

# Stability of Symplectic Leaves

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Paulette Libermann :: Héritage et Descendance  
(Institut Henri Poincaré, December 2009)

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Based on the paper:

- M. Crainic and RLF, Stability of symplectic leaves, Preprint arXiv:0810.4437 (to appear in *Inventiones Mathematicae*).

# Flows: Stability of periodic orbits

## Definition

A periodic orbit of a vector field  $X \in \mathfrak{X}(M)$  is called **stable** if every nearby vector field also has a nearby periodic orbit.

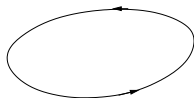
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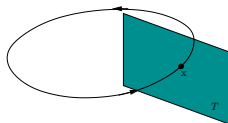


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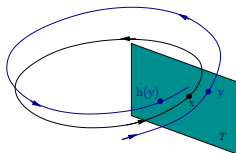


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## Flows: Stability of periodic orbits

Simple assumptions on  $d_x h$  lead to stability:

- If 1 is not an eigenvalue of  $d_x h$  then the orbit is stable.

Consider the representation  $\rho : \mathbb{Z} \rightarrow \mathrm{GL}(\nu(\mathcal{O})_x)$ , defined by

$$n \cdot v := (d_x h)^n v.$$

$H^\bullet(\mathbb{Z}, \nu(\mathcal{O})_x) \equiv$  **group cohomology** with coefficients in  $\nu(\mathcal{O})_x$ .

### Theorem

*Let  $\mathcal{O}$  be a periodic orbit of a vector field  $X$  and assume that*

$$H^1(\mathbb{Z}, \nu(\mathcal{O})_x) = 0.$$

*Then  $\mathcal{O}$  is stable: every nearby vector field has a family of nearby periodic orbits smoothly parametrized by  $H^0(\mathbb{Z}, \nu(\mathcal{O})_x)$ .*

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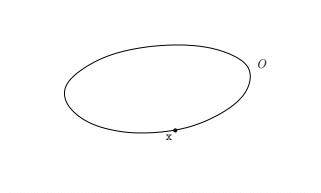
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An orbit  $\mathcal{O}$  of  $\alpha \in \text{Act}(G; M)$  is called **stable** if every nearby action in  $\text{Act}(G; M)$  has a nearby orbit diffeomorphic to  $\mathcal{O}$ .

## Group actions: stability of orbits

The stability of an orbit  $\mathcal{O}$  is controlled by the **isotropy representation**:

- $G_x := \{g \in G : g \cdot x = x\}$  **isotropy group** at  $x \in \mathcal{O}$ .
- $g \in G_x$  induces a map  $\alpha_g : M \rightarrow M, y \mapsto g \cdot y$  that fixes  $x$  and maps the orbit to the orbit.

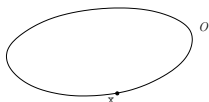


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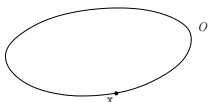


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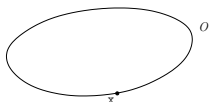


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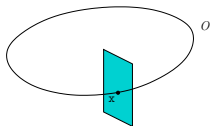


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$$\Rightarrow \rho(g) : \nu(\mathcal{O})_x \rightarrow \nu(\mathcal{O})_x.$$

## Group actions: stability of orbits

Linear normal isotropy representation:

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$H^\bullet(\mathbf{G}_x, \nu(\mathcal{O})_x)$  denotes the corresponding **group cohomology**.

Theorem (Hirsch,Stowe)

*Let  $\mathcal{O}$  be a compact orbit and assume that*

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Fix a manifold  $M$  and denote by  $\text{Fol}_q(M)$  the set of codimension  $q$  foliations. Frobenius says:

$$\text{Fol}_q(M) \longleftrightarrow \{D : M \rightarrow \text{Gr}_q(TM) \mid D \text{ is involutive}\}$$

$\implies \text{Fol}_q(M)$  has a **natural  $C^r$ -topology**

## Definition

A leaf  $L$  of a foliation  $\mathcal{F} \in \text{Fol}_k(M)$  is called **stable** if every nearby foliation in  $\text{Fol}_k(M)$  has a nearby leaf diffeomorphic to  $L$ .

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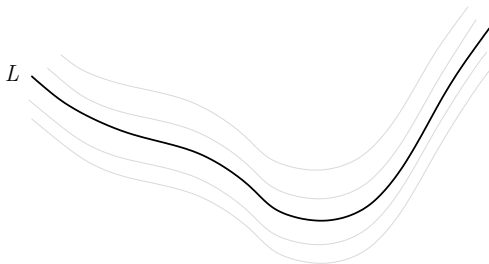
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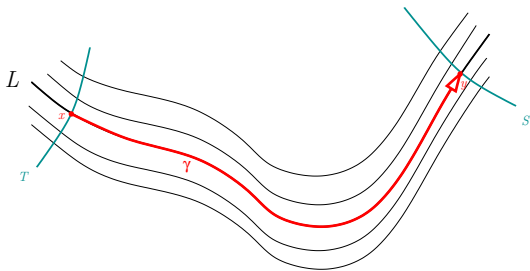
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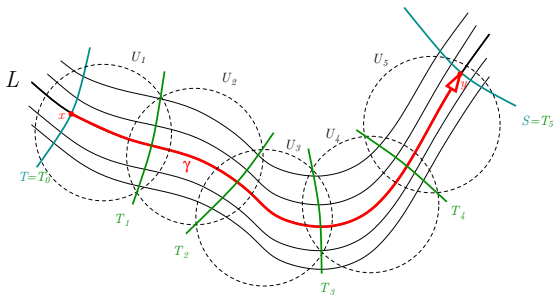
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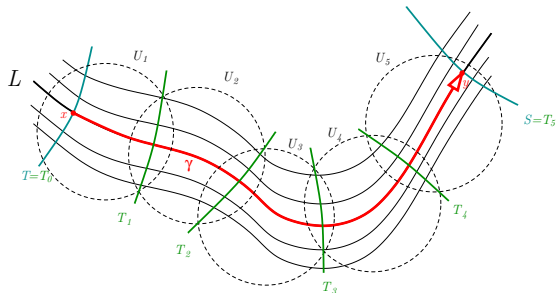
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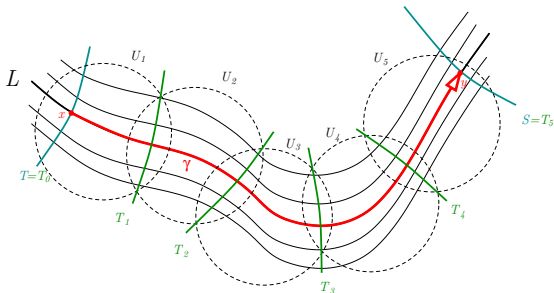


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**Note:** The Poincaré return map is a special case of this. ⏪ ⏩ ⏴ ⏵ 🔍 ↺

# Foliations: stability of leaves

Differentiating gives the **linear holonomy representation**:

$$\rho : \pi_1(L, x) \rightarrow GL(\nu(L)_x), \quad \rho := d_x \circ \text{Hol}$$

$H^*(\pi_1(L, x), \nu(L)_x)$  denotes corresponding **group cohomology**.

Theorem (Reeb, Thurston, Langevin & Rosenberg)

*Let  $L$  be a compact leaf and assume that*

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## Stability of leaves versus stability of orbits

In general, the **two theorems** are quite **different** (e.g., dimension of orbits of actions can vary).

However:

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Then the linear holonomy coincides with the linear isotropy representation, and the theorem for actions follows from the theorem for foliations.

# Poisson structures

For Poisson bracket  $\{ , \}$  we denote by  $\pi$  the associated **Poisson bivector**:

$$\pi(df, dg) := \{f, g\}.$$

$$\text{Poiss}(M) \longleftrightarrow \{ \pi : M \rightarrow \wedge^2(TM) \mid [\pi, \pi] = 0 \}.$$

$\implies$   $\text{Poiss}(M)$  has a **natural  $C^r$  topology**

## Definition

A symplectic leaf  $S$  of  $\pi \in \text{Poiss}(M)$  is called:

- **stable** if every nearby Poisson structure in  $\text{Poiss}(M)$  has a nearby leaf diffeomorphic to  $S$ .
- **strongly stable** if every nearby Poisson structure in  $\text{Poiss}(M)$  has a nearby leaf symplectomorphic to  $S$ .

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$\implies$   $\text{Poiss}(M)$  has a **natural  $C^r$  topology**

## Definition

A symplectic leaf  $S$  of  $\pi \in \text{Poiss}(M)$  is called:

- **stable** if every nearby Poisson structure in  $\text{Poiss}(M)$  has a nearby leaf diffeomorphic to  $S$ .
- **strongly stable** if every nearby Poisson structure in  $\text{Poiss}(M)$  has a nearby leaf symplectomorphic to  $S$ .

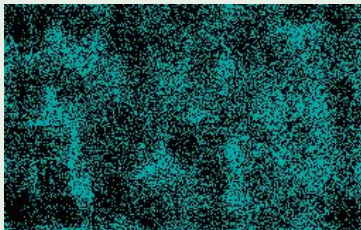


## Examples (Constant and linear Poisson structures)

■  $M = \mathbb{R}^2$  with  $\pi = 0$

→

$$\pi_\varepsilon = \varepsilon \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$$

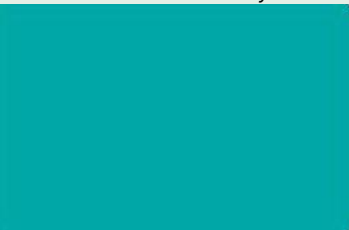
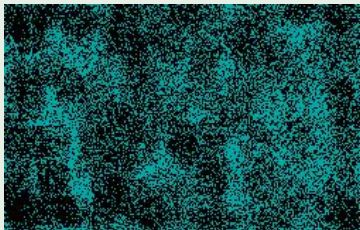


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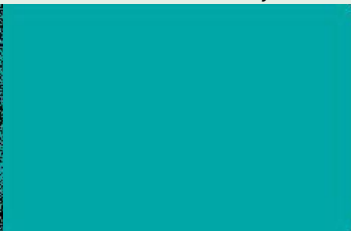
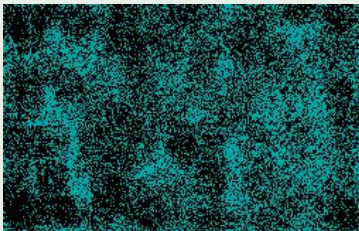
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## Examples (Constant and linear Poisson structures)

■  $M = \mathbb{R}^2$  with  $\pi = 0$   $\longrightarrow$   $\pi_\varepsilon = \varepsilon \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$



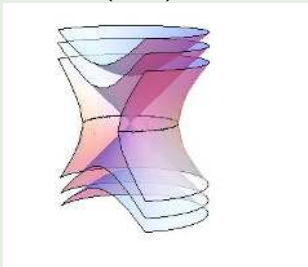
$\implies$  no leaves are stable.

## Examples (Constant and linear Poisson structures)

- $M = \mathfrak{sl}^*(2, \mathbb{R}) \simeq \mathbb{R}^3$

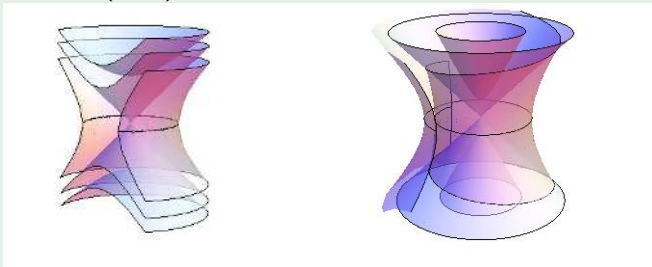
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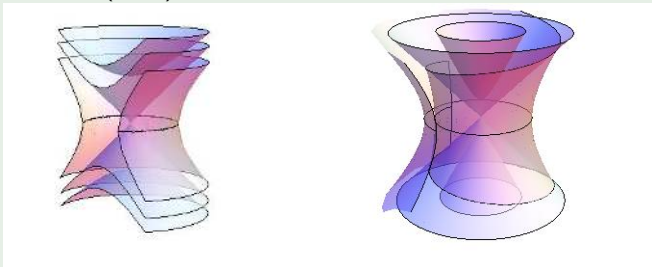
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⇒ some leaves are stable.

## Examples (Constant and linear Poisson structures)

- $M = \mathfrak{su}^*(2) \simeq \mathbb{R}^3$

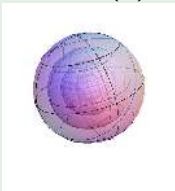
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$\implies$  all leaves are stable.

# Poisson cohomologies

Stability of a symplectic leaf  $S$  of a Poisson manifold  $(M, \pi)$  is controlled by certain cohomologies:

- **Poisson cohomology:**  $H_\pi^\bullet(M)$  is the cohomology of  $(\mathfrak{X}^k(M), d_\pi)$ , the complex of multivector fields with  $d_\pi := [\pi, \ ]$ .
- **Restricted Poisson cohomology:**  $H_{\pi,S}^\bullet(M)$  is the cohomology of  $(\mathfrak{X}_S^\bullet(M), d_\pi|_S)$ , the complex of multivector fields along  $S$ .
- **Relative Poisson cohomology:**  $H_\pi^\bullet(M, S)$  is the cohomology of the quotient complex:

$$\mathfrak{X}^\bullet(M, S) := \mathfrak{X}_S^\bullet(M) / \Omega^\bullet(S)$$

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# Stability

## Theorem (Crainic & RLF)

*Let  $S$  be a compact symplectic leaf and assume that*

$$H_{\pi}^2(M, S) = 0.$$

*Then  $S$  is stable: every nearby Poisson structure has a family of nearby diffeomorphic leaves smoothly parametrized by  $H_{\pi}^1(M, S)$ .*

- The relative Poisson cohomology  $H_{\pi}^*(M, S)$  is typically finite dimensional.
- $H_{\pi}^1(M, S)$  coincides with the space of leaves of the first jet approximation  $j_S^1\pi$  which project diffeomorphically to  $S$ .
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## Theorem (Crainic & RLF)

*Let  $S$  be a compact symplectic leaf and assume that*

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*Then  $S$  is strongly stable: every nearby Poisson structure has a family of nearby symplectomorphic leaves smoothly parametrized by the image of  $\Phi : H_{\pi, S}^1(M) \rightarrow H_{\pi}^1(M, S)$ .*

- The parameter space of strongly stable leaves is a subspace of the parameter space for stable leaves.
- It coincides with the space of leaves of  $j_S^1\pi$  which project diffeomorphically to  $S$  and which have symplectic form isotopic to  $\omega_S$ .

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The condition for strong stability (i.e.,  $H_{\pi, S}^2(M) = 0$ ) **does not** imply the condition for stability ( $H_{\pi}^2(M, S) = 0$ ).

### Conjecture

*If  $S$  is a compact symplectic leaf and the map*

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## Necessary conditions for stability

A Poisson structure  $\pi$  is said to be of **first order** around a symplectic leaf  $S$  if  $\pi$  is Poisson diffeomorphic to  $j_S^1\pi$  in some neighborhood of  $S$ .

### Theorem (Crainic & RLF)

*Let  $\pi$  be a Poisson structure which is of first order around a compact symplectic leaf  $S$ . Then:*

- (i) If  $S$  is stable, then the map  $\Phi : H_{\pi,S}^2(M) \rightarrow H_{\pi}^2(M, S)$  vanishes.*
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## Example: dual of a Lie algebra

$M = \mathfrak{g}^*$  where  $\mathfrak{g}$  is a **compact semi-simple** Lie algebra:

- Symplectic leaves are the coadjoint orbits:  $S = \mathcal{O}_\xi$ ;
- All leaves satisfy criteria for stability and strong stability:  
 $H_{\pi, S}^2(M) = H_\pi^2(M, S) = 0$ .
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**Conclusion:** Every nearby Poisson structure has a one nearby symplectic leaf symplectomorphic to  $S$  and a  $\dim Z(\mathfrak{g}_\xi)$ -family of symplectic leaves diffeomorphic to  $S$ .

## Basic problem

- Is there a **general setup** to deal with stability problems?

To answer this question one requires a setup where geometric objects such as flows, actions, foliations, Poisson structures, etc., are all on equal footing.

⇒ Lie groupoids/algebroids

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# Lie algebroids

## Definition

A **Lie algebroid** is a vector bundle  $A \rightarrow M$  with:

- (i) a Lie bracket  $[ , ]_A : \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A)$ ;
- (ii) a bundle map  $\rho : A \rightarrow TM$  (the **anchor**);

such that:

$$[\alpha, f\beta]_A = f[\alpha\beta]_A + \rho(\alpha)(f)\beta, \quad (f \in C^\infty(M), \alpha, \beta \in \Gamma(A)).$$

$\text{Im } \rho \subset TM$  is a integrable (singular) distribution



Lie algebroids have a **characteristic foliation**

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## Examples

- **Flows.** For  $X \in \mathfrak{X}(M)$ , the associated Lie algebroid is:

$$A = M \times \mathbb{R}, \quad [f, g]_A := fX(g) - gX(f), \quad \rho(f) = fX.$$

Leaves of  $A$  are the orbits of  $X$ .

- **Actions.** For  $\alpha \in \text{Act}(G; M)$ , the associated Lie algebroid is:

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- **Foliations.** For  $\mathcal{F} \in \text{Fol}_k(M)$ , the associated Lie algebroid is:

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# Universal Stability Theorem

- For a fixed vector bundle  $A$  there is a natural  $C^r$  topology on the set  $\mathbf{Algbrd}(A)$  of Lie algebroid structures on  $A$ .
- A leaf  $L$  of  $A$  is called **stable** if every nearby Lie algebroid structure in  $\mathbf{Algbrd}(A)$  has a nearby leaf diffeomorphic to  $L$ .
- The normal bundle  $\nu(L)$  carries a canonical Bott type  $A|_L$ -connection.
- One can define the restricted  $A$ -cohomology with coefficients in  $\nu(L)$ , denoted  $H^\bullet(A|_L; \nu(L))$ .

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## Theorem (Crainic & RLF)

*Let  $L$  be a compact leaf of  $A$ , and assume that  $H^1(A|_L; \nu(L)) = 0$ . Then  $L$  is stable: every nearby Lie algebroid has a family of nearby leaves smoothly parametrized by  $H^0(A|_L; \nu_L)$ .*

- The parameter space be characterized as the space of leaves of the **first jet approximation** to the Lie algebroid  $A$  along  $L$ .
- For Lie algebroid structures of **first order type** around  $L$  the condition in the theorem is also a necessary condition for stability.

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## From Lie algebr(oids) to Lie group(oids)

The **fundamental group** of  $A$  based at  $x$ :

$$G_x(A) = \frac{\{A\text{-loops based at } x\}}{A\text{-homotopies}}.$$

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$$\text{hol} : G_x(A) \longrightarrow GL(\nu(L)_x).$$

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If  $G_x(A)$  is smooth, then for any  $x \in L$ :

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# Poisson vs algebroid stability

## Examples

$M = \mathfrak{g}^*$  with  $\mathfrak{g}$  the non-abelian 2-dimensional Lie algebra:

$$\pi = x \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$$

so that  $A = T^*\mathbb{R}^2$  and

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- There are Poisson structures  $\pi$  such that  $A = T^*M$  has many closeby Lie algebroids which **are not** associated with a Poisson structure.
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