi_1 \times \pi_2 : Z_1 \times Z_2 \to X_1 \times X_2 \times X_2 \times X_3.$$

$$\Delta_2 \subset X_2 \times X_2$$

is the diagonal and

$$Z := (\pi_1 \times \pi_2)^{-1}(X_1 \times \Delta_2 \times X_3).$$

Finally, let

$$\pi : Z \to X_1 \times X_3$$

be the fibration

$$Z \to Z_1 \times Z_2 \to X_1 \times X_2 \times X_2 \times X_3 \to X_1 \times X_3$$

where the first map is the inclusion map and the last map is projection onto the first and last components.

Hamilton's theorem.

$$Z := (\pi_1 \times \pi_2)^{-1}(X_1 \times \Delta_2 \times X_3).$$

$$\pi : Z \to X_1 \times X_3.$$

Let

$$\phi : Z \to \mathbb{R}$$

be the restriction to Z of the function

$$(z_1, z_2) \mapsto \phi_1(z_1) + \phi_2(z_2).$$

Theorem

ϕ is a generating function for $\Gamma_2 \circ \Gamma_1$ with respect to the fibration $\pi : Z \to X_1 \times X_3$.

$$\phi : Z \to \mathbb{R}$$

is the restriction to Z of the function

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Theorem

ϕ is a generating function for $\Gamma_2 \circ \Gamma_1$ with respect to the fibration $\pi : Z \to X_1 \times X_3$.

Notice that X_2 has now become a factor in the *parameter space*. The function ϕ is given by

$$\phi(x_1, x_3, x_2, s, t) = \phi_1(x_1, x_2, s) + \phi_2(x_2, x_3, t).$$

We saw this in our example of multiplication of two by two matrices.

"Enhancing" the symplectic "category" with half-densities.

Suppose that M_1, M_2, and M_3 are symplectic manifolds, and that

$$\Gamma_2 \in \mathrm{Morph}(M_2, M_3) \text{ and } \quad \Gamma_1 \in \mathrm{Morph}(M_1, M_2)$$

are canonical relations which can be composed. Let ρ_1 be a $\frac{1}{2}$-density on Γ_1 and ρ_2 a $\frac{1}{2}$-density on Γ_2. There is a "follow your nose" way to define a $\frac{1}{2}$-density $\rho_2 \circ \rho_1$ on $\Gamma_2 \circ \Gamma_1$ and to show that the composition

$$(\Gamma_2, \rho_2) \times (\Gamma_1, \rho_1) \mapsto (\Gamma_2 \circ \Gamma_1, \rho_2 \circ \rho_1)$$

is associative when defined, and that the axioms for a "category" are satisfied.

Oscillatory half densities and semi-classical analysis.

Let (Λ, ψ) be an exact Lagrangian submanifold of T^*X. Let $k \in \mathbb{Z}$. One can associate to (Λ, ψ) and to k a space

$$I^k(X, \Lambda, \psi)$$

of rapidly oscillating $\frac{1}{2}$-densities on X. Roughly speaking the assignment goes as follows:

Let $\pi : Z \to X$ be a fibration and ϕ a generating function for (Λ, ψ) with respect to π. We assume that the canonical relation corresponding to π is enhanced in the sense of the preceding slide. It turns out that this enhancement is the same as giving a half density σ along the fibers of π, and hence if τ is a half-density on Z, the product $\sigma\tau$ when restricted to each fiber is a density, which (if compactly supported) can be integrated over the fiber to give a half-density on X.

We define $I^k_0(X, \Lambda, \phi)$ to be the space of all compactly supported $\frac{1}{2}$-densities on X of the form

$$\mu = \hbar^{k-\frac{d}{2}}\pi_*\left(ae^{i\frac{\phi}{\hbar}}\tau\right) \tag{10}$$

where π_* denotes integration over the fiber and $a = a(z, \hbar)$

$$a \in C^\infty_0(Z \times \mathbb{R}).$$

One shows that this class is independent of the choice of generating function (by showing that it is invariant under each of the Hörmander moves) and then obtains a class $I^k(X, \Lambda, \psi)$ of half-densities via a partition of unity.

By our usual device of using $\varsigma \times \mathrm{id}$ one then obtains a class of operators associated to every canonical relation Γ. Hamilton's theorem plays a key role in understand the calculus of such operators.