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Preface

The school, the book

This book is based on lectures given by the authors of the various chapters in a three week long CIMPA summer school, held in Sophia-Antipolis (near Nice) in July 1992.

The first week was devoted to the basics of symplectic and Riemannian geometry (Banyaga, Audin, Lafontaine, Gauduchon), the second was the technical one (Pansu, Muller, Duval, Lalonde and Sikorav). The final week saw the conclusion of the school (mainly McDuff and Polterovich, with complementary lectures by Lafontaine, Audin and Sikorav).

Globally, the chapters here reflect what happened there. Locally, we have tried to reorganise some of the material to make the book more coherent. Hence, for instance, the collective (Audin, Lalonde, Polterovich) chapter on Lagrangian submanifolds and the appendices added to some of the chapters.

Duval was not able to write up his lectures, so that genuine complex analysis will not appear in the book, although it is a very current tool in symplectic and contact geometry (and conversely). Hamiltonian systems and variational methods were the subject of some of Sikorav's talks, which he also was not able to write up. On the other hand, F. Labourie, who could not be at the school, wrote a chapter on pseudo-holomorphic curves in Riemannian geometry.

The aim and sources of the book

When we planned the summer school (during the Souriau conference of 1990), our aim was to understand in some detail the techniques and results of Gromov's 1985 paper[1]. It is a long, but very concise paper, with a lot of hard techniques, some beautiful tricks, and many spectacular results (see our introductory chapter for a small panorama of these).

Some of the authors of the present book had already written good—often very good—papers explaining such or such a page or result of Gromov, which papers, for some obscure reasons, have been lying dormant in some (necessarily obscure) drawer since 1986 or 1987. Some of these underground papers are already classics,

[1] *Pseudo-holomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), 307–347.

for instance those of Pansu[2] and Sikorav[3]. Muller, Pansu and Labourie used their respective clandestine papers as a basis for their written contribution here. Others preferred to have theirs plundered by someone else: "Corollaires symplectiques" in chapter X and Duval's very beautiful text[4] in chapters III and VIII. We hope this book will become the "official" reference replacing these clandestine papers.

Some of the contributions are really new, for example McDuff's chapter VI on the positivity of intersections.

We have tried to make this book accessible to graduate students: introductory and more or less self contained (hence the chapters I to IV, where she or he will nevertheless find some nonstandard material); with the necessary examples and techniques needed to understand the applications given in the book, for instance in the first and last chapters, but also those *not* given in the book, as in the beautiful papers of McDuff on symplectic 4-manifolds, and of course, those in all forthcoming (hopefully) papers. For the same purpose, we have included numerous exercises in some of the chapters.

The students and the question of language

In keeping with its vocation, the CIMPA organised the school for students of different ages and/or levels coming from various (mainly developing) countries, who very bravely survived the techniques, the hot weather and the accomodation in Sophia: even at the very end of the school, they were still very enthusiastically discussing the contents of the many two hours (and more) lectures of Dusa McDuff.

It was even possible to overcome the language problems: a lot of the students came from the so-called French-speaking African countries—this just means that French was already their second language and that their English was very poor. Some came from various Asian countries—same problem, just interchange French and English in the previous sentence. To tell the truth, we also had a Bilingual Canadian.

Everybody made some kind of effort and communication was indeed very good. The more unusual experiments we witnessed were, an Israeli lecturer discussing exotic structures in Russian with a Vietnamese student, and, even more exotic, a Rwandese professor explaining his lectures to a student from Burundi, in Kinyarwanda of course (yes, it works)!

After this, the reader will probably appreciate finding not only English in the book: two of the chapters are in French, none in Hebrew nor Alsatian. Moreover, the English of all contributors (except Dusa, of course) was checked by an English mathematician, Marcus Slupinski, thanks to whom the book is not completely gibberish.

[2] *Notes sur les pages* 316 *à* 323 *de l'article de M. Gromov...*
[3] *Corollaires symplectiques.*
[4] *Compacité des J-courbes : le cas simple des J-droites d'un pseudoplan.*

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Michèle Audin and Jacques Lafontaine
December 3, 1993

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Contents

Introduction

Applications of pseudo-holomorphic curves to symplectic topology

Jacques Lafontaine Michèle Audin

This chapter is an introduction to the book. First we will describe some problems in symplectic geometry, or more exactly topology, and the way to solve them using pseudo-holomorphic curves techniques. Then we describe very roughly the contents of the book. For the basic results in geometry, the reader can consult chapters I, II or III.

Unless otherwise mentioned (i.e. apart from Darboux's theorem), all theorems in this chapter are due to Gromov and come from [9] (see also [10]).

1. Examples of problems and results in symplectic topology

1.1. Flexibility and rigidity in symplectic manifolds

A *symplectic manifold* (W, ω) is a manifold W endowed with a closed nondegenerate 2-form ω. The basic examples are $\mathbf{R}^{2n}$ with the 2-form $\omega_0 = \sum dp_i \wedge dq_i$ and the 2-sphere with its usual volume form (or any orientable surface with any volume form).

Locally, such a structure is very flexible: all such structures are isomorphic. More precisely, if we call a diffeomorphism *symplectic* when it preserves the symplectic structures, the basic form of Darboux's theorem (see chapter I) is:

THEOREM 1.1.1 (Darboux). — *Let (W, ω) be a symplectic manifold of dimension $2n$. Let x be any point in W. There exists a neighborhood U of x and a symplectic diffeomorphism of $\left(U, \omega_{|U}\right)$ into an open subset of $(\mathbf{R}^{2n}, \omega_0)$.*

This shows that there exists no *local* invariant in symplectic geometry like, for instance, the curvature in Riemannian geometry: locally, all symplectic manifolds are alike, and there is no difference between suitable open subsets of the 2-sphere and the plane[1].

[1]This allows us to draw local maps of the earth in which the areas are exact if not the lengths and angles.

Of course, globally, things are quite different. For instance, if W is a $2n$-manifold, $\omega^{\wedge n}$ is a volume form (this is the nondegeneracy condition) and thus, any symplectic diffeomorphism preserves the volume: there is no symplectic diffeomorphism from a large ball into a smaller one in $\mathbf{R}^{2n}$.

A basic question is thus to understand if there is a way to distinguish between volume preserving and symplectic diffeomorphisms. Note that "volume geometry" is very flexible, even globally. On the one hand, we have the Moser theorem, which asserts that two volume forms on a closed manifold are diffeomorphic when they give the same total volume to the manifold. On the other hand, if U and V are connected open subsets of $\mathbf{R}^n$ with $\mathrm{vol}(U) < \mathrm{vol}(V)$, there is an embedding $U \to V$ which preserves the volume form. In contrast, the first result we quote in symplectic geometry is a very spectacular rigidity theorem:

THEOREM 1.1.2. — *Let (W, ω) be a symplectic manifold, and let f be a diffeomorphism of W which is a limit, in the compact-open (C^0) topology, of a sequence of symplectic diffeomorphisms. Then f is symplectic.*

Thus a volume preserving diffeomorphism which is not symplectic cannot be a limit of symplectic diffeomorphisms. According to Arnold, who invented the phrase "symplectic topology", this theorem of Gromov proves the existence of symplectic topology [1].

In the same mood, Gromov defined a *symplectic invariant*, much finer than the volume, the *width*, which prevents, for instance, the embedding of a large ball into an infinite thin cylinder. We consider a ball B_R^{2n+2} of radius R in the standard symplectic space $\mathbf{R}^{2n+2} = \mathbf{R}^2 \oplus \mathbf{R}^{2n}$ and a ball B_r^2 of radius r in the *symplectic* summand $\mathbf{R}^2$:

THEOREM 1.1.3. — *Suppose there exists a symplectic embedding of B_R^{2n+2} into the cylinder $B_r^2 \times \mathbf{R}^{2n}$. Then $R \leq r$.*

Similarly, one cannot embed two disjoint small balls in a large one (not too large, of course, but with volume bigger than the sum of the volumes of the small balls):

THEOREM 1.1.4. — *Let U be an open subset of $\mathbf{R}^{2n}$ which contains two disjoint balls of radii r_1 and r_2. Suppose there exists a symplectic embedding of U into a ball of radius R. Then $r_1^2 + r_2^2 \leq R^2$.*

Remarks.

1. C. Viterbo likes to present theorem 1.1.3 as an "uncertainty principle"... in classical mechanics [25].

2. The inequality in theorem 1.1.4 is called a *packing inequality*. For beautiful results and a discussion of packing inequalities, see [17].

3. Another famous and related problem is that of the *symplectic camel*, whose ability to pass through the eye of a needle is discussed, following St Luke[2] **18** 25, in e.g. [19], [7] and [18].

1.2. Flexibility and rigidity for Lagrangian submanifolds

The next flexibility result is Gromov's h-principle, applied here to Lagrangian immersions. Recall that an immersion $f : L \to W$ is *Lagrangian* if $f^\star\omega = 0$ and $\dim L = \frac{1}{2}\dim W$. From such an f, we can extract two topological data: the homotopy class of the map f (such that $f^\star\omega = 0$) and the Lagrangian subbundle TL of $f^\star TW$. The h-principle (see [11] and chapter X for interesting special cases and other references) asserts that the map which associates these two data with f is a weak homotopy equivalence: everything which is not excluded by homotopy theory is allowed.

Using pseudo-holomorphic curves methods, Gromov proved that the situation for Lagrangian embeddings is quite different. A typical consequence is the proof of the Arnold conjecture. Let $\lambda_0 = \sum p_i dq_i$ be the Liouville form on $\mathbf{R}^{2n}$, so that $\omega_0 = d\lambda_0$. Note that $f^\star\omega_0 = 0 \Leftrightarrow f^\star\lambda_0$ is a closed 1-form, and call *exact* any Lagrangian immersion f such that $f^\star\lambda_0$ is an exact 1-form.

THEOREM 1.2.1 (Arnold conjecture). — *There exists no closed exact Lagrangian submanifold in* $\mathbf{R}^{2n}$.

For $n = 1$, this is a consequence of the theorems of Stokes and Jordan. In higher dimensions, this rigidity theorem implies the existence of *exotic* symplectic structures on $\mathbf{R}^{2n}$ ($2n \geq 4$); it is also related to problems of intersections of Lagrangian submanifolds and of fixed points of symplectic diffeomorphisms: any symplectic diffeomorphism of (W, ω) defines a Lagrangian submanifold, namely its graph in $W \times W$ endowed with the symplectic form $\omega \oplus -\omega$.

All of these things will be discussed in great detail in chapter X, so we shall not spend more time on them here. However, we must mention that a lot of activity related to the above conjecture of Arnold—which actually belongs to a whole set of conjectures, stated by Arnold in the sixties—began a long time before theorem 1.2.1 was proved, and took its roots in "Poincaré's last geometric theorem" (or Poincaré-Birkhoff theorem). Once again, we refer the reader to the paper of Arnold [1], and also of Chaperon [3] and to chapter X and the references therein.

2. Pseudo-holomorphic curves in almost complex manifolds

Although some rigidity results can be proved by variational methods, we will emphasise here methods involving pseudo-holomorphic curves, as initiated by Gro-

[2]See also Mat. **19** 24, Mk **10** 25.

mov in [9]. It might seem surprising, but we know of no other way to prove the Arnold conjecture or the existence of exotic structures.

Basically, what happens is the following:

1. Given a symplectic manifold (W, ω), there are plenty of *tame almost complex structures*, that is: sections of $\mathrm{End}(TW)$ such that $J^2 = -\mathrm{Id}$ and such that $\omega(x, Jx)$ is positive on $TW - 0$. But (except in the case of a Kähler structure), there is no natural choice for such a J. The basic idea of Gromov is to consider the whole family of these tame almost complex structures.

2. For a generic almost complex structure, there are no holomorphic functions (even locally), but there are many local pseudo-holomorphic curves. Moreover, many classical properties of ordinary holomorphic curves still hold: see § 2.2 below and chapters V and VI respectively for analytical and geometric properties.

3. In many cases, if there are compact pseudo-holomorphic curves for *some* almost complex structure J_0 tamed by a symplectic form ω, there are such curves for *any* tame symplectic structure J. For instance, the properties of pseudo-holomorphic curves in $\mathbf{CP}^2$ or $\mathbf{CP}^1 \times \mathbf{CP}^1$ for any almost complex structure tamed by the usual (Kähler) symplectic form mimic the classical properties of algebraic curves (holomorphic for the usual complex structure): see theorems 2.2.3 and 2.2.4 below.

2.1. Almost complex structures and pseudo-holomorphic curves

As we mentioned above, an *almost complex* structure on a manifold W is a field of endomorphisms J (a section of $\mathrm{End}(TW)$) such that $J^2 = -\mathrm{Id}$. For example, a complex manifold carries a natural almost complex structure. Namely, if $(z_1, \ldots, z_n)$ is a local holomorphic chart on an open set U and if we write $z_k = x_k + iy_k$, J is given by

$$J\frac{\partial}{\partial x_k} = \frac{\partial}{\partial y_k} \text{ and } J\frac{\partial}{\partial y_k} = -\frac{\partial}{\partial x_k}.$$

However, "most" almost complex structures are not defined by a complex (or holomorphic, or analytic, or *integrable*) structure (see chapter II). In other words, an almost complex structure J is usually not integrable. To see this, define *holomorphic functions* as smooth functions $f : W \to \mathbf{C}$ such that $T_x f \circ J = i T_x f$. For a generic almost complex structure, there are *no* holomorphic functions, even locally.

On the contrary, there are always many pseudo-holomorphic *curves*. A *pseudo-holomorphic curve* f is a map from a Riemann surface S into W such that $J \circ Tf_x = Tf_x \circ i$ (the same compatibility condition as above, but now W is the target space). In local coordinates, this amounts to the differential system

$$\frac{\partial f_j}{\partial y} = \sum_{k=1}^{2n} J_{j,k}(f)\frac{\partial f_k}{\partial x} \qquad (1 \leq j \leq 2n)$$

of $2n$ equations for $2n$ unknown functions.

If J is analytic, the Cauchy-Kowalevskaya theorem implies that there are (local) analytic solutions. In the smooth case, this is still true, since the differential system is a quasi-linear elliptic system: the system is linear with respect to higher derivatives; when linearised, it gives the usual Cauchy-Riemann equations, which are elliptic. Therefore, pseudo-holomorphic curves enjoy the properties which are expected of the solutions of an elliptic system: regularity properties coming from Sobolev and/or Hölder estimates (chapter V) and also a removable singularity theorem (chapter VII).

To save on typing, we shall often simply write a *J-curve* for a pseudo-holomorphic curve.

Now, any symplectic manifold will have a lot of complex structures, related to the symplectic form in the following way: an almost complex structure J on a manifold W is *tamed* by a symplectic form ω if

$$\omega(X, JX) > 0 \quad \forall X \in TW - 0;$$

if moreover ω is J-invariant, J is said to be *calibrated.*

PROPOSITION 2.1.1. — *The space of almost complex structures on a given symplectic manifold (W,ω) which are tamed (resp. calibrated) by ω is nonempty and contractible. In particular, these spaces are connected.*

See chapter II for details and for a proof of the proposition. Note that, if J is an ω-tame almost complex structure, then

$$\mu(X,Y) = \omega(X, JY) - \omega(JX, Y)$$

is a J-invariant Riemannian metric.

The following elementary property of tame structures turns out to be crucial when studying pseudo-holomorphic curves.

PROPOSITION 2.1.2. — *Let C be a pseudo-holomorphic curve in a closed symplectic manifold (W,ω) equipped with a tame almost complex structure, and let μ be a J-invariant metric. There is a universal bound*

$$\operatorname{area}_\mu(C) \leq A(\omega, J, \mu, [C])$$

involving the geometry of W and the homology class $[C]$ of C.

Proof. — Using compactness, we see there exists some constant $M > 0$ such that

$$\mu(X, JX) \leq M\omega(X, JX) \quad \forall X \in TW.$$

Then

$$\text{area}_\mu(C) = \int_C v_\mu \leq M \int_C \omega.$$

But

$$\int_C \omega = \langle [\omega], [C] \rangle . \square$$

An important consequence of this easy proposition is that a closed pseudo-holomorphic curve is never a boundary.

Remark. — This result is a coarse generalisation of a classical property of compact Kähler manifolds, namely: any Kähler submanifold is volume minimising in its homology class (Wirtinger inequality, see chapter III).

Another basic property of holomorphic curves persists in the pseudo-holomorphic case, namely positivity of intersections. Indeed, one easily sees that, when two smooth pseudo-holomorphic curves in a 4 dimensional almost complex manifold (W, J) meet transversally, the intersection of their homology classes is positive as in the holomorphic case. M. Gromov pointed out and D. McDuff proved (see chapter VI) that the same positivity property holds without the transversality or smoothness assumptions.

2.2. Some existence theorems for pseudo-holomorphic curves

Gromov discovered a general procedure for proving the existence of J-holomorphic curves with certain properties when J is a tame almost complex structure on a manifold which carries some model structure J_0 for whose holomorphic curves the analogous properties are well understood.

Kähler structures. — A good "model" structure might be a Kähler structure: a *Kähler metric* on a manifold W is given by a Riemannian metric g, an integrable complex structure J compatible with J (i.e. J is an isometry of g), such that the skew form

$$\omega(X, Y) = g(JX, Y)$$

is closed. The form ω is then called *the Kähler form*. The condition $\omega(X, JX) > 0$ forces the nondegeneracy of ω, thus a Kähler form is symplectic and tames the complex structure. For more details, see e.g. [20], [13] and chapter II.

There are some basic examples. The first one is, of course, $\mathbf{R}^{2n}$ identified with $\mathbf{C}^n$ and equipped with its standard metric and complex structure. The Kähler form is just the standard symplectic form. Also, in dimension 2, every J is integrable (see chapter II for a discussion and references) and giving J is equivalent, once an orientation is prescribed, to giving a (pointwise) conformal class of Riemannian metrics: Kähler 2-dimensional manifolds are just Riemann surfaces.

The projective space. — The next example is the complex projective space $\mathbf{CP}^n$. Consider the Hopf fibration

$$h : S^{2n+1} \to \mathbf{CP}^n$$

which assigns to $Z = (z_0, \ldots, z_n) \in S^{2n+1} \subset \mathbf{C}^{n+1}$ the point with homogeneous coordinates Z in $\mathbf{CP}^n$. In other words, $h(Z)$ is the line in $\mathbf{C}^{n+1}$ generated by the vector Z, and $\mathbf{CP}^n$ is obtained as the quotient of S^{2n+1} by the S^1-action $(u, Z) \mapsto uZ$. There is a natural Riemannian metric on $\mathbf{CP}^n$: take the standard (induced) metric on the sphere, for which the S^1-action is by isometries, and decide that h is a Riemannian submersion.

2.2.1. Exercise. — Use Fubini's theorem to show that the volume of S^{2n+1} is $2\pi \operatorname{vol}(\mathbf{CP}^n)$. Compute $\operatorname{vol}(\mathbf{CP}^n)$ and show in particular that $\operatorname{vol}(\mathbf{CP}^1) = \pi$. Deduce that, for $n = 1$, the procedure gives the 2-sphere, equipped with a metric of constant curvature 4. Give another (direct) proof of this last fact.

Now the normal space to the sphere at Z is the real line generated by Z and the tangent space to the S^1-orbit is the real line generated by iZ, so that, taking quotients, we get an almost complex structure on $\mathbf{CP}^n$, which is, of course, the one associated with the analytic structure.

Now, let α be the 1-form on S^{2n+1} given by $\alpha_Z(X) = \langle JZ, X\rangle$. Then $d\alpha = h^\star\omega$, where ω is the 2-form on $\mathbf{CP}^n$ given by

$$\omega(X, Y) = g(JX, Y).$$

Thus we have all the necessary structures and the following exercise is straightforward.

2.2.2. Exercise. — Show that g is a Kähler metric.

Remark. — This way of describing the 2-form ω is an example of the basic *symplectic reduction* process, see chapter II.

Curves in "pseudo"-projective spaces. — We can now state two results which are derived from the basic examples of $\mathbf{CP}^2$ and $\mathbf{CP}^1 \times \mathbf{CP}^1$ equipped with the Kähler structures just described[3].

THEOREM 2.2.3. — *Let J be an almost complex structure tamed by the (standard) symplectic form ω on $\mathbf{CP}^2$. Then*

a) If x and y are two distinct points, there exists one and only one J-curve homologous to $\mathbf{CP}^1 \subset \mathbf{CP}^2$ (in other words: of degree 1) through them.

b) If $(x_1, \ldots, x_5)$ are five points such that no three of them lie on a degree 1 J-curve, there is one and only one degree 2 J-curve through them.

In other words, the situation of "pseudo-lines" or "pseudo-conics" in a "pseudo-plane" is the same as that of lines and conics in a plane. In the same way, the following result mimics existence theorems for algebraic curves in $\mathbf{CP}^1 \times \mathbf{CP}^1$ (see [22]).

THEOREM 2.2.4. — *Let J be a tame almost complex structure on $\mathbf{CP}^1 \times \mathbf{CP}^1 = S^2 \times S^2$ equipped with the standard (product) symplectic structure.*

a) Any point (x, y) is contained in a unique J-curve homologous to $S^2 \times \star$.

b) If $a \geq 0$, $b \geq 0$ and $a + b > 0$, the class $a[S^2 \times \star] + b[\star \times S^2]$ is represented by a J-curve.

Now we give a statement for higher dimensions. Here the model situation is that of $S^2 \times V$, equipped with a split symplectic form and a split almost complex structure. We assume that V is *aspherical* (for 2-spheres), namely that $\pi_2(V) = 0$.

THEOREM 2.2.5. — *Let (V, σ) be a compact aspherical symplectic manifold and let J be an almost complex structure on $\mathbf{CP}^1 \times V$ tamed by the product symplectic structure $\omega \oplus \sigma$. Then any point (x, v) of $\mathbf{CP}^1 \times V$ is contained in exactly one J-curve homologous to the fibre $\mathbf{CP}^1 \times \{v\}$ of v.*

The proofs of these results go as follows. Let (W, ω) be one of the symplectic manifolds in the statements ($\mathbf{CP}^2$, $\mathbf{CP}^1 \times \mathbf{CP}^1$ or $\mathbf{CP}^1 \times V$). Consider the set C of pairs (f, J) where J is a tame almost complex structure and f a J-sphere in W in the given homology class. Then C can be given the structure of a manifold modelled on a suitable chain of Hölder or Sobolev spaces (see [12] and [2]).

Let p be the projection of C into the space $\mathcal{J}$ of all tame complex structures. The main point is to show that p is surjective. As is usual in this kind of nonlinear problem, one shows that $p(C)$ is open, closed and nonempty (we already know from 2.1.1 that $\mathcal{J}$ is connected).

1. $p(C)$ is not empty: in the first two cases, take the standard complex structure and algebraic curves, in the third case, use any tame split structure.

[3]Of course, there is a natural Kähler structure on the product of two Kähler manifolds.

2. $p(C)$ is open: basically, one shows that the linearisation of p is surjective by combining a vanishing theorem and an index computation (see [9], [14]) involving the Riemann-Roch theorem (see chapter IV).

3. p is proper, therefore $p(C)$ is closed. As usual, this is the most difficult part. Here, one proves first an equicontinuity property (see chapters V, VII and VIII) which comes from a suitable generalisaton of the Schwarz lemma for holomorphic functions. The main difficulty is to control the possible collapsing of the curves. For instance a sequence of conics may converge to a degenerate conic (i.e. two projective lines intersecting at a point) in the same homology class. In the same way, a sequence of curves of degree[4] $(1,1)$ in $\mathbf{CP}^1 \times \mathbf{CP}^1$ may converge to $S^2 \times \{x\} \cup \{x\} \times S^2$. Indeed, this type of difficulty already occurs in the proof of the conformal mapping theorem (of which assertion (b) of theorem 2.2.4 is some kind of generalisation). It can be proved (see chapter VIII or [26]) that such degeneracies never occur for simple homotopy classes, i.e. classes which do not admit a nontrivial decomposition as a sum of classes containing J-curves: for instance, the class of a conic in $\mathbf{CP}^2$ is not simple, as it can be written as twice the class of a line, the class of the line is simple, as are the degree $(1,0)$ class in $S^2 \times S^2$ and the class of $S^2 \times V$ with the assumptions of theorem 2.2.5.

3. Proofs of the symplectic rigidity results

We apply these existence results to prove the rigidity theorems of § 1.1.

3.1. Symplectic width

We begin by the proof of theorem 1.1.3. Suppose φ is an embedding of the ball B_R^{2n+2} into the cylinder $B_r^2 \times \mathbf{R}^{2n}$. Consider the smaller concentric balls $B_{R'} = B_{R'}^{2n+2}$ for $R' < R$. Since $\varphi\left(\bar{B}_{R'}\right)$ is compact, there is a symplectic embedding of $B_{R'}$ in $B^2(r) \times T^{2n}$: take $T^{2n} = \mathbf{R}^{2n}/L\mathbf{Z}^n$ for L big enough, one infers a symplectic embedding of $B_{R'}$ into $B^2(r) \times T^{2n}$, using the

LEMMA 3.1.1. — *The ball $B^2(r)$ and the sphere of same volume with one point removed, equipped with their standard symplectic structures, are symplectomorphic.* □

Take $R'' < R'$. Let J_0 be the standard almost complex structure of $\mathbf{R}^{2n+2}$. Then, using proposition 2.1.1 we see that the almost complex structure $\varphi_\star J_0$ on $\varphi\left(\bar{B}_{R''}\right)$ can be extended to $S^2(r/2) \times T^{2n}$ as an almost complex structure tamed by $\omega_0' + \omega_0''$. Now we can use theorem 2.2.5: there exists a J-holomorphic curve C homologous

[4] A curve has degree (p,q) if its projections onto the two factors have respective degrees p and q. Equivalently, it is in the same homology class as $p\mathbf{CP}^1 \times \star + q \star \times \mathbf{CP}^1$.

to $S^2 \times \star$ and containing $\varphi(0)$. One has

$$\int_C \omega_0' + \omega_0'' = \pi r^2.$$

If $C' = C \cap \varphi\left(\bar{B}_{R''}\right)$, we have

$$\pi r^2 \geq \int_{C'} \omega_0' + \omega_0'' = \int_{\varphi^{-1}(C')} \omega_0$$

since φ is symplectic. But $\varphi^{-1}(C')$ is a J_0-holomorphic curve through 0, whose boundary is contained in $S_{R''}$. Using the monotonicity lemma for minimal surfaces (see chapter III), we have

$$\text{area}\left(\varphi^{-1}(C')\right) = \int_{\varphi^{-1}(C')} \omega_\circ \geq \pi(R'')^2.$$

Finally, since R'' and R' are arbitrary, one infers that $R \leq r$. □

The arguments for theorem 1.1.4 are very similar, but use 2.2.3. We can suppose that U is symplectically embedded in the standard complex projective space, using the following equivariant version of 3.1.1:

LEMMA 3.1.2. — *The open sets $B_r^{2n} \subset \mathbf{R}^{2n}$ and $\mathbf{CP}^n - \mathbf{CP}^{n-1}$, equipped with their standard symplectic structures, are symplectomorphic, if they have the same total volume.* □

Now we have a symplectic embedding of V into $\mathbf{CP}^2$ equipped with the $U(3)$-invariant symplectic form such that the total volume is $\pi^2 R^4/2 = \text{vol}\,(B_R^4)$. Then we can proceed just as in the previous theorem. If $B\,(a, R_1)$ and $B\,(b, R_2)$ are balls contained in U, push forward the standard complex structure J_0 of $\mathbf{R}^4$ to $(B\,(a, R_1')) \cup \varphi\,(B\,(b, R_2'))$ (for some $R_1' < R_1$ and $R_2' < R_2$, and extend it to $\mathbf{CP}^2$ as a tame almost complex structure J. Let C be a J-holomorphic curve through $\varphi(a)$ and $\varphi(b)$, homologous to $\mathbf{CP}^1$. Then area$(C) = \pi R^2$, and the same area comparison argument applies.

3.2. Rigidity of symplectic diffeomorphisms

The rigidity theorem 1.1.2 is a simple corollary of Darboux's theorem and of the next result:

THEOREM 3.2.1. — *Let $(f_k)_{k\in\mathbf{N}}$ be a sequence of symplectic embeddings of the ball $B(0, R)$ into $\mathbf{R}^{2n}$. If f_k converges for the C^0-topology to a map f which is differentiable at 0, then the linear map $f'(0)$ is symplectic.*

The proof relies on a simple but tricky linear algebra lemma, due to Y. Eliashberg:

LEMMA 3.2.2. — *Let $L \in SL(2n, \mathbf{R}^{2n})$. Equip $\mathbf{R}^{2n}$ with its standard symplectic struture. If L is neither symplectic nor anti-symplectic, there exists a linear symplectic map S and a symplectic basis $(e_1, f_1, \dots, e_n, f_n)$ such that*

1. *modulo* $\mathrm{Vect}\,(e_2, \dots, f_n)$, *one has* $(L \circ S)\,(e_1) = \lambda e_1$ *and* $(L \circ S)\,(f_1) = \lambda f_1$, *with* $|\lambda| < 1$,
2. $L \circ S$ *leaves the symplectic orthogonal to* $\mathbf{R}e_1 \oplus \mathbf{R}f_1$ *globally invariant.*

Proof. — There is a pair (e, f) of vectors such that

$$|\omega(e, f)| = 1$$

and

$$|\omega(Le, Lf)| = \lambda^2 < 1.$$

Indeed, since L est unimodular, $|\omega(Le, Lf)|$ cannot be always strictly bigger than 1.

Let S be a symplectic map such that $S(Le) = \lambda e$ and $S(Lf) = \lambda f$. Now, to get the result, just transpose everything (symplectically!). □

Proof of the theorem. — Since the f_k are symplectic embeddings, they preserve Lebesgue measure. This property is preserved by C^0-limits. We claim that $f'(0)$ is unimodular. Indeed, by composing the f_k on the left and on the right with linear unimodular maps, one is reduced to the case when $f'(0)$ is diagonal with respect to the standard basis. The claim follows by comparing f to $f'(0)$.

Now, it is enough to prove that $f'(0)$ is symplectic or anti-symplectic: using the same argument for the sequence $g_k = f_k \times \mathrm{Id}_{\mathbf{R}^2}$, we see that $f'(0) \times \mathrm{Id}_{\mathbf{R}^2}$ will be also symplectic or anti-symplectic.

Taking coordinates with respect to the symplectic basis of the lemma, and working in a neighbourhood of 0, we have:

$$(f \circ S)\,(x_1, x_2, \dots, x_{2n}) = (\lambda x_1, \lambda x_2, \dots) + o(|x|).$$

(Don't bother about the coordinates of $f \circ S$ after the second one!). Take the Euclidean metric defined by this basis. If ε is small enough, we can find $\varepsilon' < \varepsilon$ such that $f \circ S$ sends the ball $B^{2n}(0, \varepsilon)$ into $B^2(0, \varepsilon') \times \mathbf{R}^{2n-2}$. Then the same thing is true for $f_k \circ S$, provided that k is big enough. This contradicts theorem 1.1.3.

4. What is in the book... and what is not

4.1. Contents of the book

We begin with two chapters introducing symplectic geometry and almost complex manifolds. The reader will find the very basic definitions and results, a lot of examples (how can one construct symplectic manifolds?) and exercises.

The next chapter is devoted to some facts in Riemannian geometry which will be useful, either directly in the proofs below, or simply to understand what is going on. Chapter IV is a panorama of the theory of linear connections and Chern classes, culminating in the Hirzebruch-Riemann-Roch theorem.

Then we get to the technical part. Chapter V is devoted to analytic aspects of the theory of pseudo-holomorphic curves, chapter VII gives a proof of Gromov's version of the classical Schwarz lemma, which is the main tool in the proof of the Gromov compactness theorem, which is in turn the subject of chapter VIII. We have already explained above how this theorem can be used to prove the existence of (global) pseudo-holomorphic curves satisfying given properties. In chapter IX, pseudo-holomorphic curves are shown to appear also in Riemannian geometry. Chapter VI explains some geometrical aspects of pseudo-holomorphic curves: just as two holomorphic curves in analytic surfaces, two pseudo-holomorphic curves in almost complex manifolds have *positive* intersection.

Chapter X is a panoramic view of Lagrangian submanifolds and related subjects. We have tried to explain all aspects of the situation: homotopic, topologic, and "hard" (that is: using pseudo-holomorphic curves techniques) results and to give as many examples as we know. All of the results evoked in § 1.2 and their interrelationships will be proved—usually in a more general setting.

4.2. What could have been in the book

There is no complex analysis in this book. By this we mean results like filling by holomorphic discs, rational convexity of Lagrangian tori and so forth, as appear in the work of Eliashberg [6] and Duval [4], [5] for instance.

Related to this, there is no contact geometry, although there are many interesting new results in this theory. We refer the reader to the beautiful survey talk by Giroux [8] and the references he gives.

There are no infinite dimensional Morse theory techniques although results like the distinction between the thin cylinder and the sphere above (theorem 1.1.3) would have fitted very well[5] with them. Here, good references are [24] and [21].

But this is a book about pseudo-holomorphic curves. There are also some results closely related with pseudo-holomorphic curves which are also not in the book. For example, a whole set of results on the classification of 4-dimensional symplectic manifolds, which constitute one of the beautiful achievements of the theory, have been obtained by D. McDuff [15] and [16] in the last few years. They do not appear explicitly here, but we hope the reader of the present book will be able to read the original papers.

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Part 1

Basic symplectic geometry

Chapter I

An introduction to symplectic geometry

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with an appendix by M. Audin, A. Banyaga, F. Lalonde and L. Polterovich

1. Linear symplectic geometry

1.1. Symplectic vector spaces

A *symplectic form* on a vector space V is a skew-symmetric bilinear form $\alpha : V \times V \to \mathbf{R}$ such that $\widetilde{\alpha} : V \to V^\star : \widetilde{\alpha}(x)(y) = \alpha(x, y)$ is an isomorphism. Here $V^\star$ denotes the dual of V.

The couple (V, α) is called a *symplectic vector space*. Two symplectic vector spaces (V, α) and (U, β) are *isomorphic* if there exists a linear mapping $T : V \to U$ such that $T^\star\beta = \alpha$. We will prove that all finite dimensional symplectic vector spaces are even dimensional and that any two symplectic vector spaces of the same dimension are isomorphic. Let us start with a few examples:

1. Let V be a $2n$-dimensional vector space. Choose a basis $e = (e_1, e_2, \ldots, e_{2n})$ of V and denote by $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{2n})$ the dual basis: $\varepsilon_i \in V^\star$ are defined by:

$$\varepsilon_i(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Recall that if ω_1 and ω_2 are linear forms on V, we define a skew-symmetric bilinear form $\omega_1 \wedge \omega_2$ by

$$(\omega_1 \wedge \omega_2)(v_1, v_2) = \omega_1(v_1) \cdot \omega_2(v_2) - \omega_1(v_2) \cdot \omega_2(v_1).$$

On the vector space V, consider the 2-form:

$$\alpha_s = (\varepsilon_1 \wedge \varepsilon_{1+n}) + (\varepsilon_2 \wedge \varepsilon_{2+n}) + \cdots + (\varepsilon_n \wedge \varepsilon_{2n}).$$

The matrix of $\widetilde{\alpha}_s : V \to V^\star$ within the bases e and ε is:

$$J = \left(\begin{array}{c|c} 0 & -I_n \\ \hline I_n & 0 \end{array} \right)$$

where I_n is the identity $(n \times n)$-matrix.

Hence α_s is a symplectic form on V. If $V = \mathbf{R}^{2n}$ and $e = (e_1, \dots, e_{2n})$ is the standard basis, then α_s above is called the "standard" symplectic form on $\mathbf{R}^{2n}$.

2. Let U be an n-dimensional vector space and let $V = U \oplus U^\star$. Define a 2-form α by:
$$\alpha(u \oplus \rho, u' \oplus \rho') = \rho'(u) - \rho(u').$$
This is a symplectic form which looks much like the one in the preceeding example (with obvious change of notations).

3. If A is any skew-symmetric invertible $2n \times 2n$-matrix, the form α_A defined on $\mathbf{R}^{2n}$ by
$$\alpha_A(X, Y) = \langle X, AY \rangle,$$
where $\langle \ , \ \rangle$ is the usual inner product, is a symplectic form.

For instance, if a_1, a_2, a_3 are three real numbers with $a_3 \neq 0$, the form α_A defined on $\mathbf{R}^4$ by the matrix
$$A = \begin{pmatrix} 0 & 1 & 0 & -a_1 \\ -1 & 0 & 0 & -a_2 \\ 0 & 0 & 0 & -a_3 \\ a_1 & a_2 & a_3 & 0 \end{pmatrix}$$
was used by Zehnder in [21] (to construct examples of Hamiltonian systems without periodic orbits).

Our first theorem asserts that any symplectic vector space is isomorphic to $(\mathbf{R}^{2n}, \alpha_s)$.

THEOREM 1.1.1. — *Let (V, α) be a symplectic vector space. There exists a basis $e = (u_1, \dots, u_n, v_1, \dots, v_n)$ of V such that, for all i, j,*
$$\begin{aligned} \alpha(u_i, u_j) &= \alpha(v_i, v_j) = 0 \\ \alpha(u_i, v_j) &= \delta_{ij}. \end{aligned}$$
If $(\varepsilon_1, \dots, \varepsilon_n, \eta_1, \dots, \eta_n)$ is the basis dual to e, then α assumes the expression:
$$\alpha = \varepsilon_1 \wedge \eta_1 + \varepsilon_2 \wedge \eta_2 + \cdots + \varepsilon_n \wedge \eta_n.$$

A basis satisfying these properties is called a *canonical* or *symplectic basis*. The theorem is a consequence of the following lemma.

LEMMA 1.1.2. — *Let V be an n-dimensional vector space and α a skew-symmetric bilinear form. There exists a basis $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ of $V^\star$ and an integer $q \leq n/2$ depending on α such that*
$$\alpha = \varepsilon_1 \wedge \varepsilon_2 + \varepsilon_3 \wedge \varepsilon_4 + \cdots + \varepsilon_{2q-1} \wedge \varepsilon_{2q}.$$

Proof. — Let $b = (b_1, \dots, b_n)$ be any basis of V and let $\beta = (\beta_1, \dots, \beta_n)$ be the dual basis. Then $\alpha = \sum_{i<j} a_{ij}\beta_i \wedge \beta_j$. If $\alpha = 0$, then $q = 0$, and the lemma is proved. Otherwise, there exists at least one a_{ij} which is non zero. Assume $a_{12} \neq 0$. Let

$$\begin{aligned} \varepsilon_1 &= \beta_1 - \left(\frac{a_{23}}{a_{12}}\beta_3 + \frac{a_{24}}{a_{12}}\beta_4 + \cdots + \frac{a_{2n}}{a_{12}}\beta_n\right) \\ \varepsilon_2 &= a_{12}\beta_2 + a_{13}\beta_3 + \cdots + a_{1n}\beta_n. \end{aligned}$$

It is easy to check that $\alpha_1 = \alpha - \varepsilon_1 \wedge \varepsilon_2$ does not contain β_1 and β_2. Observe that $(\varepsilon_1, \varepsilon_2, \beta_3, \dots, \beta_n)$ are linearly independent. If $\alpha_1 = 0$, we are done. Otherwise, we redo the argument with α_1 as a bilinear 2-form on the space with dual basis $\beta_3, \dots, \beta_n$. Each time, the number of arguments decreases by 2 until the form vanishes.

Finally we have a basis $(\varepsilon_1, \dots, \varepsilon_n)$ of $V^\star$ in which $\alpha = \varepsilon_1 \wedge \varepsilon_2 + \cdots + \varepsilon_{2q-1} \wedge \varepsilon_{2q}$. □

Proof of the theorem. — Let $E(\alpha) \subseteq V^\star$ be the image of $\tilde{\alpha} : V \to V^\star$. Then $E(\alpha)$ is spanned by $\varepsilon_1, \dots, \varepsilon_{2q}$. Hence the integer $2q$ above is the rank of $\tilde{\alpha}$. If α is a symplectic form, $\tilde{\alpha}$ is an isomorphism and $2q = \dim(V)$. □

1.2. Subspaces of Symplectic Vector Spaces

Let (V, α) be a symplectic vector space and $W \subseteq V$ a subspace of V. The α-orthogonal complement $W^\perp$ of W is defined as $\{v \in V \mid \alpha(v, w) = 0, \quad \forall w \in W\}$. A subspace W is said to be:

— isotropic if $W \subseteq W^\perp$
— coisotropic if $W^\perp \subseteq W$
— symplectic if $W^\perp \cap W = \{0\}$
— Lagrangian if $W = W^\perp$

The restriction of α to an isotropic subspace is identically zero. The α-orthogonal complement of a coisotropic subspace is an isotropic subspace. A Lagrangian subspace is both isotropic and coisotropic. It is obvious that any 1-dimensional subspace is isotropic and any codimension 1 subspace is coisotropic.

For any subspace W, we have:

$$\dim(W^\perp) = \dim V - \dim(\tilde{\alpha}(W))$$

But since $\tilde{\alpha}$ is an isomorphism, $\dim(\tilde{\alpha}(W)) = \dim(W)$. Therefore $\dim(W^\perp) = \dim V - \dim(W)$, i.e.:

$$\dim(W^\perp) + \dim(W) = \dim V.$$

If W is isotropic, then $2\dim W \leq \dim W + \dim W^\perp = \dim V$, hence $\dim W \leq \dim V/2$.

If W is coisotropic, then $\dim V = \dim W + \dim W^{\perp} \leq 2 \dim W$, hence $\dim W \geq \dim(V)/2$. Thus isotropic subspaces have dimension $\leq \frac{1}{2} \dim V$ and coisotropic have dimension $\geq 1/2 \dim V$. Therefore Lagrangian subspaces have exactly half the dimension of V.

Examples. — Let $\{u_1, \ldots, u_n, \ v_1, \ldots, v_n\}$ be a canonical basis of (V, α). The subspaces L_k spanned by $\{u_1, \ldots, u_k\}$ and M_k spanned by $\{v_1, \ldots, v_k\}$, $1 \leq k \leq n$ are isotropic and L_n, M_n are Lagrangian. The subspaces spanned by the $n + \ell$ vectors $\{u_1, \ldots, u_n, \ v_1, \ldots, v_\ell\}$ or $\{u_1, \ldots, u_\ell, v_1, \ldots, v_n\}$, $1 \leq \ell \leq n$ are coisotropic.

THEOREM 1.2.1. — *Any isotropic subspace is contained in a Lagrangian subspace.*

Proof. — Suppose $W \subseteq V$ is an isotropic subspace which is not Lagrangian, then $W \subset W^{\perp}$ (strict inclusion). Let $u \in W^{\perp} - W$ and let W_u be the space spanned by $\{w_1, \ldots, w_k, u\}$ where $\{w_1, \ldots, w_k\}$ is a basis of W. Then $W \subset W_u$ (strict inclusion); moreover W_u is isotropic: indeed if $v_i = x_i + \lambda_i u$, $(i = 1, 2)$, are elements in W_u, $(x_i \in W)$, then:

$$\alpha(v_1, v_2) = \alpha(x_1, x_2) + \lambda_1 \alpha(u, x_2) + \lambda_2 (x_1, u) + \lambda_1 \lambda_2 \alpha(u, u).$$

The first term in the right hand side is zero since $x_i \in W$ (which is isotropic), the next two terms are zero since $u \in W^{\perp}$, $x_i \in W$ and the last term is zero since α is skew-symmetric.

In this way we have enlarged W to a new isotropic subspace W_u containing W. If $W_u = W_u^{\perp}$ we are done. Otherwise repeat the same construction... until we get a Lagrangian space L, i.e. $L = L^{\perp}$. □

For instance, we may start with $\{0\}$ and get flags of isotropic subspaces

$$\{0\} \subset W_1 \subset W_2 \subset \cdots \subset W_n$$

where W_n is Lagrangian.

Two Lagrangian subspaces L, $H \subseteq V$ are said to be transversal if $L \cap H = \{0\}$. In that case $V = L \oplus H$.

Let $L \subseteq V$ be a Lagrangian subspace. The natural map $\tilde{\alpha} : L \to (V/L)^{\star}$ defined by

$$(\tilde{\alpha}(x))\,(y + L) = \alpha(x, y) \text{ for all } x \in L, y + L \in V/L$$

is an isomorphism.

Let now $(e_1, \ldots, e_n)$ be a basis of L and $(\varepsilon_1, \ldots, \varepsilon_n)$ its dual basis, then if $T_i = ({}^t\tilde{\alpha})^{-1}(\varepsilon_i)$, $\{T_1, \ldots, T_n\}$ generate a Lagrangian subspace H such that $V = L \oplus H$ and it is easy to check that $\{e_1, \ldots, e_n, \ T_1, \ldots, T_n\}$ is a canonical basis of V.

1.3. Euclidean and complex structures associated with symplectic vector spaces

Let (V, α) be a symplectic vector space and let $B = \{u_1, \ldots, u_n,\ v_1, \ldots, v_n\}$ be a canonical basis. We define a linear mapping $J_B : V \to V$ and an inner product $\langle\ ,\ \rangle_B$ on V by:

$$J_B(u_i) = v_i, J_B(v_i) = -u_i, i = 1, \ldots n,$$

and

$$\langle u_i, u_j \rangle_B = \langle v_i, v_j \rangle_B = \delta_{ij}, \langle u_i, v_j \rangle_B = 0 \quad \forall\, i, j.$$

To simplify notation, we write J for J_B and $\langle\ ,\ \rangle$ for $\langle\ ,\ \rangle_B$ (but have in mind that J and $\langle\ ,\ \rangle$ depend on the choice of the canonical basis).

The following identities are clear:

(i) $J \circ J = -$ identity (i.e. J is a complex structure)

(ii) $\langle Ju, Jv \rangle = \langle u, v \rangle$, (i.e. J is an isometry)

(iii) $\alpha(u, v) = \langle Ju, v \rangle$, for all $u, v \in V$

As a consequence, we have

(iv) $J^\star \alpha = \alpha$

Indeed:

$$(J^\star \alpha)(u, v) = \alpha(Ju, Jv) = \langle J^2 u, Jv \rangle = -\langle u, Jv \rangle = -\langle Jv, u \rangle = -\alpha(v, u) = \alpha(u, v).$$

DEFINITION 1.3.1. — *A Kählerian vector space is a symplectic vector space (V, α) with a complex structure J satisfying: $J^\star \alpha = \alpha$.*

We just observed that a symplectic vector space is a Kählerian vector space (these notions will be developed in chapter II).

It is very convenient to use the symplectic structure, the inner product and the complex structure together. In this way, one easily gets the results of the following exercises:

1.3.2. Exercise. — Show that a subspace W of V is Lagrangian if and only if JW is the orthogonal of W with respect to the Euclidean structure.

1.3.3. Exercise. — Consider (V, J) as a complex vector space. Show that $\langle\ ,\ \rangle - i\alpha(\ ,\ )$ is a Hermitian structure on (V, J).

2. Symplectic manifolds and vector bundles

2.1. Symplectic vector bundles

A vector bundle $\pi : E \to M$ over a smooth manifold M is called a *symplectic vector bundle* if each fiber $E_x = \pi^{-1}(x)$ carries a symplectic form α_x, which varies smoothly with x. The rank of a symplectic vector bundle must be even. The collection $\alpha = (\alpha_x)$ is called a symplectic structure on E.

The following example is inspired by the linear theory. Let TM and $T^\star M$ be respectively the tangent and cotangent bundles of a smooth manifold M, then if $E = TM \oplus T^\star M$ is their Whitney sum, then $\pi : E \to M$ is a symplectic vector bundle. We will see that not all symplectic vector bundles are obtained in this way.

If $\pi : E \to M$ is a symplectic vector bundle, a subbundle $L \to M$ is said to be *isotropic* if each fiber over $x \in M$ is an isotropic subspace of E_x. Definitions for *coisotropic*, *Lagrangian* subbundles are analogous. Observe that to the contrary of the linear theory an arbitrary symplectic vector bundle $\pi : E \to M$ does not admit a Lagrangian subbundle L. Indeed, this would imply that $E \cong L \oplus L^\star$.

If E is the tangent bundle of a smooth manifold M, then a symplectic structure $\alpha = (\alpha_x)$ on E is a (smooth) differential form Ω of degree 2 on M:

$$\Omega(x)(\xi, \eta) = \alpha_x(\xi_x, \eta_x)$$

(for all vector fields ξ, η on M). This 2-form has the property that the mapping $\widetilde{\Omega}$ assigning to a vector field ξ the 1-form $i(\xi)\Omega$ defined by:

$$i(\xi)\Omega(\eta) = \Omega(\xi, \eta)$$

is an isomorphism from the space $\mathcal{X}_M$ of vector fields on M to the space $\Omega^1(M)$ of 1-forms on M. The form Ω above is called sometimes an *almost symplectic form* or an almost symplectic structure.

An almost complex structure on a smooth manifold M is a 1-1 tensor field $J : TM \to TM$ such that $J^2 = -I$. This says that there exists in each fiber T_xM a complex structure J_x which varies smoothly with x.

An almost Hermitian structure is a couple (g, J) where J is an almost complex structure and g is a Riemannian metric such that $g(JX, JY) = g(X, Y)$ for all vector fields X, Y.

Starting with an almost complex structure J and any Riemannian metric g_0, one can form an almost Hermitian structure (g, J) where $g(X, Y) = g_0(X, Y) + g_0(JX, JY)$. Given an almost Hermitian structure (g, J) on a smooth manifold M, we construct an almost symplectic form Ω on $M : \Omega(X, Y) = g(JX, Y)$.

Conversely starting with an almost symplectic structure Ω, we can find an almost Hermitian structure (g, J) such that $\Omega(X, Y) = g(JX, Y)$.

Indeed let g_0 be any Riemannian metric and denote by $\widetilde{g}_0 : TM \to T^\star M$ the Riemannian duality: $\widetilde{g}_0(X)(Y) = g_0(X, Y)$. Let

$$K = \widetilde{g}_0^{-1} \circ \widetilde{\Omega} : TM \to TM.$$

This is a skew-symmetric tensor field such that

$$\begin{aligned} \widetilde{g}_0(K(X))(Y) &= g_0(K(X), Y) \\ &= \widetilde{\Omega}(X)(Y) = \Omega(X, Y). \end{aligned}$$

The polar decomposition of K_x (in each fiber) yields two tensor fields R, J such that $K = RJ$ where $R = \sqrt{KK^t}$ is a positive definite symmetric tensor field. One checks that $J^2 = -I$ (i.e. J is an almost complex structure) and that $\Omega(JX, JY) = \Omega(X, Y)$.

Moreover, letting $g(X,Y) = g_0(RX,Y)$, we have $g(JX,JY) = g(X,Y)$ and $g(JX,Y) = \Omega(X,Y)$.

For more details, we refer to the basic texts [14], [15], [20] and to chapter II of the present book.

2.2. Symplectic Manifolds

If an almost symplectic form Ω is closed, then we say that Ω is a *symplectic form*. The couple (M, Ω) of a smooth manifold M and a symplectic form Ω on it is called a *symplectic manifold.*

The basic example of a symplectic manifold is Euclidean space $\mathbf{R}^{2n}$ equipped with the constant symplectic form Ω defined by the linear symplectic form α_s in § 1.1. For $x \in \mathbf{R}^{2n}$,

$$\Omega_s(x)(\xi, \eta) = \alpha_s(\xi(x), \eta(x)),$$

where $\xi, \eta : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ are vector fields on $\mathbf{R}^{2n}$. If $x = (x_1, \dots, x_n, y_1, \dots, y_n)$ are coordinates on $\mathbf{R}^{2n}$, we see that:

$$\Omega_s = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \cdots + dx_n \wedge dy_n.$$

This shows that $d\Omega_s = 0$, and therefore Ω_s is a symplectic form.

The next basic examples are provided by the cotangent bundles $M = T^\star X$ of smooth n-manifolds X. Let $\pi : T^\star X \to X$ be the natural projection. An element $\underline{a} \in T^\star X$ is a couple (a, θ) where $a = \pi(\underline{a})$ and $\theta \in T^\star_a X$. For $\xi \in T_{\underline{a}}(T^\star X)$, $(T_{\underline{a}}\pi)(\xi) \in T_a X$, where $T_{\underline{a}}\pi : T_{\underline{a}}(T^\star X) \to T_a X$ is the tangent map of π. Therefore we can define a 1-form—the *Liouville form*— λ_X on $T^\star X$ by:

$$\lambda_X(\underline{a})(\xi) = \theta((T_{\underline{a}}\pi)(\xi))$$

where $\underline{a} \in T^\star X$ and $\xi \in T_{\underline{a}}(T^\star X)$.

Let $(x_1, \dots, x_n)$ be a local coordinate system in a neighbourhood U of a point in X. On $T^\star U = \pi^{-1}(U)$ we get a coordinates system $(x_1, \dots, x_n, y_1, \dots, y_n)$ such that $(T\pi)(\partial/\partial x_i) = \partial/\partial x_i$ and $(T\pi)(\partial/\partial y_i) = 0$. In these coordinates, the 1-form λ_X assumes the expression:

$$\lambda_X = \sum_{i=1}^n y_i dx_i.$$

Let $\Omega_X = d\lambda_X$; in the above coordinates

$$\Omega_X = \sum_{i=1}^n dy_i \wedge dx_i.$$

This shows that Ω_X is a symplectic form on $T^\star X$.

Remarks.

1. An oriented surface is a symplectic manifold, any volume form being a symplectic form.

2. The symplectic form α_s on $\mathbf{R}^{2n}$ is invariant by translations $x_i \mapsto x_i + 2\pi$, $y_i \mapsto y_i + 2\pi$. Therefore Ω_s descends to a 2-form on the torus $T^{2n} = \mathbf{R}^{2n}/\mathbf{Z}^{2n}$, which is a symplectic form Ω.

Note that the symplectic form $\Omega_s = \sum dx_i \wedge dy_i$ on $\mathbf{R}^{2n}$ is exact: $\Omega = d\lambda_1 = d\lambda_2$ where

$$\lambda_1 = \sum x_i dy_i \qquad \lambda_2 = \frac{1}{2}\sum(x_i dy_i - y_i dx_i).$$

However the corresponding symplectic form on T^{2n} is not exact. Indeed we note the following:

PROPOSITION 2.2.1. — *No symplectic form on a closed manifold can be exact.*

Proof. — If a symplectic form Ω on the $2n$-dimensional compact manifold M (without boundary) is exact, i.e. $\Omega = d\omega$, then $\Omega^n = d(\omega \wedge \Omega^{n-1})$ and by Stokes theorem

$$\int_M \Omega^n = \int_{\partial M} \omega \wedge \Omega^{n-1} = 0$$

which is absurd since $\int_M \Omega^n$ is a multiple of the total volume of M. □

Remark. — A symplectic form on a smooth manifold is of course an almost symplectic form, but there are almost symplectic manifolds which carry no symplectic structure. Indeed by the proposition above, the sphere S^6 cannot carry a symplectic form, since any closed two form on S^6 is exact. However S^6 carries an almost complex structure and hence almost Hermitian structures and in turn almost symplectic structures (see also chapters II and IV). In these introductory lectures, we will not consider the problem of existence of symplectic structures.

The classification problem is the following: given two symplectic manifolds (M_1, Ω_1) and (M_2, Ω_2), does there exists a diffeomorphism $\varphi : M_1 \to M_2$ such that $\varphi^\star\Omega_2 = \Omega_1$? Such a diffeomorphism is called a symplectic diffeomorphism. The classification problem is still unsolved. However, a local classification is achieved by Darboux theorem which asserts that locally, all symplectic manifolds look alike. More precisely, we have the following

THEOREM 2.2.2 (Darboux). — *For any point x in the symplectic manifold (M, Ω), there exists an open neighbourhood U of x and a local chart $\varphi : U \to \mathbf{R}^{2n}$ with $\varphi(x) = 0$ and $\varphi^\star\Omega_s = \Omega|_U$, where Ω_s is the standard symplectic structure on $\mathbf{R}^{2n}$.*

This theorem is classical. Weinstein's proof [20] of this theorem (which was inspired by a theorem of Moser and also pointed out by Martinet) extends to give a more general theorem:

THEOREM 2.2.3 (Darboux-Weinstein). — *Let N be a submanifold of a smooth manifold M equipped with two symplectic forms Ω_1 and Ω_2 such that $\Omega_2|_N = \Omega_1|_N$, there exists an open neighbourhood U of N in M and a diffeomorphism $\varphi : U \to \varphi(U) \subseteq M$ such that $\varphi(x) = x$, $\forall\, x \in N$ and $\varphi^\star\Omega_2 = \Omega_1$ where we wrote $\Omega_i = \Omega_i|_{\varphi(U)}$.*

Moreover if a compact Lie group G acts on M preserving Ω_1 and Ω_2 and leaving N invariant, then φ can be chosen to commute with the G-action.

Proof. — Let g be a G-invariant Riemannian metric on M (if there is no group acting, $G = \{e\}$ is the trivial group). Take U'' to be a tubular neighbourhood of N in M. Using the exponential map of g on the normal bundle of N, we get a smooth retraction $\rho_t : U'' \to U''$ such that $\rho_0 : U'' \to N$ $\rho_1 =$ identity and $\rho_t(x) = x$ when $x \in N$. The retraction ρ_t commutes with G.

Now, for each $x \in U''$, let $\dot\rho_t(x) \in T_{\rho_t(x)}M$ be the vector tangent to the curve $t \mapsto \rho_t(x)$ at the point $\rho_t(x)$. Hence $\dot\rho_t : U'' \to TM$ is a vector field along U''. For any p-form α on M we define a family of $(p-1)$-forms denoted $\rho_t^\star(i(\dot\rho_t)\alpha)$ on U'' by:

$$[\rho_t^\star(i(\dot\rho_t)\alpha)](x)\ (X_1, \ldots, X_{p-1}) = \alpha(\rho_t(x))(\dot\rho_t(x)), ((\rho_t)_\star X_1, \ldots, (\rho_t)_\star X_{p-1}).$$

Denote by $I(\alpha)$ the $(p-1)$-form on U''

$$I(\alpha) = \int_0^1 (\rho_t^\star i(\dot\rho_t)\alpha)dt.$$

2.2.4. Exercise. — Check the following formula of advanced calculus (see [13] or [14]):

$$\frac{d}{dt}(\rho_t^\star\alpha) = \rho_t^\star(i(\dot\rho_t)d\alpha) + d(\rho_t^\star i(\dot\rho_t)\alpha)$$

and deduce that

$$\rho_1^\star\alpha - \rho_0^\star\alpha = \int_0^1 (\rho_t^\star i(\dot\rho_t)d\alpha)dt + d\int_0^1 (\rho_t^\star i(\dot\rho_t)\alpha)dt.$$

Therefore the operator $I : \Omega^p(U'') \to \Omega^{p-1}(U'')$ is a "homotopy" operator: i.e.

$$\alpha - \rho_0^\star\alpha = d\,I(\alpha) + I(d\alpha).$$

Because ρ_t commutes with G so does I, i.e.

$$I(a^\star\alpha) = a^\star(I\alpha), \quad \forall a \in G.$$

Apply now the facts above to $\alpha = \Omega_2 - \Omega_1$ since $\alpha|_N = 0$, $\rho_0^\star\alpha = 0$ and hence

$$\alpha = d(I\alpha) + I(d\alpha) = d(I\alpha) \text{ since } d\alpha = 0.$$

We have found a 1-form $\beta = I(\alpha)$ on the neighbourhood U'' of N such that $\beta|_N = 0$ and

$$\Omega_2 = \Omega_1 + d\beta$$

on U''. Since the restriction of $\sigma_t = \Omega_1 + t(d\beta)$ to N is the symplectic structure Ω_1 for all t, there exists a smaller neighbourhood $U' \subset U''$ on which σ_t is a symplectic form, i.e. $\forall\, y \in U'$, $\tilde{\sigma}_t(y) : T_yM \to T_y^\star M$ is an isomorphism. Let X_t be the family of vector fields on U' such that $i(X_t)\sigma_t = -\beta$.

2.2.5. Exercise. — Recall the Cartan formula

$$L_X\omega = i(X)d\omega + d(i(X)\omega)$$

for the Lie derivative L_X. Apply it to get:

$$L_{X_t}\sigma_t + \frac{\partial \sigma_t}{\partial t} = 0.$$

Since $X_t(x) = 0$ for all $x \in N$, there exists a smaller neighbourhood U of N, $U \subseteq U'$ on which X_t can be integrated to a time dependent embedding $\varphi_t : U \to M$ with $\varphi_t(x) = x$, $\forall x \in N$. We have

$$\frac{d}{dt}(\varphi_t^\star \sigma_t) = \varphi_t^\star \left(\frac{\partial \sigma_t}{\partial t} + L_{X_t}\sigma_t \right) = 0.$$

This means that $\varphi_1^\star\sigma_1 = \varphi_0^\star\sigma_0 = \sigma_0$ on U i.e. $\varphi_1^\star\Omega_2 = \Omega_1$ on U.

Observe that φ_t is G-equivariant, since X_t was G-invariant (because β was and Ω_0, Ω_1 are). □

Proof of Darboux theorem. — In the preceeding theorem, replace the submanifold N by a point $\{x\}$. Choose a G-invariant Riemannian metric, and let U'' be a domain of a geodesic chart, (identifying a neighbourhood of 0 in T_xM with $\mathbf{R}^{2n}$, running along geodesics). The linear mapping, constructed in § 1.1, identifying the 2-form $\Omega(x)$ in $T_xM \simeq \mathbf{R}^{2n}$ and the canonical symplectic form Ω_s can be chosen to be G-equivariant; now apply the Darboux-Weinstein theorem to get a G-equivariant Darboux theorem. □

Relative Poincaré lemma. — Let N be a closed submanifold of a smooth manifold M and Ω a closed k-form on M such that $\Omega|_N = 0$. There exists an open neighbourhood U of N in M and a $(k-1)$-form β such that $\beta|_N = 0$ and $\Omega = d\beta$ on U. □

As we said earlier, the proof of the Darboux-Weinstein theorem given above is inspired by the proof of the following result of Moser [17]:

THEOREM 2.2.6. — *Let Ω_0, Ω_1 be two symplectic forms on a compact manifold M such that there exists a smooth family ω_t of symplectic forms with $\omega_0 = \Omega_0$, $\omega_1 = \Omega_1$ and for each t, $\dot{\omega}_t = \partial\omega_t/\partial t$ is exact. Then there exists a diffeomorphism φ of M, isotopic to the identity such that $\varphi^\star\Omega_1 = \Omega_0$.*

Proof. — By the Hodge-de Rham theorem, there exists a smooth family β_t of 1-forms such that $\dot{\omega}_t = -d\beta_t$. Let X_t be the family of vector fields defined by: $i(X_t)\omega_t = -\beta_t$, then: $L_{X_t}\omega_t + \partial\omega_t/\partial t = 0$. As above, if φ_t is the family of diffeomorphisms obtained by integrating the time dependent vector fields X_t, we have

$$\frac{d}{dt}(\varphi_t^\star\omega_t) = \varphi_t^\star(L_{X_t}\omega_t + \dot{\omega}_t) = 0.$$

This implies that $\varphi_1^\star\omega_1 = \varphi_1^\star\Omega_1 = \varphi_0^\star\omega_0 = \omega_0 = \Omega_0$. □

2.3. Submanifolds of symplectic manifolds

Let (M, Ω) be a symplectic manifold. The tangent bundle $TM \to M$ of M is a symplectic vector bundle. For any smooth manifold P and $f : P \to M$ a smooth map, the pull-back $E = f^\star(TM)$ is a symplectic vector bundle over P. The mapping f is called an *isotropic*, *coisotropic*, *Lagrangian* or *symplectic* map if $(Tf)(TP)$ is an isotropic, coisotropic, Lagrangian or symplectic subbundle of $E = f^\star(TM)$: this means that for each $x \in P$, $(T_xf)(T_xP)$ is an isotropic, coisotropic, Lagrangian or symplectic subspace of $T_{f(x)}M$.

A submanifold P of a symplectic manifold is qualified as an *isotropic*, *coisotropic*, *Lagrangian* or *symplectic* submanifold if the corresponding embedding qualifies for these terminologies. For instance any 1-dimensional submanifold of (M, Ω) is an isotropic submanifold and any codimension 1 submanifold is a coisotropic submanifold. For instance if $f : M \to \mathbf{R}$ is any smooth map and $a \in \mathbf{R}$ is a regular value of f, then $P = f^{-1}(a)$ is a coisotropic submanifold.

Here are a few examples of Lagrangian submanifolds. Let (M, Ω) be the cotangent bundle of a certain manifold: $M = T^\star X$, $\Omega = d\lambda_X$. The restriction of Ω to the zero section is identically zero. Therefore $i : X \to T^\star X$ the inclusion of X as the zero section is a Lagrangian embedding. The canonical 1-form λ_X on $T^\star X$ verifies the following property: for any 1-form β on X, i.e. β is a section $\beta : X \to T^\star X$ of the cotangent bundle, then:

$$\beta^\star\lambda_X = \beta.$$

Hence if β is a closed form ($d\beta = 0$), then

$$\beta^\star\Omega = \beta^\star(d\lambda_X) = d(\beta^\star\lambda_X) = d\beta = 0,$$

which means that the mapping $\beta : X \to T^\star X$ is a Lagrangian mapping. Since β is an embedding, we see that its image, $\beta(X) = P$, i.e. the graph of β, is a Lagrangian submanifold. For instance, any smooth function $S : X \to \mathbf{R}$ defines a Lagrangian submanifold $(dS)(X) \subseteq T^\star X$ (this Lagrangian submanifold is said to be generated by S).

PROPOSITION 2.3.1. — *The Lagrangian submanifolds of $T^\star X$ which project diffeomorphically onto X are in 1-1 correspondence with closed 1-forms on X.* □

The diagonal trick. — Another way of getting Lagrangian submanifolds is to consider the graph of symplectic diffeomorphisms. A symplectic diffeomorphism of a symplectic manifold (M, Ω) is a C^∞ diffeomorphism $\varphi : M \to M$ such that $\varphi^\star\Omega = \Omega$. We will study the space of those mappings in § 2.4. The graph of a symplectic diffeomorphism φ is the submanifold

$$\Gamma_\varphi = \{(x, y) \in M \times M \mid y = \varphi(x)\}.$$

On $M \times M$ we have the "product" symplectic form $\underline{\Omega} = \pi_1^\star\Omega - \pi_2^\star\Omega$ where $\pi_i : M \times M \to M$ is $\pi_i(x_1, x_2) = x_i$. Let $j : M \to \Gamma_\varphi \subseteq M \times M$ be the inclusion of the graph in $M \times M$ then:

$$j^\star\underline{\Omega} = j^\star\pi_1^\star\Omega - j^\star\pi_2^\star\Omega = \Omega - \varphi^\star\Omega = 0.$$

Therefore Γ_φ is a Lagrangian submanifold of $M \times M$. This trick will be very useful in chapter X: it allows to relate fixed points of symplectic diffeomorphisms and intersections of Lagrangian submanifolds.

Symplectic reduction (see also chapter II). — Let P be a submanifold of a symplectic manifold (M, Ω), and assume that $L = (TP) \cap (TP)^\perp$ is a subbundle of TP. The symplectic form Ω induces a symplectic structure $\overline{\Omega}$ on the vector bundle TP/L: $\forall x \in P$, define:

$$\overline{\Omega}_x(\xi + L_x, \eta + L_x) = \Omega(x)(\xi, \eta)$$

where $\xi, \eta \in T_xP$ and L_x are any elements in $(T_xP) \cap (T_xP)^\perp$. It is easy to see that $\overline{\Omega}_x$ is a symplectic form on $T_xP/(L_x)$.

PROPOSITION 2.3.2. — *The subbundle $L = (TP) \cap (TP)^\perp$ is involutive.*

Proof. — Let ξ_1, ξ_2 be two sections of L, i.e. $\xi_i(x) \in T_xP \cap (T_xP)^\perp$. Let $j : P \to M$ be the embedding defining the submanifold P, and denote: $\Omega_P = j^\star\Omega$. Since

$$d\Omega_P = dj^\star\Omega = j^\star d\Omega = j^\star d\Omega = 0$$

we have for any vector field η on P:

$$\begin{aligned} 0 &= (d\Omega_P)(\xi_1, \xi_2, \eta) \\ &= \xi_1 \cdot \Omega_P(\xi_2, \eta) - \xi_2 \cdot \Omega_P(\xi_1, \eta) + \eta \cdot \Omega_P(\xi_1, \xi_2) \\ &\quad -\Omega_P([\xi_1, \xi_2], \eta) + \Omega_P([\xi_1, \eta], \xi_2) - \Omega_P([\xi_2, \eta], \xi_1) \end{aligned}$$

If u is any vector field on P, $\Omega_P(u,\xi_i) = \Omega_P(\xi_i,u) = 0$ since $\xi_i(x) \in (T_xP)^\perp$. Hence in the sum above all the terms except possibly $\Omega_P([\xi_1,\xi_2],\eta)$ vanish: therefore we have

$$0 = (d\Omega_p)(\xi_1,\xi_2,\eta) = -\Omega_P([\xi_1,\xi_2],\eta).$$

Since η was an arbitrary vector field on P, the equation above means that $[\xi_1,\xi_2]$ is a section of $(TP)^\perp$ therefore $[\xi_1,\xi_2]$ is a section of $L = TP \cap (TP)^\perp$. □

Let $\mathcal{F}$ be the foliation defined on P by $L = TP \cap (TP)^\perp$ (Frobenius theorem). The 2-form Ω_P is constant along the leaves of $\mathcal{F}$. Indeed, if ξ is a tangent vector to the leaves of $\mathcal{F}$, i.e. if ξ is a section of $TP \cap (TP)^\perp$, then

$$L_\xi \Omega_P = d\,i(\xi)\Omega_P + i(\xi)d\,\Omega_P = d\,i(\xi)\Omega_P.$$

But $i(\xi)\Omega_P = 0$ since ξ is a section of $(TP)^\perp$. Therefore $L_\xi\Omega_P = i(\xi)\Omega_P = 0$, which means that Ω_P is an $\mathcal{F}$-basic form.

If the leaf space $P/\mathcal{F} = N$ is a smooth manifold and $\pi : P \to N$ is the natural projection, there exists a 2-form Ω_N on N such that $\pi^\star\Omega_N = \Omega_P$. Clearly $TN \cong TP/L$ and $\Omega_N(\overline{x}) = \Omega_p(x)$ for an $x \in P$ projecting on $\overline{x} \in N$. Hence Ω_N is a symplectic form on N. The symplectic manifold (N,Ω_N) is said to be obtained by reduction from (M,Ω) [20]. For instance if P is a coisotropic submanifold of a symplectic manifold (M,Ω), then $(TP)^\perp \subset TP$ and $(TP)^\perp \cap (TP) = (TP)^\perp$ is a subbundle. The reduction machinery applies.

Example. — Let (M,Ω) be a symplectic manifold and $P \subseteq M$ a hypersurface. Then P is a coisotropic submanifold, and $(TP)^\perp$ is a one dimensional subbundle of $T_PM = j^\star TM$, where $j : P \to M$ is the embedding.

This 1-dimensional subbundle $(TP)^\perp$ is the kernel of the 2-form $j^\star\Omega = \Omega_P$. The subbundle $(TP)^\perp$ is called the *characteristic* subbundle of P. If P is oriented, so is $(TP)^\perp$: it then admits a global section Z called the characteristic vector field of P. If P is the level surface of a smooth function $H : M \to \mathbf{R}$ on M, then Z is proportional to the Hamiltonian vector field X_H. (See § 2.4).

2.4. Symplectic transformations and Poisson brackets

If (V,α) is a symplectic vector space, we denote by $\mathrm{Sp}(V,\alpha)$ the set of all linear endomorphisms $T : V \to V$ such that $T^\star\alpha = \alpha$, i.e. $(T^\star\alpha)(u_1,u_2) = \alpha(Tu_1,Tu_2) = \alpha(u_1,u_2)$. An element of $\mathrm{Sp}(V,\alpha)$ is called a linear symplectic transformation or a linear symplectomorphism. The set $\mathrm{Sp}(V,\alpha)$ is a group called the linear symplectic group of (V,α).

Let $\mathrm{Sp}(n,\mathbf{R})$ denote the linear symplectic group of $(\mathbf{R}^{2n},\alpha_s)$. It is clear from § 1.1 that if (V,α) is any $2n$-dimensional symplectic vector space, then $\mathrm{Sp}(V,\alpha)$ is isomorphic to $\mathrm{Sp}(n,\mathbf{R})$.

The group $\mathrm{Sp}(n,\mathbf{R})$ is a closed subgroup of $GL(2n,\mathbf{R})$, hence it is a Lie group.

2.4.1. Exercise. — Show that $\mathrm{Sp}(n, \mathbf{R})$ has dimension $n(2n+1)$.

We saw that for all $x, x' \in \mathbf{R}^{2n}$, $\alpha_s(x, x') = \langle Jx, x' \rangle$ where J is the skew-symmetric matrix

$$J = \left(\begin{array}{c|c} 0 & -I_n \\ \hline I_n & 0 \end{array} \right).$$

From this we see that if A is a matrix representing a symplectic transformation, that ${}^tA \cdot J \cdot A = J$.

Let us identify $\mathbf{R}^{2n}$ with $\mathbf{C}^n$ by:

$$u = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbf{R}^{2n} \longmapsto z = (z_1, \ldots, z_n) \in \mathbf{C}^n$$

where $z_k = x_k + i\, y_k$ and let $(\ ,\ )$ be the usual Hermitian inner product on $\mathbf{C}^n$

$$(z, z') = z_1 \overline{z}'_1 + \cdots + z_n \overline{z}'_n.$$

Since $z_k \overline{z}'_k = (x_k x'_k + y_k y'_k) - i(x_k y'_k - x'_k y_k)$ we see that, if $u, u' \in \mathbf{R}^{2n}$ are identified with $z, z' \in \mathbf{C}^n$ as above,

$$(z, z') = \langle u, u' \rangle - i\, \alpha_s(u, u').$$

The standard symplectic form α_s is minus the imaginary part of the Hermitian inner product. The unitary group $U(n)$ is the group of all invertible $n \times n$-matrices A with complex entries such that $(Az, Az') = (z, z')$. Hence an element $A \in U(n)$ preserves α_s and $\langle\ ,\ \rangle$, i.e. $U(n) \subseteq \mathrm{Sp}(n, \mathbf{R})$ and $U(n) \subseteq O(2n)$ where

$$O(2n) = \{A \in GL(2n, \mathbf{R}) \mid \langle Ax, Ax' \rangle = \langle x, x' \rangle\}.$$

Applying the polar decomposition to $A \in \mathrm{Sp}(n, \mathbf{R})$ we find that A can be written in a unique way as

$$A = B \exp(C \circ a)$$

where C is a symmetric complex matrix, $a : \mathbf{C}^n \to \mathbf{C}^n$ is the complex conjugation, and $B \in U(n)$. This shows that $U(n)$ is a deformation retract of $\mathrm{Sp}(n, \mathbf{R})$ by the path $t \mapsto B \exp(t(C \circ a))$. We will come back to the study of these groups in the next section.

Let now (M, Ω) be a symplectic manifold. A symplectic diffeomorphism (also called a symplectomorphism) is a $\mathcal{C}^\infty$ diffeomorphism $\varphi : M \to M$ such that $\varphi^\star \Omega = \Omega$. This means that for each $x \in M$, the linear mapping $T_x\varphi : T_xM \to T_{\varphi(x)}M$ pulls $\Omega_{(\varphi(x))}$ back to $\Omega(x)$. The set $\mathrm{Diff}^\infty_\Omega(M)$ of all symplectic diffeomorphisms of (M, Ω) forms a group which we think of as the non-linear version of $\mathrm{Sp}(V, \alpha) \cong \mathrm{Sp}(n, \mathbf{R})$.

In contrast with $\mathrm{Sp}(n, \mathbf{R})$ which is a finite dimensional Lie group, $\mathrm{Diff}^\infty_\Omega(M)$, in any reasonable sense, is an infinite dimensional space, and its structure is much more complicated (see for instance [7]). We also saw that if (V_1, α_1) and (V_2, α_2) are any two $2n$-dimensional vector spaces, then $\mathrm{Sp}(V_1, \alpha_1) \cong \mathrm{Sp}(V_2, \alpha_2)$. For symplectic manifolds (M_i, Ω_i), $i = 1, 2$, satisfying some additional conditions, $\mathrm{Diff}_{\Omega_1}(M_1)$ is isomorphic to $\mathrm{Diff}_{\Omega_2}(M_2)$ if and only if there exists a $\mathcal{C}^\infty$ diffeomorphism $h : M_1 \to M_2$ such that $h^\star \Omega_2 = \lambda\, \Omega_1$, for some constant λ (see [8]).

An easy way of getting diffeomorphisms is to integrate vector fields. We therefore consider the set $\mathcal{L}_\Omega(M)$ of vector fields on M whose local 1-parameter group belongs to $\mathrm{Diff}^\infty_\Omega(M)$. If f_t is a local flow of $X \in \mathcal{L}_\Omega(M)$, then $f_t^\star\Omega = \Omega$. Differentiating, we get:

$$0 = \frac{d}{dt}(f_t^\star\Omega) = f_t^\star(L_X\Omega).$$

Hence we see that $\mathcal{L}_\Omega(M)$ consists of vector fields X on M such that $L_X\Omega = 0$. Since $d\Omega = 0$, this is exactly the set of vector fields X such that $i(X)\Omega$ is a closed 1-form.

With the bracket of vector fields, $\mathcal{L}_\Omega(M)$ is a Lie algebra. Indeed it is clear that $\mathcal{L}_\Omega(M)$ is a linear subspace of the space $\mathcal{X}_M$ of all vector fields on M. We just need to show that if $X_i \in \mathcal{L}_\Omega(M)$, $i = 1, 2$, then $[X_1, X_2] \in \mathcal{L}_\Omega(M)$.

2.4.2. Exercise. — Check that $i([X_1, X_2])\Omega = d(\Omega(X_2, X_1))$ and hence that $i([X_2, X_2])\Omega$ is closed being exact.

Let $H^1(M, \mathbf{R})$ be the first de Rham cohomology group of M, i.e. the quotient of the space of closed 1-forms by the subspace of exact 1-forms. The mapping $s : \mathcal{L}_\Omega(M) \to H^1(M, \mathbf{R})$: $X \mapsto [i(X)\Omega]$ (where $[\alpha] \in H^1(M, \mathbf{R})$ is the equivalence class of a closed form modulo exact ones), vanishes on the ideal $[\mathcal{L}_\Omega(M), \mathcal{L}_\Omega(M)] \subseteq \mathcal{L}_\Omega(M)$ generated by the brackets $[X_i, X_j]$, $X_i, X_j \in \mathcal{L}_\Omega(M)$.

Given a smooth function $f : M \to \mathbf{R}$ on a symplectic manifold (M, Ω), we define the *Hamiltonian vector field* X_f of f as the unique vector field defined by the equality:

$$i(X_f)\Omega = -df$$

Recall that Ω induces an isomorphism between the space of vector fields on M and the space of 1-forms. Clearly, $X_f \in \operatorname{Ker} s \subseteq \mathcal{L}_\Omega(M)$.

Given $f, g \in C^\infty(M)$ (the space of smooth functions on M), we define $\{f, g\} \in C^\infty(M)$ by:

$$\{f, g\} = \Omega(X_f, X_g)$$

called the *Poisson bracket* of f and g.

2.4.3. Exercise. — Use exercise 2.4.2 to show that $[X_f, X_g] = X_{\{f,g\}}$.

PROPOSITION 2.4.4. — *The space $C^\infty(M)$, with the Poisson bracket, is a Lie algebra and the mapping $f \mapsto X_f$ is a surjective Lie algebra homomorphism from $C^\infty(M)$ to* $\operatorname{Ker} s \subseteq \mathcal{L}_\Omega(M)$, *with kernel* $\mathbf{R}$. *If M is connected, we have the exact sequence*

$$0 \longrightarrow \mathbf{R} \longrightarrow C^\infty(M) \longrightarrow \mathcal{L}_\Omega(M) \xrightarrow{\ s\ } H^1(M, \mathbf{R}) \longrightarrow 0.$$

Proof. — The only thing we still have to prove is that $(\mathcal{C}^\infty(M), \{\ ,\ \})$ is a Lie algebra. The equation $[X_f, X_g] = X_{\{f,g\}}$ shows that $f \mapsto X_f$ is a Lie algebra homomorphism. Observe that

$$\begin{aligned} \{f,g\} &= \Omega(X_f, X_g) \\ &= -i(X_f)(i(X_g)\Omega) \\ &= -i(X_f)[-dg] \\ &= X_f \cdot g. \end{aligned}$$

Hence if $X_i = X_{f_i}$, $i = 1, 2, 3$

$$X_i \cdot \Omega(X_j, X_k) = X_i \cdot \{f_j, f_k\} = \{f_i, \{f_j, f_k\}\}$$

and

$$\Omega([X_i, X_j], X_k) = \Omega(X_{\{f_i, f_j\}}, X_{f_k}) = \{\{f_i, f_j\}, f_k\}.$$

Since $d\Omega = 0$, we have

$$\begin{aligned} 0 = (d\Omega)(X_1, X_2, X_3) &= X_1 \cdot \Omega(X_1, X_3) - X_2 \cdot \Omega(X_1, X_3) \\ &\quad + X_3 \cdot \Omega(X_1, X_2) - \Omega([X_1, X_2], X_3) \\ &\quad + \Omega([X_1, X_3], X_2) - \Omega([X_2, X_3], X_1). \end{aligned}$$

Hence

$$\begin{aligned} 0 &= \{f_1, \{f_2, f_3\}\} - \{f_2, \{f_1, f_3\}\} + \{f_3, \{f_1, f_2\}\} \\ &\quad - \{\{f_1, f_2\}, f_3\} + \{\{f_1, f_3\}, f_2\} - \{\{f_2, f_3\}, f_1\} \\ &= 2(\{f_1, \{f_2, f_3\}\} + \{\{f_1, f_3\}, f_1\} + \{f_3, \{f_1, f_2\}\}) \end{aligned}$$

and the Jacobi identity is satisfied. □

The results mentioned here show that the group $\mathrm{Diff}^\infty_\Omega(M)$ is extremely large. Suppose for instance M is compact, any function $f : M \to \mathbf{R}$ yields a 1-parameter group $\varphi_t \in \mathrm{Diff}^\infty_\Omega(M)$ obtained by integrating X_f. The set $\mathcal{F}$ of diffeomorphisms $\varphi \in \mathrm{Diff}^\infty_\Omega(M)$ which are time 1 maps of flows φ_t coming from X_f generates a normal subgroup G of $\mathrm{Diff}^\infty_\Omega(M)_0$, the identity component (for the $\mathcal{C}^\infty$ topology) in $\mathrm{Diff}^\infty_\Omega(M)$.

If $H^1(M, \mathbf{R}) = 0$, it was proved in [7] that $G = \mathrm{Diff}^\infty_\Omega(M)_0$ i.e. any element of $\mathrm{Diff}^\infty_\Omega(M)_0$ can be written as a finite composition of time 1 diffeomorphisms obtained by integrating Hamiltonian vector fields X_f, $f \in \mathcal{C}^\infty(M)$.

To close this section, let us mention a very simple property of the flow φ_t of X_f, $f \in \mathcal{C}^\infty(M)$. Since $X_f \cdot f = \Omega(X_f, X_f) = 0$, we see that

$$\frac{d}{dt}(\varphi_t^\star f) = \varphi_t^\star(X_f \cdot f) = 0,$$

i.e. f is constant on the orbits of φ_t. In particular φ_t maps a level set $P = f^{-1}(a_0)$ into itself.

Suppose a_0 is a regular value, then P is a codimension 1 submanifold and X_f is tangent to it. Let $j : P \to M$ be the inclusion of P in M and $\Omega_P = j^\star\Omega$. The kernel of Ω_P is $(TP)^\perp \subseteq (TP)$, hence it is a 1-dimensional subbundle and the nowhere vanishing vector field X_f on P is a section of $(TP)^\perp$. If ξ is a tangent vector at x to P represented by a curve $\gamma : (-\varepsilon, \varepsilon) \to P$ with $\gamma(0) = x$, then

$$(i(X_f)\Omega)(x)(\xi) = -(df)(\xi)(x) = -\frac{d}{dt}f(\gamma(t))|_{t=0} = 0$$

since $f(\gamma(t)) = a_0$ for all t. Therefore $X_f(x) \in \text{Ker } \Omega_P(x)$.

We may apply the reduction scheme to this situation: the 2-form Ω_P is invariant under the flow φ_t and if the orbit space $N = M/\{\text{orbits of } \varphi_t\}$ is a manifold, and if $\pi : M \to N$ is the natural projection, there is a symplectic form Ω_N on N such that $\pi^\star\Omega_N = \Omega_P$.

For instance if $M = \mathbf{R}^{2n+2}$ with the symplectic form $\Omega_s = \sum_{i=1}^{n+1} dx_i \wedge dy_i$ and $f : \mathbf{R}^{2n+2} \to \mathbf{R}$ is the distance function

$$f(x, y) = \frac{1}{2}(x_1^2 + \cdots + x_{n+1}^2 + y_1^2 + \cdots + y_{n+1}^2)$$

then $X_f = \sum_{i=1}^{n+1}(y_i\partial/\partial x_i - x_i\partial/\partial y_i)$ is the Hopf field. The flow φ_t of X_f is obtained by solving the Hamilton equations:

$$\begin{cases} \dot{x}_k &= y_k \\ \dot{y}_k &= -x_k, \end{cases}$$

which can be rewritten as

$$\frac{dz_k}{dt} = -iz_k$$

with $z_k = x_k + iy_k$.

Hence $z_k(t) = z_k(0)e^{-it}$. Thus the flow φ_t of X_f generates a free action of the circle S^1 on the level set $S^{2n+1} = f^{-1}(1/2)$. Therefore the quotient space $\mathbf{CP}^n$ has a natural symplectic structure as the reduction of the symplectic manifold $(\mathbf{R}^{2n+2}, \Omega_s)$.

Appendix: the Maslov class, by M. Audin, A. Banyaga, F. Lalonde and L. Polterovich

Although the Maslov class is a very classical topic, we shall collect some more or less well known facts in this appendix. We follow the classical papers of Arnold [2] and Duistermaat [11]. Some books mention or study the Maslov class, for instance [3], [5] and [19]. In chapter X, we shall give some constraints on the Maslov class of a Lagrangian submanifold (rigidity theorems) using methods from the theory of pseudo-holomorphic curves. In this appendix, we content ourselves with the simplest properties and definitions.

A.1. The Lagrangian Grassmannian

Let Λ_n be the set of all Lagrangian subspaces of the symplectic vector space $(\mathbf{R}^{2n}, \Omega_s)$. The most convenient here is to consider that $\mathbf{R}^{2n} = \mathbf{C}^n$ with its canonical Hermitian structure $(\ ,\ )$ the imaginary part of which is $-\Omega_s$, the real part being the canonical scalar product. Recall that a subspace L is Lagrangian if and only if $L^\perp = iL$.

A.1.1. Exercise. — Show that Λ_1 is diffeomorphic to S^1.

PROPOSITION A.1.2. — *The space Λ_n is diffeomorphic to the homogeneous space $U(n)/O(n)$ and $\pi_1(\Lambda_n) \cong \mathbf{Z}$ (hence $H^1(\Lambda_n, \mathbf{Z}) \cong \mathbf{Z}$).*

Proof. — Let $L \in \Lambda_n$. Because $L^\perp = iL$, any orthonormal basis of L as a real subspace is a unitary basis of $\mathbf{C}^n$ as a complex vector space. Conversely, if $(e_1, \ldots, e_n)$ is a unitary basis of $\mathbf{C}^n$, the real subspace spanned by the n vectors is Lagrangian.

This shows that the unitary group $U(n)$ acts transitively on Λ_n, the stabiliser of the Lagrangian supspace $\mathbf{R}^n \subset \mathbf{C}^n$ being the orthogonal group $O(n)$. Hence $\Lambda_n \cong U(n)/O(n)$. Remember that $\pi_1(U(n))$ is isomorphic to $\mathbf{Z} = \pi_1 S^1$ thanks to the determinant mapping $U(n) \to S^1$ and that the fiber $SU(n)$ is simply connected.

Of course the determinant does not define a map on $U(n)/O(n)$ but, as it takes values in ± 1 on $O(n)$, its square does. Thus there is a well defined mapping

$$\det{}^2 : \Lambda_n \longrightarrow S^1 = \{z \in \mathbf{C} \mid |z| = 1\},$$

which is a fibration with fiber $SU(n)/SO(n)$.

Since $SU(n)$ is simply connected and $SO(n)$ is connected, $SU(n)/SO(n)$ is simply connected. The homotopy exact sequence of the fibration $\Lambda_n \to S^1$ gives

$$0 = \pi_1(SU(n)/SO(n)) \to \pi_1(\Lambda_n) \to \pi_1(S^1) \to \pi_0(SU(n)/SO(n)) = 0.$$

Therefore $\pi_1(\Lambda_n) \cong \pi_1(S^1) \cong \mathbf{Z}$. □

A.1.3. Exercise. — Deduce that Λ_n is a compact manifold of dimension $n(n+1)/2$.

The tangent space to the Lagrangian Grassmann manifold (see [11]*).* — Fix a Lagrangian subspace $\lambda \in \Lambda_n$. As we shall now show one can naturally identify the tangent space $T_\lambda \Lambda_n$ with the space $S(\lambda)$ of all symmetric bilinear forms on λ. Indeed, each tangent vector, say $v \in T_\lambda \Lambda$ can be represented as $\frac{d}{dt} A(t)\lambda_{|t=0}$, where $A(t)$ is a path of linear symplectic transformations of $\mathbf{C}^n$ with $A(0) = id$. We attach to such a vector a form $S_v \in S(\lambda)$ given by

$$S_v(\xi, \eta) = \omega(\xi, \frac{d}{dt} A(t)\eta_{|t=0}) \text{ for all } \xi, \eta \in \lambda\ .$$

A.1.4. Exercise. — Show that the map $v \mapsto S_v$ is a well-defined isomorphism.

A.2. The universal Maslov class

Let $d\theta$ be a 1-form representing the fundamental class of S^1, i.e. $[d\theta] = 1 \in \mathbf{Z} \cong H^1(S^1, \mathbf{Z})$. The class

$$\mu_n = (\det^2)^\star [d\theta] \in H^1(\Lambda_n, \mathbf{Z})$$

is a generator of $H^1(\Lambda_n, \mathbf{Z})$ called the *universal Maslov class.*

The reader interested in higher cohomology classes is invited to consult [12], [6] or [16] for instance.

A.2.1. Exercise. — Show that the value of the Maslov class on the cycle $\{e^{2\pi it} \cdot \mathbf{R}^n\}$, $t \in [0,1]$ is equal to $2n$.

Geometry of the universal Maslov class (see [2], [11], [4], [10]*).* — Here we give a geometric description of the universal Maslov class. Fix a Lagrangian subspace, say $\alpha \in \Lambda_n$. Let Λ_n^α be the set of all Lagrangian subspaces which are *not transversal* to α. It turns out that Λ_n^α is a *stratified* submanifold of the form $\bigcup\limits_{j=1}^{n} \Lambda_{n,j}^\alpha$, where codim $\Lambda_{n,1}^\alpha = 1$ and codim $\Lambda_{n,j}^\alpha \geq 3$ for $j > 1$.

A.2.2. Exercise. — Show that the set of all Lagrangian subspaces which are *transversal* to α (the complementary of Λ_n^α) is diffeomorphic to the vector space of *symmectric* bilinear maps $S : \alpha \to \alpha$ (Hint: any n-plane transversal to α is the graph of a linear map $f : i\alpha \to \alpha$). Deduce that $\Lambda_n - \Lambda_n^\alpha$ is contractible.

Define a transversal orientation of Λ_n^α as follows. A tangent vector $v \in T_\lambda \Lambda_n$ is called *α-positive* if $\lambda \in \Lambda_n^\alpha$ and S_v is positive definite on $\lambda \cap \alpha$. One can show that all α-positive vectors form an open convex cone in $T_{\Lambda_n^\alpha}\Lambda_n$. Moreover each such vector is transversal to Λ_n^α.

It turns out that the submanifold Λ_n^α together with the transversal orientation by α-positive vectors represents a *singular cycle* which is Poincaré-dual to the universal Maslov class.

The Maslov class for a Lagrangian immersion. — If we have a continuous map $g : L \to \Lambda_n$, we may pull back μ_n to L by g and get $g^\star(\mu_n) \in H^1(L, \mathbf{Z})$.

For instance if $f : L \to \mathbf{R}^{2n}$ is a Lagrangian immersion, i.e. for all $x \in L$, $(T_x f)(T_x L) \in \Lambda_n$, then $x \mapsto (T_x f)(T_x L)$ is a mapping $g = \overline{f} : L \to \Lambda_n$ (the *Gauss map*) and the class $\overline{f}^\star(\mu_n) \in H^1(L, \mathbf{Z})$ is called the Maslov class of the immersion j.

Example. — For $n = 1$, the Maslov class of a Lagrangian immersion $S^1 \to \mathbf{C}$ is nothing other than the degree of the tangent mapping.

A.2.3. Exercise. — Show that any even integer is the Maslov class of some (Lagrangian) immersion of S^1 into $\mathbf{C}$. Use the Whitney turning tangents theorem (see [9] for instance) to show that any (Lagrangian) *embedding* of S^1 into $\mathbf{C}$ has Maslov class ± 2.

A.2.4. Exercise. — Let $f : L \longrightarrow \mathbf{C}^n$ be a Lagrangian immersion. Let γ be a smooth loop on L. Assume that the image of the tangent vector to γ by the differential of the Gauss map is always α-positive (for some $\alpha \in \Lambda_n$). Show that the Maslov class of f, evaluated on γ, is positive.

When M is the cotangent bundle $T^\star T^n$ (p, q mod 1) of the torus and $\omega = \sum dp \wedge dq$, we identify each tangent space $T_x M$ with the standard symplectic linear space $\mathbf{C}^n$. Thus any Lagrangian immersion into $T^\star T^n$ has a Gauss map $\overline{f} : L \longrightarrow \Lambda_n$ and a Maslov class μ_f, equal to $\overline{f}^\star \mu_0 \in H^1(L; \mathbf{Z})$. We shall also use the notation μ_L where $L \subset M$ is a Lagrangian submanifold and $f : L \longrightarrow M$ is the inclusion.

Recall that the group $H^1(L; \mathbf{Z})$ is a free $\mathbf{Z}$-module. An element of $H^1(L; \mathbf{Z})$ is called *primitive* if it can be included into some basis. For every $\nu \in H^1(L; \mathbf{Z}) - \{0\}$ we denote by $\|\nu\|$ the unique positive integer such that $\nu = \|\nu\| \cdot$ (primitive element). We set $\|0\| = 0$.

A.2.5. Exercise. — Let $f, g : L \longrightarrow M$ be two Lagrangian immersions with $\|\mu_f\| \neq \|\mu_g\|$. Show that f and g belong to different connected components of the space of all Lagrangian immersions.

A.2.6. Exercise. — Show that $\|\mu_f\|$ is even for a Lagrangian immersion f of an *orientable* manifold.

A.2.7. Exercise. — Show that $\|\mu_L\| = 2$ for the standard Lagrangian torus $L = \{p_j^2 + q_j^2 = 1; \quad j = 1, \ldots, n\} \subset \mathbf{C}^n$.

A.2.8. Exercise (see [18]*).* — Find a Lagrangian embedding $f : S^{2k-1} \times S^1 \longrightarrow \mathbf{C}^{2k}$ with $\|\mu_f\| = 2$.

The Maslov class of a pair of two Lagrangian subbundles. — Now we look at the case of a symplectic vector bundle $\pi : E \longrightarrow M$ over a smooth manifold M and two Lagrangian subbundles L_0, L_1 in it. Choose a vector bundle λ_0 such that $L_0 \oplus \lambda_0$ is a trivial bundle of rank m. Then $L_1 \oplus \lambda_0$ and $L_0 \oplus \lambda_0$ are both Lagrangian subbundles of the trivial symplectic bundle $E \oplus (\lambda_0 \otimes \mathbf{C})$. For each $x \in M$, there exists $E_x \in U(m)$ such that $(L_1 \oplus \lambda_0)_x = E_x((L_0 \oplus \lambda_0)_x)$ which is well defined modulo $O(m)$. This way we get a mapping $f_{L_0,L_1} : M \longrightarrow \Lambda_m$.

We will call $\mu_{L_0,L_1} = f^\star_{L_0,L_1}(\mu_m) \in H^1(M,Z)$ the *Maslov class of the pair* L_0, L_1 of Lagrangian subbundles. It measures how much one of the bundles rotates around the other one.

A.2.9. Exercise (additivity of the Maslov class). — Let L_0, L_1 and L_2 be three Lagrangian subbundles of a symplectic bundle E. Prove that

$$\mu_{L_0,L_2} = \mu_{L_0,L_1} + \mu_{L_1,L_2}.$$

A.2.10. Exercise (naturality of the Maslov class). — Let Φ be a morphism of symplectic bundles

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & F \\ \downarrow & & \downarrow \\ V & \xrightarrow{\varphi} & W \end{array}$$

(Φ is a symplectic linear isomorphism $E_x \to F_{\varphi(x)}$). Let L_0, L_1 be two Lagrangian subbundles of F. Show that

$$\mu_{\varphi^\star L_0, \varphi^\star L_1} = \varphi^\star \mu_{L_0,L_1}.$$

Note that, for a symplectic bundle E to be able to contain a Lagrangian subbundle L, it is necessary (and sufficient) that E is isomorphic to $L \otimes \mathbf{C}$ (as a symplectic or as a complex vector bundle). Of course, the complex vector bundles which happen to be complexifications of real vector bundles are rather special. Anyway, there are interesting examples. For instance, let $f : M \to T^\star N$ be a Lagrangian immersion. Then $E = f^\star(T(T^\star N)) \to M$, the pull back by f of the tangent bundle $T(T^\star N) \to T^\star N$ is a symplectic vector bundle over M. Let L_0 be the tangent bundle of M and $L_1 = f^\star(V)$ where $V \to T^\star M$ is the subbundle of $T(T^\star M)$ consisting into vectors tangent to the fibers of $T^\star M \to M$. Then L_0, L_1 are both Lagrangian subbundles of E and $\mu_{L_0,L_1} \in H^1(M,Z)$ is an object depending only on the Lagrangian immersion, the significance of which is explained in the following exercise.

A.2.11. Exercise. — Let $\pi : E \to M$ be a symplectic vector bundle and suppose L_0, L_1 are two Lagrangian subbundles such that $(L_0)_x$ and $(L_1)_x$ are transversal for any $x \in M$. Show that the map $f_{L_0,L_1} : M \to \Lambda_m$ is homotopic to a constant map.

Thus the Maslov class μ_{L_0,L_1} appears as an obstruction to the transversality of the Lagrangian subbundles L_0, L_1.

A.2.12. Exercise. — Let $f : M \to T^\star N$ be a Lagrangian immersion. Express in terms of the composition

$$M \xrightarrow{f} T^\star N \xrightarrow{\pi} N$$

the fact that the two subbundles L_0, L_1 defined above are transversal. So what?

A.3. The composition of two Lagrangian immersions

If $f : M \to T^{\star}N$ is a Lagrangian immersion, we shall simply call the above μ_{L_0,L_1} the Maslov class of f. Now, let X, Y and Z be three manifolds of the same dimension n. Suppose $f : X \to T^{\star}Y$ and $g : Y \to T^{\star}Z$ are two Lagrangian immersions.

A.3.1. Exercise. — Use the Darboux-Weinstein theorem to show that g extends to a symplectic immersion $\tilde{g}$ of a neighbourhood of the zero section of $T^{\star}Y$ into $T^{\star}Z$. Let f' be the Lagrangian immersion of X into this neighbourhood obtained from f by a suitable homothety in the fibers. Check that $\tilde{g} \circ f'$ is a Lagrangian immersion, whose regular homotopy class does not depend on the choices made.

We shall write $g \circ f$ for any $\tilde{g} \circ f'$. Figure 1 shows the composition of two immersions.

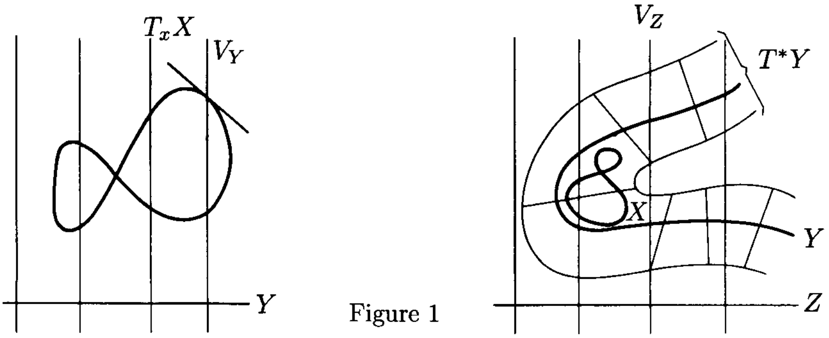

Figure 1

A.3.2. Exercise. — In the symplectic bundle $\tilde{g}^{\star}T\,(T^{\star}Z) \to T^{\star}Y$, we have two Lagrangian subbundles, the tangent bundle L_0 along the fibers of $T^{\star}Y$ and that of $T^{\star}Z$ (say L_1), thus a class $\mu_{L_0,L_1} \in H^1\,(T^{\star}Y; \mathbf{Z})$. Let π be the projection $T^{\star}Y \to Y$. Show that $\mu_{L_0,L_1} = \pi^{\star}\mu_g$ (Hint: in $\pi^{\star}T\,(T^{\star}Z)$ we have three Lagrangian subbundles. Use the results of exercises A.2.10, A.2.9 and A.2.11 to conclude).

Now, as is easily understood from figure 1, the Maslov class of the composition of two Lagrangian immersions, which measures the way the tangent bundle of X rotates with respect to the fibers of $T^{\star}Z$, must be the sum of the two Maslov classes, more precisely we have the following simple *summation formula*, which will be very useful in chapter X.

PROPOSITION A.3.3. — *Let X, Y, Z be three n-dimensional manifolds, and*

$$f : X \to T^{\star}Y \qquad g : Y \to T^{\star}Z$$

be two Lagrangian immersions. Then

$$\mu(g \circ f) = \mu(f) + (\pi \circ f)^\star(\mu(g))$$

where $\pi : T^\star Y \to Y$ *is the projection.*

Proof. — By definition, the Maslov class $\mu(f)$ compares TX and the tangent bundle V_Y along the fibers in $f^\star T(T^\star Y) \to X$: $\mu(f) = \mu_{TX,V_Y} \in H^1(X;\mathbf{Z})$. Now the symplectic immersion $\widetilde{g} : T^\star Y \to T^\star Z$ gives a morphism φ $(= T\widetilde{g})$

$$\begin{array}{ccc} f^\star T(T^\star Y) & \xrightarrow{\ \varphi\ } & \widetilde{g}^\star T(T^\star Z) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\ f\ } & T^\star Y \end{array}$$

thus in $f^\star T(T^\star Y)$ we have three Lagrangian subbundles: TX and the tangents along the fibers of $T^\star Y$ and $T^\star Z$. So

$$\mu(g \circ f) = \mu_{TX,V_Y} + \mu_{V_Y,V_Z} = \mu_f + f^\star \mu_{L_0,L_1}$$

using the notations of exercise A.3.2 which also gives the proof. □

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Chapter II

Symplectic and almost complex manifolds

Michèle Audin

with an appendix by P. Gauduchon

The aim of this chapter is to introduce the basic problems and (soft!) techniques in symplectic geometry by presenting examples—more exactly series of examples—of almost complex and symplectic manifolds: it is obviously easier to understand the classification of symplectic ruled surfaces if you have already heard of Hirzebruch surfaces for instance.

The first § explains why there are so many almost complex structures on a symplectic manifold, an elementary, but very important tool of Gromov's theory. Then, I will explain what the complex ruled surfaces look like (Hirzebruch surfaces) and give some systematic constructions of symplectic manifolds, either by symplectic reduction or by surgery: here I will explain which surgeries cannot work... and which *do* (for instance the ones used in the recent constructions of Gompf). The last § is an appendix, written by P. Gauduchon, in which he gives a construction of an almost complex structure on a certain jet space which will be useful to state and prove Gromov's Schwarz lemma in chapter VII.

I have not tried to give a complete list of references for the classical material of this chapter. Obviously, one should have a look at [1], [19], [21] and [29]. The other references given are either more technical but unavoidable, as is obviously [12], or well known to me. All of this chapter, and specially § 1.1 is an improvement of the lectures given at CIMPA due to the clever and patient help of Emmanuel Giroux and Bruno Sévennec whom I am very pleased to thank.

1. Almost complex structures

The aim of this § is to show that there *are* almost complex structures related to symplectic forms, that there are in fact many of them, but that in some sense not too many: they form a contractible set. I shall also *very* briefly discuss integrability and Kählerness.

An *almost complex structure* on a manifold W is a section J of the bundle $\operatorname{End} TW$ such that $J_x^2 = -\operatorname{Id}_{T_xW}$ for all x in W. For instance, if W is a complex manifold (i.e. with holomorphic change of local coordinates), its tangent space T_xW at any point has a natural structure of complex vector space and multiplication by i is an almost complex structure. It is the situation we are trying to mimic, J plays the role of multiplication by i. Note that J gives to TW the structure of a complex vector bundle, each T_xW being a complex vector space by

$$(a+ib)\cdot\xi = a\xi + bJ\xi.$$

Let X be an oriented surface, endowed with a Riemannian metric, in such a way that we have a notion of rotation by $+\pi/2$ in every tangent plane. The family of all such rotations defines an almost complex structure on X, thus any oriented surface admits almost complex structures. Of course, not any manifold, even of even dimension, even oriented, may be endowed with an almost complex structure. For instance, we shall see below that it is not the case for the 4-sphere.

1.1. The linear case: almost complex structures, almost complex structures related to a non-degenerate skew-symmetric 2-form

In $\mathbf{R}^{2n}$, a *linear* almost complex structure is an endomorphism J such that $J^2 = -\operatorname{Id}$. The linear group $GL(2n,\mathbf{R})$ acts on the space of these structures by conjugation: $g\cdot J = gJg^{-1}$.

1.1.1. Exercise. — Use a J-complex basis to show that this action is transitive and that the stabiliser of a given almost complex structure J_0 is the group of all *complex* (for J_0) automorphisms.

In other words, the set of all almost complex structures can be identified with

$$\mathcal{J}_n = GL(2n,\mathbf{R})/GL(n,\mathbf{C}).$$

In $\mathbf{C}^n$, the standard Hermitian product decomposes into its real and imaginary parts:

$$\langle u,v\rangle = (u,v) - i\omega(u,v)$$

the Euclidean scalar product of $\mathbf{R}^{2n}$ and its symplectic form, respectively. One easily checks that $\omega(u,iv)$ is the Euclidean scalar product and moreover that $\omega(iu,iv) = \omega(u,v)$: multiplication by i is an isometry of ω. We now try to mimic this situation in the following definitions.

Let ω be a non-degenerate skew-symmetric bilinear form on $\mathbf{R}^{2n}$. A linear almost complex structure J is *tamed* by ω if the quadratic form $\omega(x,Jx)$ is positive definite. It is *calibrated* by ω if, moreover, it is an isometry of ω: $\omega(Jx,Jy) = \omega(x,y)$ for all x, $y \in \mathbf{R}^{2n}$.

1.1.2. Exercise. — Show that, in this case, the bilinear form $(x, y) \mapsto \omega(x, Jy) = (x, y)_J$ is a scalar product on $\mathbf{R}^{2n}$, for which one has

$$\begin{cases} E^\perp = (JE)^\circ \\ E^\circ = (JE)^\perp \\ JE \cap E^\circ = 0, \end{cases}$$

$^\circ$ and $^\perp$ denoting orthogonality respectively for ω and $(\ ,\ )_J$.

1.1.3. Exercise. — Let L be any Lagrangian subspace. Check that $JL \in \Lambda_L$ (the space of all Lagrangian subspaces transversal to L) and that it is $L^\perp$ (for $(\ ,\ )_J$). Deduce that there exists a basis $(x_1, \ldots, x_n, Jx_1, \ldots, Jx_n)$ of $\mathbf{R}^{2n}$ which is symplectic.

Let Ω_n be the set of all non-degenerate skew-symmetric bilinear forms on $\mathbf{R}^{2n}$, and, for $\omega \in \Omega_n$, let $\mathcal{J}_c(\omega)$ be the set of almost complex structures calibrated by ω. The symplectic group is denoted by Sp_{2n} ($= \mathrm{Sp}(n, \mathbf{R})$ in the notation of chapter I).

PROPOSITION 1.1.4. — *$\Omega_n \cong GL(2n, \mathbf{R})/Sp_{2n}$, $\mathcal{J}_c(\omega) \cong Sp_{2n}/U(n)$.*

Proof. — The group $GL(2n, \mathbf{R})$ acts on Ω_n by $(g, \omega) \mapsto \omega(g^{-1}\cdot, g^{-1}\cdot)$. The action is transitive, due to the existence of symplectic bases, and the stabiliser of ω is its group of isometries Sp_{2n}.

Also, if ω is fixed, Sp_{2n} acts on $\mathcal{J}_c(\omega)$ by $(g, J) \mapsto gJg^{-1}$:

$$\begin{aligned} \omega\left(gJg^{-1}x, gJg^{-1}y\right) &= \omega\left(Jg^{-1}x, Jg^{-1}y\right) && g \text{ is an isometry of } \omega \\ &= \omega\left(g^{-1}x, g^{-1}y\right) && J \in \mathcal{J}_c(\omega) \\ &= \omega(x, y) && g^{-1} \text{ is an isometry of } \omega \end{aligned}$$

the action is transitive due to the existence of "complex symplectic bases" (exercise 1.1.3) and the stabiliser of a given J is $Sp_{2n} \cap GL(n, \mathbf{C}) = U(n)$ (see 1.1.5). □

1.1.5. Exercise. — Let $\mathbf{C}^n = \mathbf{R}^{2n}$ be endowed with the standard symplectic structure. Consider the subgroups $O(2n)$, Sp_{2n}, $GL(n, \mathbf{C})$, $U(n)$ and compute the intersections of any two of them.

Denote by $\mathcal{J}_t(\omega)$ the set of all almost complex structure tamed by ω. Consider now $\mathbf{C}^n$ endowed with its canonical structures (J_0, ω and the scalar product). I learned the next proposition from B. Sévennec:

PROPOSITION 1.1.6. — *The map $J \mapsto (J + J_0)^{-1} \circ (J - J_0)$ is a diffeomorphism from $\mathcal{J}_t(\omega)$ (resp. $\mathcal{J}_c(\omega)$) onto the open unit ball in the vector space of matrices (resp. symmetric matrices) S such that $J_0 S + S J_0 = 0$.*

COROLLARY 1.1.7. — *The spaces $\mathcal{J}_c(\omega)$ and $\mathcal{J}_t(\omega)$ are connected, and even contractible.* □

Remark. — The fact that $\mathcal{J}_c(\omega)$ is contractible is both well-known and easy (see below in exercise 1.1.11 the classical proof). That $\mathcal{J}_t(\omega)$ is contractible is a statement taken from [12]. The original proof of Gromov required some agility in the use of fibrations. The argument of Sévennec we use here is both nicer and more elementary.

Proof of the proposition. — Note first that $J + J_0$ is invertible: if $x \neq 0$, $\omega(x, (J + J_0)(x)) > 0$ thus $\operatorname{Ker}(J + J_0) = \{0\}$ and the map is well defined. Write

$$S = (J + J_0)^{-1} \circ (J - J_0) = (A + \mathrm{Id})^{-1} \circ (A - \mathrm{Id})$$

where $A = J_0^{-1} \circ J$. We now prove that $\|S\| < 1$, that is, that $\|Ax - x\|^2 < \|Ax + x\|^2$ for all $x \neq 0$:

1.1.8. Exercise. — Check that

$$\|Ax + x\|^2 - \|Ax - x\|^2 = 4\omega(x, Jx) > 0.$$

Now let S be a matrix such that $\|S\| < 1$, which implies that $\mathrm{Id} - S$ is invertible and that the endomorphism

$$J = J_0 \circ (\mathrm{Id} + S) \circ (\mathrm{Id} - S)^{-1}$$

is well defined.

1.1.9. Exercise. — Check that J is an almost complex structure ($J^{-1} = -J$) if and only if $J_0 S + S J_0 = 0$ and that it is tamed by ω.

Obviously the map $S \mapsto J$ is the inverse of the one we are considering, so that we have proved the proposition for tamed structures. For calibrated structures, we just have to remark that J is calibrated if and only if S is symmetric:

1.1.10. Exercise. — Check that J is calibrated if and only if for any x, $y \in \mathbf{C}^n$,

$$\omega((\mathrm{Id} - S)x, (\mathrm{Id} - S)y) = \omega((\mathrm{Id} + S)x, (\mathrm{Id} + S)y)$$

and that this is equivalent to

$$\omega(Sx, y) + \omega(x, Sy) = 0.$$

Deduce that J is calibrated if and only if S is symmetric.

This ends the proof of the proposition. □

The next exercise gives "the" classical proof that $\mathcal{J}_c(\omega)$ is contractible.

1.1.11. Exercise. — Let L be a Lagrangian subspace. If $J \in \mathcal{J}_c(\omega)$, $JL \in \Lambda_L$ and $(\ ,\ )_J$ defines a positive definite symmetric bilinear form on L.

1. Conversely, given $L' \in L$ and a scalar product g on L, construct an element of $\mathcal{J}_c(\omega)$: for a non zero vector $x \in L$, consider its g-orthogonal $x^\perp \subset L$ as a subspace of E. Now $\left(x^\perp\right)^\circ$ is an $n+1$-dimensional subspace of E. Show that $\dim \left(x^\perp\right)^\circ \cap L' = 1$. Define Jx to be the unique vector of this line such that $\omega(x, Jx) = 1$ and check that this defines the required element of $\mathcal{J}_c(\omega)$.
2. Deduce that there is a one-to-one correspondence between $\mathcal{J}_c(\omega)$ and $\Lambda_L \times \mathcal{Q}(L)$, where $\mathcal{Q}(L)$ is the set of all positive definite quadratic forms on L.
3. Prove that Λ_L can be identified with the vector space[1] of all symmetric $n \times n$ matrices.
4. Deduce that $\mathcal{J}_c(\omega)$ is contractible.
5. Consider $\mathcal{C}_n = \{(\omega, J) \mid J \text{ is calibrated by } \omega\}$. Show that $\mathcal{C}_n$ can be identified with the homogeneous space $GL(2n, \mathbf{R})/U(n)$. Show that the first projection $\mathcal{C}_n \to \Omega_n$ is a homotopy equivalence.

1.2. Tamed and calibrated almost complex structures on a symplectic manifold

Let W be a $2n$-dimensional manifold endowed with a symplectic form ω. With its tangent bundle are associated the bundles

$$\begin{array}{rcll} \mathcal{J}(TW) & \longrightarrow & W & \text{with fibre } \mathcal{J}_n \\ \mathcal{J}_c(TW, \omega) & \longrightarrow & W & \text{with fibre } \mathcal{J}_c(\omega) \\ \mathcal{J}_t(TW, \omega) & \longrightarrow & W & \text{with fibre } \mathcal{J}_t(\omega) \end{array}$$

With an obvious definition of an almost complex structure tamed (resp. calibrated) by ω, such an object is nothing other than a *section* of the bundle $\mathcal{J}_t(TW, \omega)$ (resp. $\mathcal{J}_c(TW, \omega)$). These two bundles having contractible fibres, one gets

PROPOSITION 1.2.1. — *The space of almost complex structures tamed (or calibrated) by ω is non empty and contractible.* □

Of course, the existence of a non degenerate 2-form calibrating or taming the complex structure is essential: there *are* oriented even dimensional manifolds which do not admit any almost complex structure, the 4-sphere, for instance (see § 1.5).

There is also a relative version of 1.2.1: you can extend any section already defined along a closed subset of W. It is often[2] used in the following form.

[1]See the appendix to chapter I. There, we also prove that this vector space can be identified with the tangent space of Λ_n.

[2]For instance in the proofs of the classification theorems of Dusa McDuff.

PROPOSITION 1.2.2. — *Let $\Sigma \subset W$ be a symplectic submanifold, and let J_0 be an almost complex structure defined along Σ (i.e. an endomorphism of $TW|_\Sigma$ of square $-\mathrm{Id}$) and tamed (resp. calibrated) by ω. There exists an ω-tamed (resp. ω-calibrated) almost complex structure J on W which extends J_0.* □

1.2.3. Exercise. — If V is an almost complex submanifold of the symplectic manifold W for an almost complex structure calibrated by the symplectic form, show that it is a symplectic submanifold.

Remarks.

1. I must mention here that I have never used the fact that ω is a *closed* form.
2. There can be a lot of symplectic forms taming or calibrating the same almost complex structure. For instance this is the case for all forms $\omega \oplus \lambda\omega$ on $S^2 \times S^2$ (ω a volume form, $\lambda > 0$) and the usual (product) complex structure.
3. It is legitimate to wonder whether it is really useful to consider *tamed* almost complex structures. I know of at least one place where one is forced to use tamed almost complex structures. This is in the study of the so-called CR-structures (see the beginning of [7]).

1.3. Integrable complex structures (a few words)

As I have already mentioned, an *integrable* complex structure (that is, the structure of a complex analytic manifold) is the basic example of an almost complex structure.

Of course, there are more examples of almost complex structures than just integrable structures. There even exist almost complex manifolds which do not have any complex structure: this is the case for instance for the connected sum of three (or any odd number—except 1) copies of $\mathbf{P}^2(\mathbf{C})$, see [5] and § 1.5 below.

The way the almost complex structure J is related to the differentiable structure of the manifold is described by the famous Nijenhuis tensor (I understand that it was introduced by Ehresmann, see [21]):

$$N(X,Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y].$$

In the appendix to this part, P. Gauduchon will use the Nijenhuis tensor to discuss a natural almost complex structure on the space of 1-jets of pseudo-holomorphic mappings between two almost complex manifolds. In this §, we evoke the relation of N with integrability.

1.3.1. Exercise.

1. Check that, if $X \mapsto [X, Y]$ is $\mathbf{C}$-linear (in the sense that it commutes with J), then $N \equiv 0$.

2. Show that N is actually a tensor, that is: $N(X,Y)$ depends only on the values $X(x)$, $Y(x)$ and not really on the vector fields X and Y.

3. Compute $N(X,JX)$ and deduce that, if W is a *surface*, N vanishes identically (on vectors) and everywhere (on W).

1.3.2. Exercise. — For any map $f : \mathbf{R}^{2n} \to \mathbf{C}$ and any vector field X on $\mathbf{R}^{2n}$, recall that $X \cdot f = X \cdot (f_1 + if_2) = X \cdot f_1 + iX \cdot f_2$ so that $f \mapsto X \cdot f$ is a complex linear map.

1. Let $\mathbf{R}^{2n}$ be endowed with an almost complex structure J and suppose f is a *holomorphic* function, that is $df \circ J = i\, df$. Show that $df\,(N(X,Y)) = 0$ for all X, Y.

2. Suppose there exist n holomorphic functions on $\mathbf{R}^{2n}$, which are independent at some point x. Show that N identically vanishes at x.

3. Assume W is a complex manifold and J is its (integrable) complex structure. Show that the Nijenhuis tensor vanishes identically and everywhere on W.

Remarks.

1. What was used in the previous exercise is that an integrable complex structure is one with many holomorphic functions. In general, an almost complex manifold has *no* holomorphic functions at all. On the other hand, it has *a lot* of holomorphic curves (maps $f : \mathbf{C} \to W$ such that $df \circ i = J \circ df$)—they will be the main tool in this book.

2. It is a hard theorem (in its full generality) [27] that the converse is also true. For an introduction to these questions, see [20] for instance.

3. Any almost complex structure on a *surface* is integrable (a result already known to Gauss in the analytic case). In the smooth case, it is of course a consequence of what we just said, but there exist simpler proofs, see [28] for instance.

1.4. Kähler structures

If M is a complex manifold endowed with a *Hermitian* metric, the imaginary part of this metric is skew-symmetric and thus can be viewed as a 2-form ω which calibrates the complex structure J (ω is said to be a $(1,1)$-form). Of course, ω is non-degenerate.

When ω is closed, that is when it is a symplectic form, it is called a *Kähler* form and M is a Kähler manifold.

1.4.1. Exercise. — Calling d', d'' the Dolbeault operators defined by

$$d'f = \sum \frac{\partial f_j}{\partial dz_j} dz_j \qquad d''f = \sum \frac{\partial f_j}{\partial d\bar{z}_j} d\bar{z}_j,$$

consider the function $f : \mathbf{C}^n \to \mathbf{R}$ defined by $f(v) = \log\left(1 + \|v\|^2\right)$. Let ω be the 2-form $\omega = id'd''f$. Show that ω is a Kähler form on $\mathbf{C}^n$.

Kähler manifolds are very common: any complex submanifold of a Kähler manifold is Kähler (for the induced $(1,1)$-form), thus all projective complex algebraic manifolds are Kähler—it suffices to prove that the projective space is, and it is: in this chapter, there will be a lot of definitions of symplectic forms on $\mathbf{P}^n(\mathbf{C})$. All will coincide (up to a scalar factor) and will be Kähler. The next exercise gives a first construction.

1.4.2. Exercise. — Let $\varphi_k : U_k \to \mathbf{C}^n$ $(0 \le k \le n)$ be the usual charts in $\mathbf{P}^n(\mathbf{C})$:

$$U_k = \{[x_0, \ldots, x_n] \mid x_k \neq 0\} \qquad \varphi_k([x_0, \ldots, x_n]) = \left(\frac{x_i}{x_k}\right)_{i \neq k} \in \mathbf{C}^n = \left\{(X_k)_{i \neq k}\right\}.$$

Put $\omega_k = \varphi_k^\star \omega$, where ω is the Kähler form of exercise 1.4.1 so that

$$\omega_k = id'd'' \log \frac{\sum |x_i|^2}{|x_k|^2}.$$

Show that, on $U_k \cap U_l$, ω_k and ω_l coincide. Deduce that the ω_k $(0 \le k \le n)$ define a Kähler form on $\mathbf{P}^n(\mathbf{C})$.

Kähler manifolds satisfy the Hodge duality, in particular they have $\dim H^{p,q} = \dim H^{q,p}$ (see [11] for instance). Note that it implies that their odd Betti numbers must be even.

It was only in 1976 that Thurston gave the first example of a non-Kähler symplectic manifold: a clever quotient of $\mathbf{R}^4$ by a group preserving the symplectic form... but such that the quotient has $b_1 = 3$ (see [29]). Then in 1984, D. McDuff gave examples of simply connected non-Kähler symplectic manifolds [24] of rather large dimensions. Cleverly combining a very easy construction (see § 5.3 below) with sophisticated examples, Gompf has recently produced 4-dimensional symplectic non-Kähler manifolds which are simply connected [10].

1.5. Some general remarks

As already mentioned, there are many more almost complex structures than just complex integrable structures. In this section, we shall concentrate on oriented 4-manifolds. Let W be such a manifold. It has two obvious topological invariants, its

Euler characteristic χ and the signature σ of its intersection form (see chapter IV). Now suppose J is an almost complex structure on W. Then (TW, J) is a complex vector bundle and, as such, it has Chern classes $c_1(J)$, $c_2(J)$. The latter is the Euler class of TW and does not depend on J. The first does depend on J and is related to the topological invariants by

$$\langle c_1^2, [W]\rangle = 3\sigma + 2\chi.$$

For instance if $W = S^4$, $H^2(W) = 0$ thus $\sigma = 0$. On the other hand $\chi = 2$ thus $3\sigma + 2\chi = 4 \neq 0$ and of course there is no element c_1 in $H^2(S^4)$ such that $\langle c_1^2, [S^4]\rangle = 4$, thus S^4 does not admit any almost complex structure[3].

Note also that the mod 2 reduction w of $c_1(J)$ depends only on W: it was shown by Wu in [31] that w is a characteristic element for the quadratic form, that is, it satisfies

$$\forall x \in H^2(W; \mathbf{Z}/2), \qquad w \smile x = x \smile x.$$

It is also a result of Wu in [31] that

PROPOSITION 1.5.1. — *Homotopy classes of almost complex structures on W are in one-to-one correspondence with integral classes $c \in H^2(W; \mathbf{Z})$ which lift w and such that $\langle c^2, [W]\rangle = 3\sigma + 2\chi$.*

As a result one finds as I already mentioned, that there are almost complex structures on the connected sum of 3 copies of $\mathbf{P}^2(\mathbf{C})$, but this manifold has *no* integrable complex structure (and no symplectic form either). Other examples can be found for instance in [5].

1.5.2. Exercise. — Consider the connected sum W of two copies of $\mathbf{P}^2(\mathbf{C})$ both with the canonical orientation. Show that $H^2(W) = H^2(\mathbf{P}^2(\mathbf{C})) \oplus H^2(\mathbf{P}^2(\mathbf{C})$ and thus is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$. Compute $\sigma(W)$ and $\chi(W)$. Assume W can be endowed with an almost complex structure J and show that the first Chern class of J must decompose into two odd numbers in the previous splitting of H^2. Deduce that W has no almost complex structure[4].

Of course, two complex structures can be homotopic among almost complex structures without being isomorphic. We shall see in the next § (see § 2) examples of complex structures that have the same c_1 but are not isomorphic.

[3]It is not much harder to prove, using the integrality of the Chern character (see [18] for instance), that the sphere S^{2n} has no almost complex structure, except for $2n = 2$ or 6. It is also a consequence of the results of Wu in [31] that S^6 has an almost complex structure. An explicit one can be described using the Cayley octonions (see [20] for instance).

[4]More or less the same proof gives that if W_1 and W_2 are 4-dimensional manifolds which can be endowed with almost complex structures, then their connected sum cannot (see [5]).

2. Hirzebruch surfaces

In this §, I present the Hirzebruch surfaces [15] (another good reference is [17]). In addition to being relevant examples for the discussion almost complex/complex, they are basic examples for symplectic geometry too: for instance they play a major role in the classification of Hamiltonian S^1-actions [3] and in the work of Dusa McDuff on symplectic 4-dimensional manifolds [26].

2.1. Definition

Consider a complex line bundle L over the projective line $\mathbf{P}^1(\mathbf{C}) = S^2$ and add to it a *section at infinity*, in other words, consider the bundle

$$\mathbf{P}\left(L \oplus \mathbf{1}\right) = \left\{(x, l) \mid x \in \mathbf{P}^1(\mathbf{C}), l \text{ is a line in } L_x \oplus \mathbf{C}\right\},$$

a bundle over $\mathbf{P}^1$ with fibre $\mathbf{P}^1$. The zero section is given by $l = 0 \oplus \mathbf{C}$, the section at infinity by $l = L_x \oplus 0$.

If D is any other line bundle over $\mathbf{P}^1$, it is clear that $\mathbf{P}(L\oplus\mathbf{1}) \cong \mathbf{P}((L\oplus\mathbf{1})\otimes D)$ (by the map $(x, l) \mapsto (x, l \otimes D_x)$) thus the bundles $\mathbf{P}(L \oplus \mathbf{1})$ describe all the $\mathbf{P}(L_1 \oplus L_2)$. Moreover, using $D = L^\star$, one gets an isomorphism $\mathbf{P}(L \oplus \mathbf{1}) \cong \mathbf{P}(L^\star \oplus \mathbf{1})$ which exchanges the zero section and the section at infinity[5].

It is easy to prove that, from the topological viewpoint, all the rank 2 complex vector bundles over $\mathbf{P}^1$ split. This property is still true if one considers *holomorphic bundles*[6] (see [14]).

The isomorphism class of a (topological, or differentiable, or holomorphic) complex line bundle $L \to \mathbf{P}^1$ is well defined by an integer, its Euler class: decompose the sphere into two hemispheres and trivialise the bundle on both. To get the bundle, glue the two trivialisations along the equator S^1 by a map $\varphi : S^1 \to \mathbf{C}^\star$. Identify $\pi_1(\mathbf{C}^\star)$ with $\mathbf{Z}$, associating an integer, the Euler class, to the bundle.

2.1.1. Exercise. — Let B be any orientable surface and let D be a disc in B. Use a decomposition of B as a polygon to show that the complement of D has the homotopy type of a wedge of circles. Let $L \to B$ be a complex line bundle. Show that it is trivialisable both on D and on its complement. Deduce that the isomorphism type of L is given by an integer, its Euler class (see e.g. [4]).

For $k \in \mathbf{Z}$, call $\mathcal{O}(k)$ the bundle over $\mathbf{P}^1$ with Euler class k:

- $\mathcal{O}(0) \to \mathbf{P}^1$ is the trivial bundle.
- $\mathcal{O}(-1) \to \mathbf{P}^1$ is the *tautological* bundle: the fibre at $d \in \mathbf{P}^1$ is the line d of $\mathbf{C}^2$ itself.

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[5]Recall that if $L^\star$ is the dual of L, $L \otimes L^\star$ has a canonical trivialisation by $(x, \varphi) \mapsto \varphi(x)$.

[6]The analogous property is definitely false if one replaces $\mathbf{P}^1$ by a Riemann surface of genus > 0, see [2].

- $\mathcal{O}(1)$ is the *canonical* bundle, the dual of the previous one, and
- $\mathcal{O}(k) = \mathcal{O}\left(k/\left|k\right|\right)^{\otimes |k|}$.

Call, for $k \in \mathbf{N}$, $W_k = \mathbf{P}\left(\mathcal{O}(-k) \oplus \mathbf{1}\right)$.

2.1.2. Examples.

1. For $k = 0$, $W_0 = \mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$.
2. For $k = 1$,

$$W_1 = \left\{(l, d) \mid d \text{ is a line in } \mathbf{C}^2 \text{ and } l \text{ a line in } d \oplus \mathbf{C} \subset \mathbf{C}^2 \oplus \mathbf{C}\right\}.$$

In addition to the structure of a bundle over $\mathbf{P}^1$, this manifold is endowed with a projection π to $\mathbf{P}^2(\mathbf{C})$: to the point (d, l), associate $l \subset \mathbf{C}^3$. Notice that any point in $\mathbf{P}^2$ is the image of a unique point of W_1: d is the projection of l on $\mathbf{C}^2$... except for the line $d = 0 \oplus \mathbf{C}$ which is the image of all points in the $\mathbf{P}^1$ of lines d in $\mathbf{C}^2$.

The projection $\pi : W_1 \to \mathbf{P}^2(\mathbf{C})$ is the *blow up* of $\mathbf{P}^2(\mathbf{C})$ at the point $[0, 0, 1]$. One often denotes W_1 by $\widetilde{\mathbf{P}^2}(\mathbf{C})$.

2.1.3. Exercise. — Consider the standard ball in $\mathbf{P}^2(\mathbf{C})$:

$$B = \left\{[x, y, 1] \mid |x|^2 + |y|^2 \leq R^2\right\}.$$

Check that $\pi^{-1}(B)$ is a tubular neighbourhood of the zero section of $\mathcal{O}(-1)$, its boundary a 3-sphere, and its complement a tubular neighbourhood of the section at infinity.

It is easy to represent W_k as an algebraic submanifold of $\mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^2(\mathbf{C})$:

2.1.4. Exercise. — By definition,

$$W_k = \left\{(\ell, d) \in \mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^N(\mathbf{C}) \mid \ell \subset \mathbf{C}^2, d \subset \ell^{\otimes k} \oplus \mathbf{C} \subset \left(\mathbf{C}^2\right)^{\otimes k} \oplus \mathbf{C}\right\}$$

where $N = \dim\left(\mathbf{C}^2\right)^{\otimes k}$. Choose a basis of $\mathbf{C}^2$ and show that

$$W_k = \left\{[a, b], [ua^k, ua^{k-1}b, \ldots, uab^{k-1}, ub^k, v]\right\} \subset \mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^{k+1}(\mathbf{C})$$

(notice that this does *not* mean that $W_k \cong \mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$). Similarly, show that the "projection"

$$[z_1, \ldots, z_{k+1}, t] \longmapsto [z_1, z_{k+1}, t]$$

although not well defined on $\mathbf{P}^{k+1}(\mathbf{C})$ defines an embedding

$$W_k = \left\{[a, b], [ua^k, ub^k, v]\right\} \subset \mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^2(\mathbf{C}).$$

Alternatively $W_k = \left\{([a,b],[x,y,z]) \mid a^k y - b^k x = 0\right\}$ and any Kähler form over $\mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^2(\mathbf{C})$ then defines a Kähler form over W_k.

2.2. Topology

In fact, there are only two classes of projectivised $\mathbf{P}^1$-bundles over $\mathbf{P}^1$. Let $W \to S^2$ be such a bundle. Over the hemisphere $S^2_\pm$, W can be trivialised as $S^2_\pm \times \mathbf{P}^1$. The gluing is given by a map $S^1 \to PGL(2,\mathbf{C})$ and there are only two homotopy classes: $\pi_1(PGL(2,\mathbf{C})) \cong \mathbf{Z}/2$.

2.2.1. Exercise. — Projectivising a bundle $L \oplus \mathbf{1}$ may be achieved by a map $\pi_1(\mathbf{C}^\star) \to \pi_1(PGL(2,\mathbf{C}))$. Show that it can be identified with mod 2 reduction $\mathbf{Z} \to \mathbf{Z}/2$. Deduce that $\mathbf{P}(L \oplus \mathbf{1})$ is trivialisable if and only if the Euler class of L is even.

Using the examples above, one then gets

PROPOSITION 2.2.2. — *W_k is diffeomorphic to $S^2 \times S^2$ if k is even and to $\widetilde{\mathbf{P}^2}(\mathbf{C})$ if k is odd.* □

One deduces easily from this the cohomology of W_k, which, of course depends only on k mod 2. It may nevertheless be convenient to express it in terms of k and of the projection π over $\mathbf{P}^1$: from the inclusion $W_k \hookrightarrow \mathbf{P}^1 \times \mathbf{P}^2$, one gets two classes u, $v \in H^2(W_k;\mathbf{Z})$, restrictions of the generator of $H^2(\mathbf{P}^1;\mathbf{Z})$, $H^2(\mathbf{P}^2;\mathbf{Z})$ resp. It is classical (see [18] or the definition of Chern classes in chapter IV for instance) that

PROPOSITION 2.2.3. — *The ring $H^\star(W_k;\mathbf{Z})$ is isomorphic to the ring of polynomials $\mathbf{Z}[u,v]/(u^2, v^2 - kuv)$.* □

2.2.4. Exercise. — Check that the isomorphism type of this ring actually depends only on k mod 2.

2.3. Symplectic forms

Consider a symplectic form induced by the standard symplectic (Kähler) form on $\mathbf{P}^1 \times \mathbf{P}^2$ (as defined on each factor in 1.4.2 for instance), rescaled such that its cohomology class is $\lambda u + \mu v$ and compute the volumes of the following spheres $i : \mathbf{P}^1 \hookrightarrow W_k$:

- the zero section $i([a,b]) = ([a,b],[0,0,1])$: $i^\star(\lambda u + \mu v) = \lambda x$, it has volume λ,
- the section at infinity $i([a,b]) = ([a,b],[a^k,b^k,0])$: $i^\star(\lambda u + \mu v) = (\lambda + k\mu)x$, it has volume $\lambda + k\mu$,
- the fibre $i([a,b]) = ([1,0],[a,0,b])$: $i^\star(\lambda u + \mu v) = \mu x$, it has volume μ.

(x denotes the generator of $H^2(\mathbf{P}^1;\mathbf{Z})$).

I find it very convenient to represent W_k and the class of the Kähler form under consideration by a trapezium, the lengths of its edges being the volumes of the corresponding spheres as in figure 2.

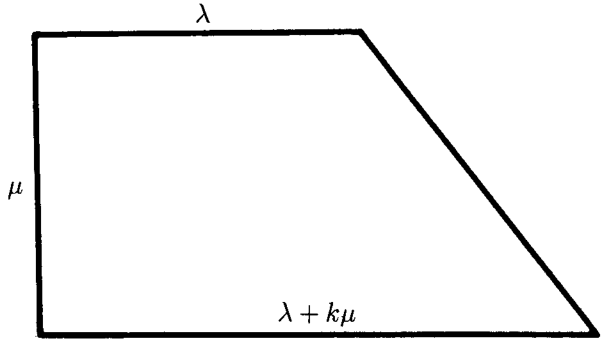

Figure 2

Remark. — In fact, this trapezium is not only a good way to represent the cohomology classes, but it is actually the image of the moment map of a Hamiltonian T^2-action on W_k.

2.3.1. Exercise. — Consider the S^1-action on W_k induced by the action

$$t \cdot ([a,b],[x,y,z]) = ([a,b],[x,y,tz])$$

on $\mathbf{P}^1 \times \mathbf{P}^2$. Check that its fixed points are the sections at zero and at infinity and that $H = \mu |z|^2 / 2 \left(|x|^2 + |y|^2 + |z|^2\right)$ is a Hamiltonian for this action. Any $r \in]0, \mu[$ is then a regular value of H and S^1 acts on $H^{-1}(r)$. Check that the quotient sphere $H^{-1}(r)/S^1$ gets a natural "reduced" symplectic structure (see § 4) of volume $\lambda + k(\mu - r)$.

Remark. — This volume is an affine function of r, with slope $\pm k$. This is represented on the trapezium, and is a special case of the Duistermaat-Heckman formula [9]. Note that, for $k > 0$, the volume is *decreasing* when starting from the zero section of $\mathcal{O}(k)$ and *increasing* when starting from the zero section of $\mathcal{O}(-k)$.

2.3.2. Exercise. — Using 2.2.3, show that there does not exist a symplectic form on W_k which gives the same volume to the sections at zero and at infinity ($k \neq 0$).

All these Kähler forms belong to "exotic complex structures" on $S^2 \times S^2$ or $\widetilde{\mathbf{P}}^2(\mathbf{C})$. Fix the parity of k, say k even. The *complex* manifold W_k is $S^2 \times S^2$, with a complex structure which depends on k. The almost complex structures are homotopic due to 1.5.1, but the complex manifolds are not isomorphic (see [15]).

If one agrees to use intersection in homology and the positivity of intersections of complex curves, this result can be made into an exercise: suppose $\varphi : W_k \to W_l$ $(0 \le k \le l)$ is a holomorphic isomorphism. Let Σ_0 be a complex curve[7] with self intersection k in W_k, then $\varphi(\Sigma_0)$ will be a complex curve in W_l. Call (F, S_0) a basis of $H_2(W_l; \mathbf{Z})$ such that F is the class of the fibre and S_0 that of a section such that $S_0 \cdot S_0 = l$.

2.3.3. Exercise. — Write the homology class of $\varphi(\Sigma_0)$ as $aF + bS_0$. Show that $a \ge 0$, $b \ge 0$, and that $k = 2ab + b^2 l$. Deduce that $b = 0$ and $k = 0$. Let Φ be a fibre in W_0. Write the homology class of $\varphi(\Phi)$ as $xF + yS_0$. Use the fact that (Φ, Σ_0) is a basis of $H_2(W_0; \mathbf{Z})$ to prove that $y \ne 0$, then show that $y > 0$ and $x \ge 0$. Compute the self intersection of $\varphi(\Phi)$ and get $x = y = 0$, a contradiction.

3. Coadjoint orbits (of $U(n)$)

The simplest possible example of a compact symplectic manifold is the sphere S^2 endowed with a volume form. Any oriented surface would also work: in dimension 2, a symplectic form is simply a volume form (this is the non degeneracy condition, the condition of being closed is automatically fulfilled).

It turns out that S^2 is the basic example in two families of symplectic manifolds: it is a coadjoint orbit—of $SO(3)$ as well as of $U(2)$—and it is a symplectic reduction.

In this §, I shall briefly explain the symplectic structure of coadjoint orbits and then in §4 the symplectic reduction process (this will also be used to construct Lagrangian submanifolds in chapter X). We will then have at hand a lot of examples and will be in a position to try to manufacture new ones out of these old ones. I shall focus attention on dimension 4, and explain in §5 why taking the connected sum is not a symplectic process while blowing up and down are, and how these remarks can be generalised to give the symplectic fibre sum recently used by Gompf [10] to construct his new examples.

The vector space $\mathbf{C}^n$ is endowed with its canonical Hermitian structure. A matrix A is *Hermitian* if ${}^t\overline{A} = A$.

3.1. Description of the manifolds

Fix a vector $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{R}^n$ and consider, in the space of all $n \times n$ complex matrices, the subspace

$$\mathcal{H}_\lambda = \{\text{Hermitian matrices with spectrum } \lambda\}.$$

It is well known that Hermitian matrices are diagonalisable in a unitary basis: $A = gDg^{-1}$, D diagonal and $g \in U(n)$. In other words, $\mathcal{H}_\lambda$ is an orbit of the $U(n)$-action by conjugation.

[7] Thinking of W_l, W_k as $\mathbf{P}^1$-bundles over $\mathbf{P}^1$, one can choose S_0 and Σ_0 to be the zero sections.

Call $\mu_1 < \cdots < \mu_k$ the distinct values of the λ_i and n_i the multiplicity of μ_i. It is clear that the stabilisers of the elements of $\mathcal{H}_\lambda$ are conjugate to $U(n_1)\times\cdots\times U(n_k) \subset U(n)$, thus

PROPOSITION 3.1.1. — *The space $\mathcal{H}_\lambda$ is a manifold, diffeomorphic to the homogeneous space $U(n)/U(n_1) \times \cdots \times U(n_k)$.*

Remark. — Note that the diffeomorphism type depends only on the multiplicities of the eigenvalues, not on their values.

To any element in $\mathcal{H}_\lambda$, one can associate the sequence $Q_1, Q_2, \ldots, Q_k$ of its eigenspaces. They are pairwise orthogonal and $\dim Q_i$ is the multiplicity n_i of μ_i. Equivalently, one can consider the *flag*

$$0 \subset P_1 \subset \cdots \subset P_k = \mathbf{C}^n$$

with $P_j = Q_1 \oplus \cdots \oplus Q_j$. The manifold $\mathcal{H}_\lambda$ is a *flag manifold.*

Generic orbits. — Generic (biggest) orbits correspond to simple (multiplicity 1) eigenvalues, the stabiliser type is the torus T^n of diagonal unitary matrices. In this case Q_i is a line and $\mathcal{H}_\lambda$ is the manifold of *complete* flags in $\mathbf{C}^n$,

$$\mathcal{H}_\lambda \cong \{(0 \subset P_1 \subset \cdots \subset P_n = \mathbf{C}^n) \mid \dim P_j = j\}.$$

The dimension of $\mathcal{H}_\lambda$ is $n^2 - n = 2\frac{n(n-1)}{2}$ in this case.

Small orbits. — The small orbits, excepting the trivial case where $k = 1$ and where $\mathcal{H}_\lambda$ is a point, are obtained for 2 distinct eigenvalues μ_1 with multiplicity n_1, and μ_2 with multiplicity n_2. Using again the eigenspaces, one identifies $\mathcal{H}_\lambda$ with the Grassmann manifold $G_{n_1}(\mathbf{C}^n)$ (or $G_{n_2}(\mathbf{C}^n)$) of n_1-planes in $\mathbf{C}^n$, a $2n_1n_2$-dimensional manifold. It is the homogeneous space $U(n)/U(n_1) \times U(n_2)$ or the quotient of the Stiefel manifold $V_{n_1}(\mathbf{C}^n)$ of unitary n_1-frames in $\mathbf{C}^n$ by the group $U(n_1)$.

An interesting subcase is $n_1 = 1$, in which case the Grassmann manifold is the complex projective space $\mathbf{P}^{n-1}(\mathbf{C})$.

3.2. Coadjoint orbits and their symplectic structures

The mapping $h \mapsto ih$ is an isomorphism of real vector spaces from the space $\mathcal{H}$ of all Hermitian matrices onto the space $\mathfrak{u}(n)$ of skew-Hermitian matrices[8].

The assignment $(X, Y) \mapsto \operatorname{tr}(XY)$ defines a (negative) definite ($\mathbf{R}$-)bilinear form on $\mathfrak{u}(n)$, which is invariant under conjugation. This allows us to identify $\mathfrak{u}(n)$ with the dual vector space $\mathfrak{u}(n)^\star$. With these identifications, the $U(n)$-action on $\mathfrak{u}(n)$ or $\mathcal{H}$ by conjugation is the *adjoint* or *coadjoint* action. The flag manifolds $\mathcal{H}_\lambda$ are

[8]It is of course the Lie algebra of the Lie group $U(n)$: the equation ${}^t\overline{A}A = \operatorname{Id}$ gives ${}^t\overline{X} + X = 0$ when you differentiate it at the point Id.

thus described as coadjoint orbits of a Lie group. This is why they have a natural symplectic structure, called the Kirillov structure [19]: in all of this §, you can replace $U(n)$ by a Lie group G and $\mathcal{H}$ or $\mathfrak{u}(n)^\star$ by the dual $\mathfrak{g}^\star$ of its Lie algebra.

First define, for any $h \in \mathcal{H}$, a skew-symmetric bilinear form ω_h on $\mathfrak{u}(n)$ by

$$\omega_h(X,Y) = \operatorname{tr}([X,Y]\,ih) = \operatorname{tr}(X\,[Y,ih]).$$

As $(X,Y) \mapsto \operatorname{tr}(XY)$ is non-degenerate, the kernel of ω_h is the subspace of $\mathfrak{u}(n)$

$$K_h = \{Y \in \mathfrak{u}(n) \mid [Y,ih] = 0\},$$

which is obviously the Lie algebra of the stabiliser of h for the $U(n)$-action.

The tangent space to the orbit $\mathcal{H}_\lambda$ at the point h is the subspace of $\mathcal{H}$ generated by the *fundamental vector fields* of the $U(n)$-action: one considers the orbit map defined by h:

$$\begin{array}{ccc} U(n) & \xrightarrow{f_h} & \mathcal{H} \\ g & \longmapsto & g\cdot h = ghg^{-1} \end{array}$$

and $T_h\mathcal{H}_\lambda$ is the image of $T_1 f_h$, that is of

$$\begin{array}{ccc} \mathfrak{u}(n) & \xrightarrow{T_1 f_h} & \mathcal{H} \\ X & \longmapsto & [X,h] \end{array}$$

in particular, $K_h = \operatorname{Ker} T_1 f_h$, and thus ω_h defines, via $T_1 f_h$, a *non-degenerate* skew-symmetric bilinear form on $T_1 f_h$:

$$\widetilde{\omega}_h([X,h],[Y,h]) = \operatorname{tr}([X,Y]\,ih)$$

for X, $Y \in \mathfrak{u}(n)$ or $[X,h]$, $[Y,h] \in T_h\mathcal{H}_\lambda$.

Thus we get a non-degenerate 2-form $\widetilde{\omega}$ on the orbit $\mathcal{H}_\lambda$. The next claim is, of course, that $\widetilde{\omega}$ is closed. You just need to prove that

$$d\widetilde{\omega}_h([X,h],[Y,h],[Z,h]) = 0 \text{ for all } X,Y,Z \in \mathfrak{u}(n).$$

Call $\underline{X}$ the fundamental vector field associated with X: $\underline{X}_h = [X,h]$. Then

$$\begin{aligned} 3d\widetilde{\omega}(\underline{X},\underline{Y},\underline{Z}) = {} & \underline{X}\cdot\widetilde{\omega}(\underline{Y},\underline{Z}) - \underline{Y}\cdot\widetilde{\omega}(\underline{X},\underline{Z}) + \underline{Z}\cdot\widetilde{\omega}(\underline{X},\underline{Y}) \\ & +\widetilde{\omega}([\underline{X},\underline{Y}],\underline{Z}) + \widetilde{\omega}([\underline{Y},\underline{Z}],\underline{X}) + \widetilde{\omega}([\underline{Z},\underline{X}],\underline{Y}). \end{aligned}$$

By definition, $\widetilde{\omega}$ is invariant and the vector fields under consideration are fundamental vector fields for the action, thus the first three terms vanish. What is left also vanishes due to the Jacobi identity.

The $\mathcal{H}_\lambda$ are thus compact symplectic manifolds. Note that, although the topology of $\mathcal{H}_\lambda$ depends only on the multiplicities in λ, the symplectic structure does depend on the λ_j.

3.2.1. Exercise. — Assume $n = 2$, $k = 2$ and $n_1 = 1$ (thus $\mathcal{H}_\lambda$ is a projective line). Compute $\int_{\mathcal{H}_\lambda} \widetilde{\omega}$ in terms of μ_1 and μ_2.

3.2.2. Exercise. — Consider the group $SO(3)$ of rotations of 3-dimensional Euclidean space. Show that its Lie algebra $\mathfrak{so}(3)$ is the vector space of skew symmetric matrices. Identify it with $\mathbf{R}^3$ by

$$\varphi : (x, y, z) \mapsto \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}.$$

Show that φ is an isomorphism of Lie algebras (the Lie algebra structure on $\mathbf{R}^3$ being given by the vector product), that the action of $SO(3)$ by conjugation on $\mathfrak{so}(3)$ corresponds to the action by rotations. Thus the scalar product is invariant, which allows to identify $\mathfrak{so}(3)$ with its dual, and the adjoint action with the coadjoint. Show that the coadjoint orbits are 2-spheres, thus finding once again a family of symplectic structures on S^2.

3.3. Almost complex structures

Once the $\mathcal{H}_\lambda$ are described are flag manifolds of $\mathbf{C}^n$, they clearly have a natural additional structure: they are *complex* manifolds. As above (proposition 3.1.1) the complex diffeomorphism type depends only on the multiplicities. Let us describe the underlying almost complex structure. One can write $\mathfrak{u}(n) = \mathfrak{u}(n_1) \oplus \cdots \oplus \mathfrak{u}(n_k) \oplus \mathfrak{m}$ where $\mathfrak{m}$ is a *complex* vector space. One simply decomposes in blocks a matrix A in an orbit with multiplicities $n_1, \ldots, n_k$:

$$A = \begin{pmatrix} A_1 & X_{1,2} & X_{1,3} & \cdots \\ -{}^t\overline{X_{1,2}} & A_2 & X_{2,3} & \cdots \\ \cdots & & & \\ & & & X_{k-1,k} \\ & & & A_k \end{pmatrix}$$

where $A_i \in \mathfrak{u}(n_i)$ and $X_{i,j}$ is a *complex* matrix. The complex structure on $\mathfrak{m}$ is given by the complex structure of the spaces where the blocks $X_{i,j}$ live, not by the complex structure of the space of big matrices: if

$$X = \begin{pmatrix} 0 & X_{1,2} & X_{1,3} & \cdots \\ -{}^t\overline{X_{1,2}} & \cdots & \cdots & \cdots \\ -{}^t\overline{X_{2,3}} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \in \mathfrak{m}$$

call $j(X)$ the same matrix with each $X_{k,l}$ replaced by $iX_{k,l}$: this is the complex structure of $\mathfrak{m}$.

Of course, $\mathfrak{m}$ is not a Lie subalgebra of $\mathfrak{u}(n)$, but $\mathfrak{g} = \mathfrak{u}(n_1) \oplus \cdots \oplus \mathfrak{u}(n_k)$ is.

Now $\mathfrak{m}$ is a good representant of the tangent space to $U(n)/U(n_1) \times \cdots \times U(n_k)$ at the point image of $I \in U(n)$, and it has a natural complex structure. Everything is invariant enough to define something on $\mathcal{H}_\lambda$.

Fix a diagonal matrix $D = (\mu_1, \ldots, \mu_1, \ldots, \mu_k, \ldots, \mu_k)$ in the orbit $\mathcal{H}_\lambda$ with $\mu_1 < \cdots < \mu_k$ (so that the stabiliser of D is actually $U(n_1) \times \cdots \times U(n_k)$. Any element in $T_d\mathcal{H}_\lambda$ can be writen in a unique way as $[X, D]$ for an $X \in \mathfrak{m}$ (the tangent space to the orbit is generated by fundamental vector fields, unicity comes from the specification $X \in \mathfrak{m}$, this subspace being a supplementary of the Lie algebra of the stabiliser).

Define now $J_D[X, D] = [j(X), D]$ in such a way that J_D is a complex structure on $T_D\mathcal{H}_\lambda$. Let now $h = gDg^{-1}$ be any element in the orbit. Any element in $T_h\mathcal{H}_\lambda$ can be writen uniquely (once g is chosen) as $[gXg^{-1}, h]$ with $X \in \mathfrak{m}$. Put

$$J_h\left[gXg^{-1}, h\right] = \left[gj(X)g^{-1}, h\right]$$

thus defining an almost complex structure J on the orbit $\mathcal{H}_\lambda$.

3.3.1. Exercise. — Show that $\tilde{\omega}(JX, JY) = \tilde{\omega}(X, Y)$: the complex structure is an isometry of $\tilde{\omega}$. Compute $\tilde{\omega}(X, JY)$ in terms of the blocks in X and Y and in terms of the μ_j. Deduce that (if the μ_j are as above) $\tilde{\omega}(X, JY)$ is a Riemannian metrics on $\mathcal{H}_\lambda$. Thus J is calibrated by $\tilde{\omega}$.

4. Symplectic reduction

Symplectic reduction is a very simple and very clever process, due to Marsden and Weinstein [23], already mentionned in chapter I, which can be used for different purposes, such as constructing symplectic quotients, i.e. symplectic manifolds obtained from group actions on large symplectic manifolds (see 4.2), Lagrangian immersions into the reduced symplectic manifolds starting from Lagrangian immersions into the large one or Lagrangian submanifolds of the large symplectic manifold starting from Lagrangian submanifolds of the small one (see chapter X). Here we discuss some classical examples.

4.1. The projective space, dissection

The complex projective space $\mathbf{P}^n(\mathbf{C})$ is the quotient $\mathbf{C}^{n+1} - \{0\}/z \sim \lambda z$ for $\lambda \in \mathbf{C}^\star$, or, what is almost the same, think $S^{2n+1}/z \sim \lambda z$ for $\|\lambda\| = 1$: any line contains unitary vectors! Let us look at that trivial remark more closely: let

$$S_R^{2n+1} = \left\{z \in \mathbf{C}^{n+1} \mid \|z\|^2 = R^2\right\}$$

be the radius R sphere. We are considering the quotient by the S^1 action $(\lambda, z) \mapsto \lambda \cdot z$.

There is a relation between the function $H = \frac{1}{2}\|z\|^2$ (of which S_R^{2n+1} is a level set) and the S^1-action we consider: the fundamental vector field for the action is $\underline{X}_z = iz\ldots$ and this is the symplectic gradient[9] of H:

$$\omega_z(\underline{X}_z, Y) = \mathrm{Im}\langle iz, Y\rangle = \mathrm{Re}\langle z, Y\rangle = dH(Y).$$

[9]Of course, $\mathbf{C}^{n+1}$ is endowed with the standard symplectic form.

What happens is the following: the tangent space to $\mathbf{P}^n(\mathbf{C})$ at the point $[z]$ is the (isomorphic) image of the quotient of $\mathbf{C}^{n+1}$ by the subspace generated by z and iz, in particular it is $\mathbf{C}^{n+1}/\mathbf{C}\cdot z$, thus a complex vector space, which is not surprising: the quotient is a complex manifold.

What happens in the symplectic framework is exactly the same: of course, if you restrict it to S_R^{2n+1}, the symplectic form is degenerate; its kernel at z is generated by iz and it is exactly what one has to kill to get the tangent space to $\mathbf{P}^n(\mathbf{C})$.

4.1.1. Exercise. — Compute the volume of a projective line $\mathbf{P}^1(\mathbf{C}) \subset \mathbf{P}^n(\mathbf{C})$ in terms of R.

4.2. Symplectic reduction: general presentation

Symplectic reduction is a list of consequences of a straightforward linear algebra lemma:

LEMMA 4.2.1. — *If (E,ω) is a symplectic vector space and $F \subset E$ is any coisotropic subspace, then ω defines a non-degenerate 2-form on F/F°.* □

In the $\mathbf{P}^n$ case, $E = \mathbf{C}^{n+1}$, $F = T_z S_R^{2n+1}$, $F^\circ = \mathbf{R}\cdot iz$ and $F/F^\circ = T_{[z]}\mathbf{P}^n(\mathbf{C})$.

4.2.2. Exercise. — Let $\mathbf{R}^{n+1}$ be endowed with its usual Euclidean structure, and $\mathbf{R}^{n+1}\times\mathbf{R}^{n+1}$ be endowed with the symplectic form $\omega\left((p,q),(p',q')\right) = p\cdot q' - p'\cdot q$. Consider the Hamiltonian (function) $H(p,q) = \frac{1}{2}\|p\|^2$. Compute the Hamiltonian vector field (symplectic gradient) X_H and its flow φ_t. Show that X_H is complete and write down the corresponding $\mathbf{R}$-action. What are the orbits? Deduce a symplectic structure on the set of oriented affine lines of $\mathbf{R}^{n+1}$. Show that this manifold is diffeomorphic to $T^\star S^n$. What about the symplectic structures?

I present here a very general setting as a list of exercises (see [21]). Let (W^{2N},ω) be a symplectic manifold (to replace $\mathbf{C}^{n+1}$ endowed with an action of a Lie group G (to replace S^1) with a *moment mapping* $\mu : W \to \mathfrak{g}^\star$ (to replace $H : \mathbf{C}^{n+1} \to \mathbf{R}$) such that

$$\langle T_w\mu(X), Y\rangle = \omega_w(X,\underline{Y}) \text{ for } X \in T_wW, \quad Y \in \mathfrak{g}$$

and such that μ preserves the Poisson brackets: W has a natural Poisson bracket given by its symplectic form, and $\mathfrak{g}^\star$ has also a natural one coming from the Lie bracket on $\mathfrak{g}$ and described in the following exercise.

4.2.3. Exercise. — Let f, $g \in \mathcal{C}^\infty(\mathfrak{g}^\star)$ and put

$$\{f,g\}(\xi) = [df(\xi), dg(\xi)]$$

where $df(\xi)$ is considered as an element of $\mathfrak{g}$ using biduality. Show that this defines a Poisson bracket on $\mathfrak{g}^\star$—that is, a Lie algebra structure on $\mathcal{C}^\infty(\mathfrak{g}^\star)$. Let $\mathcal{O}$ be a coadjoint orbit in $\mathfrak{g}^\star$. Show that the induced Poisson bracket coincides with the one defined by the symplectic structure described in § 3.

4.2.4. Exercise. — Show that the moment map μ is equivariant ($\mathfrak{g}$ is endowed with the coadjoint action).

Assume $\xi \in \mathfrak{g}^\star$ is a regular value of μ. Then $V = \mu^{-1}(\xi)$ is a submanifold of W and

$$\forall x \in V \qquad T_x V = \operatorname{Ker} T_x \mu.$$

Let G_ξ be the stabiliser of ξ for the coadjoint action (it is all of S^1 in the projective space example—as it is G in any example where G is abelian).

4.2.5. Exercise.

1. Show that the G-action on W induces a G_ξ-action on V.

2. Let $j : V \hookrightarrow W$ be the inclusion, then

$$\operatorname{Ker}(j^\star\omega)_x = T_x(G_\xi \cdot x).$$

As ξ is a regular value, we know that the G-action is locally free in the neighbourhood of any $x \in V$. Assume moreover that the G_ξ-action is *free*. Then one can apply lemma 4.2.1 to obtain the unique symplectic form σ on V/G_ξ such that, in the diagram

$$\begin{array}{ccc} V & \stackrel{j}{\hookrightarrow} & W \\ \pi \big\downarrow & & \\ V/G_\xi & & \end{array}$$

$\pi^\star\sigma = j^\star\omega$. The symplectic manifold $(V/G_\xi, \sigma)$ is a *symplectic reduction* of V.

The next § will give some examples.

4.3. Construction of symplectic manifolds

Hirzebruch surfaces, first version. — Consider the hypersurface H in $(\mathbf{C}^2 - 0) \times (\mathbf{C}^3 - 0)$ defined by the equation $a^k y - b^k x = 0$. Being complex, it is symplectic (see 1.2.3). It is endowed with the product action of T^2 by $(s,t) \cdot (a,b,x,y,z) = (sa, sb, tx, ty, tz)$ whose moment mapping is

$$\begin{array}{rcl} \mu : H & \longrightarrow & \mathbf{R}^2 \\ (a,b,x,y,z) & \longmapsto & \left(|a|^2 + |b|^2, |x|^2 + |y|^2 + |z|^2\right) \end{array}$$

if ξ is in the positive quadrant of $\mathbf{R}^2$, $\mu^{-1}(\xi) = (S^3 \times S^5) \cap H$ and the quotient $W_k \subset \mathbf{P}^1 \times \mathbf{P}^2$ appears as a symplectic reduction.

Hirzebruch surfaces, second version. — There is a more economical—in terms of dimensions—way to obtain Hirzebruch surfaces as symplectic reductions, starting from $\mathbf{C}^4$. Consider $\mathcal{U} = (\mathbf{C}^2 - 0) \times (\mathbf{C}^2 - 0) \subset \mathbf{C}^4$, with the action of $\mathbf{C}^\star \times \mathbf{C}^\star$ by

$$(s,t) \cdot (z_1, z_2, z_3, z_4) = \left(s^k t z_1, t z_2, s z_3, s z_4\right). \tag{4.3.1}$$

The quotient is identified to W_k by

$$(z_1, z_2, z_3, z_4) \longmapsto \left([z_3, z_4], [z_3^k z_2, z_4^k z_2, z_1]\right).$$

Let $\mu : \mathbf{C}^4 \to \mathbf{R}^2$ be the moment map for the action of $T^2 \subset \mathbf{C}^\star \times \mathbf{C}^\star$ by (4.3.1):

$$\mu(z_1, z_2, z_3, z_4) = \frac{1}{2}\left(k\left|z_1\right|^2 + \left|z_3\right|^2 + \left|z_4\right|^2, \left|z_1\right|^2 + \left|z_2\right|^2\right).$$

Let $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$. Consider $\mu^{-1}(\xi)$. I want to show that, under certain hypotheses on ξ, $\mu^{-1}(\xi)/T^2 = \mathcal{U}/\mathbf{C}^\star \times \mathbf{C}^\star$.

So let $z \in \mathcal{U}$. I am looking for $Z \in \mu^{-1}(\xi)$ in the same $\mathbf{C}^\star \times \mathbf{C}^\star$ orbit and more precisely such that there exist a, $b > 0$ with

$$(Z_1, Z_2, Z_3, Z_4) = \left(a^k b z_1, b z_2, a z_3, a z_4\right).$$

Eliminating b and letting $x = a^2$, it is necessary that the equation

$$\begin{aligned} x^{k+1}\left|z_1\right|^2\left(\left|z_3\right|^2 + \left|z_4\right|^2\right) + x^k\left(k\xi_2\left|z_1\right|^2 - \xi_1\left|z_1\right|^2\right) \\ + x\left|z_2\right|^2\left(\left|z_3\right|^2 + \left|z_4\right|^2\right) - \xi_1\left|z_2\right|^2 = 0 \end{aligned} \tag{4.3.2}$$

has a unique positive root. Then $b^2 = \xi_2/\left(x^k\left|z_1\right|^2 + \left|z_2\right|^2\right)$ has a unique positive solution when $\xi_2 > 0$. The $k = 0$ case being obvious, assume for simplicity that $k \geq 1$. Call $f(x)$ the left hand side of (4.3.2), and compute the derivatives f' and f''. One gets

$$f''(x) = Ax^{k-2}\left[(k+1)\,x + \frac{(k-1)\,(k\xi_2 - \xi_1)}{\left|z_3\right|^2 + \left|z_4\right|^2}\right]$$

where A is positive. Assume that $k\xi_2 - \xi_1 > 0$, then $f''(x) > 0$ for $x > 0$ and f' is increasing on $[0, +\infty[$. As $f'(0) > 0$ and $f(0) < 0$ there is a unique positive root x:

PROPOSITION 4.3.3. — *If $\xi_1 > 0$, $\xi_2 > 0$ and $k\xi_2 - \xi_1 > 0$ then $\mu^{-1}(\xi)/T^2 = W_k$, and W_k is obtained by symplectic reduction starting from $\mathbf{C}^4$.* □

Remark. — This is a particular case of a more general construction: W_k is a *toric manifold* (see [4] for this construction, which basically comes from [8], although there is a (slight!) mistake in [4] as I learned from Y. Karshon and S. Tolman).

Remark. — We shall see later (chapter X) why it is interesting to have W_k as a symplectic reduction starting from the smallest possible complex vector space.

Grassmannians. — Let us now describe the complex Grassmann manifold $G_n\left(\mathbf{C}^{n+k}\right)$ as a symplectic reduction. It is well known (see 3.1) that it is a quotient of a part of a numerical space: let $V_k\left(\mathbf{C}^{n+k}\right) \subset \left(\mathbf{C}^{n+k}\right)^k$ be the Stiefel manifold of all unitary k-frames in $\mathbf{C}^{n+k}$. It is clear that $G_k\left(\mathbf{C}^{n+k}\right) = V_k\left(\mathbf{C}^{n+k}\right)/U(k)$.

My aim is now to show that this quotient is also a symplectic reduction. Write $\left(\mathbf{C}^{n+k}\right)^k = M_{(n+k)\times k}(\mathbf{C})$, the space of all complex matrices with $n+k$ rows and k columns. This space is endowed with its canonical symplectic form $\omega = \operatorname{Im}\operatorname{tr}\left({}^t\overline{X}Y\right)$. Giving a matrix $A \in M_{(n+k)\times k}(\mathbf{C})$ is equivalent to giving k vectors in $\mathbf{C}^{n+k}$, its columns. These k vectors form a unitary k-frame exactly when ${}^t\overline{A}A = \mathrm{Id} \in M_{k\times k}(\mathbf{C})$, thus $V = V_k\left(\mathbf{C}^{n+k}\right)$ is the "level Id" of

$$\begin{array}{rccc} \mu: & M_{(n+k)\times k}(\mathbf{C}) & \longrightarrow & \mathcal{H} \subset M_{k\times k}(\mathbf{C}) \\ & A & \longmapsto & {}^t\overline{A}A \end{array}$$

where now $\mathcal{H}$ denotes as in § 3 the real vector space of Hermitian matrices.

Of course $U(k)$ acts on $M_{(n+k)\times k}(\mathbf{C})$ by $g \cdot A = gA$. On the other hand, we are not afraid to identify $\mathcal{H}$ with $\mathfrak{u}(k)^\star$. The next assertion is that μ is the moment mapping for that action.

4.3.4. Exercise. — For X in $M_{(n+k)\times k}(\mathbf{C})$ and Y in $\mathfrak{u}(k)$, compute the number $\langle T_A\mu(X), Y\rangle$ and show it is $\pm\omega(X, \underline{Y})$.

We are precisely in the situation of the previous §. Moreover, the stabiliser G_ξ is the whole group $U(k)$ because $\xi = \mathrm{Id}$. The quotient $G_k\left(\mathbf{C}^{n+k}\right)$ is a symplectic reduction starting from a complex vector space.

Remark. — It is not necessary to do the exercises of 4.2 in full generality to get this result. Let A_0 be the matrix of the k first vectors in the canonical basis.

$$T_{A_0}\mu(X) = {}^t\overline{A_0}X + {}^t\overline{X}A_0 = 0 \Leftrightarrow$$

the upper $k \times k$ square in X is a Hermitian matrix. In other words

$$\begin{aligned} T_{A_0}V &= \left\{ X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \middle| X_1 \in \mathfrak{u}(k), X_2 \in M_{n\times k}(\mathbf{C}) \right\} \\ &= \mathfrak{u}(k) \oplus M_{n\times k}(\mathbf{C}) \\ &= \text{isotropic} \oplus \text{symplectic} \end{aligned}$$

is a coisotropic subspace F whose orthogonal is exactly $\mathfrak{u}(k)$, thus $F/F^\circ \cong M_{n\times k}(\mathbf{C})$.

5. Surgery?

A natural idea, to construct new symplectic manifolds out of old ones is to perform surgeries.

5.1. Connected sums

Speculative remarks. — Let us try[10] to make a symplectic connected sum $W = W_1 \# W_2$. A natural idea is to remove a small ball in a Darboux chart of both manifolds and to try to glue what is left along the boundary S^{2n-1}. The reason why you cannot manufacture a symplectic form agreeing with the standard ones outside the sphere is that you are trying to glue together two *outsides* of this sphere. I mean that there is a symplectic way of distinguishing the *inside* from the *outside* of the sphere: there exists a vector field X on $\mathbf{C}^n$, which is transversal to the sphere, and such that $\mathcal{L}_X\omega = \omega$.

5.1.1. Exercise. — Check that the radial field $X(p) = \frac{1}{2}p$ has this property.

5.1.2. Exercise. — If (W, ω) is a symplectic manifold and if X is a vector field such that $\mathcal{L}_X\omega = \omega$, show that X dilates the symplectic form, that is, show that its flow satisfies $\varphi_t^\star\omega = e^t\omega$.

Let us come back to the $(2n-1)$-sphere in $\mathbf{C}^n$. The dilatation is very explicit when $n \geq 2$: the volume of the reduced symplectic manifolds $\mathbf{P}^{n-1}(\mathbf{C})$ is *increasing* in the direction of X.

5.1.3. Exercise. — Let (W, ω) be a symplectic manifold of dimension $2n \geq 4$, and let S be a compact hypersurface. Assume that, in the neighbourhood of S, there exists a vector field X, transversal to S and such that $\mathcal{L}_X\omega = \omega$. Show that $i_X\omega$ defines (a primitive of ω and) a contact form[11] on S. Conversely, assume that α is a contact form on S such that $d\alpha$ is the restriction of ω. Show that a transverse dilating vector field X exists on a neighbourhood of S.

One says that such a hypersuface S has *contact type* (see [30] for this notion... and the solutions of the exercises if needed).

5.1.4. Exercise. — Let Σ be a Riemannian manifold. Endow $T^\star\Sigma$ with the canonical symplectic structure. Show that the sphere bundle (with respect to the given metric) $S(T^\star\Sigma)$ has contact type.

Rigorous statement and proof. — There is *no* symplectic connected sum except for surfaces. Here is a proof (valid except in dimension 6). Suppose (W_1, J_1) and (W_2, J_2) are two almost complex manifolds. Remove balls B_1 and B_2, perform the surgery by gluing the boundaries and assume there exists a J on W such that $J|_{W_i - B_i}$ is homotopic to J_i.

[10]The remarks here *are* speculative, but the statements in the exercises are rigorous mathematics—see [16].

[11]A *contact form* on a $2n-1$-manifold is a 1-form α such that $\alpha \wedge (d\alpha)^{n-1}$ is a volume form.

Around B_i, TW_i is a trivialisable complex bundle. Fix a trivialisation. The existence of J is equivalent to that of a map $\varphi : S^{2n-1} \to GL(n, \mathbf{C})$ to be used to glue together the two trivialisations to construct the (complex) tangent bundle TW. Such a map φ might also be used to glue the trivialisations of the bundles over B_1 and B_2... to give an almost complex structure on the tangent bundle to $B_1 \cup_S B_2 = S^{2n}$.

It is well known that, if $2n \neq 2, 6$, this *cannot* exist. We already mentioned this result[12] in 1.5, where we even proved it for $2n = 4$. It is very easy to make symplectic connected sums of surfaces (and more generally of volume forms) as in the following exercise.

5.1.5. Exercise.

1. Let σ be the usual ("$d\theta$") volume form on S^1, and let $\varphi : \mathbf{R} \to \mathbf{R}$ be a *positive* smooth function such that $\varphi(t) = t^2$ for $|t| > 1$. Check that the 2-form $\omega = \varphi(t)dt \wedge \sigma$ on $\mathbf{R} \times S^1$ is a symplectic form which induces the standard form on $\mathbf{R}^2$–unit disc on *both* sides $]-\infty, -1] \times S^1$ and $[1, +\infty[$ via the diffeomorphisms $(t, z) \mapsto tz$.

2. Deduce a symplectic connected sum of symplectic surfaces. Generalise to volume forms in arbitrary dimensions. Why is there no staightforward generalisation to symplectic forms in higher dimensions?

In dimension 6, any almost complex structure on S^6 allows us to construct almost complex connected sums, but this does not work in the symplectic category, the only proof I know uses pseudo-holomorphic curves (the result is in [12], and a proof, due to D. McDuff is written in [5]).

Now we have spent enough time constructing *non*symplectic manifolds, let us try to understand what actually works. For simplicity, let us concentrate on dimension 4.

5.2. Blowing up and down

Let us come back to the standard ball in Darboux coordinates. Do we know something that we can glue to it, that is, do we know something which is symplectic and whose boundary looks like S^3... but from the inside?

In the spirit of § 2.3, that is, of the Duistermaat-Heckman formula [9], represent the complement of B_1 in W_1 as on the left part of figure 3. The slope is 1 due to the fact that the volume of the 2-sphere obtained by reducing the 3-sphere of radius R (vertical segment) is precisely R^2. It is now clear that we can glue the right part of the same figure, that is, the total space of the tautological line bundle

[12] Of course the speculative consideration of increasing volumes has no significance in dimension 2.

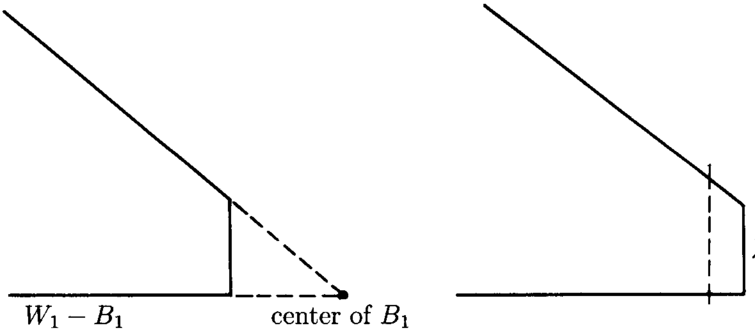

Figure 3

$\mathcal{O}(-1) \to \mathbf{P}^1(\mathbf{C})$: its boundary is actually S^3, and if we endow it with one of the symplectic forms of § 2.3 (The total space of $\mathcal{O}(-1)$ is the complement of the section at infinity in $\mathbf{P}(\mathcal{O}(-1) \oplus \mathbf{1})$), its boudary is actually seen from where we want. The volume considerations are equivalent to this.

Figure 3 is so accurate that it even shows what we are really allowed to do in the symplectic category: we can glue, provided that the volume λ of the zero section in $\mathcal{O}(-1)$ is less than the square of the radius R of the ball we remove: there must exist an $r > 0$ such that $R^2 = \lambda + r^2$.

This is the *symplectic blowing up* construction of D. McDuff [24], [25]: one replaces a large ball (of radius R) by a (neighbourhood of) a sphere of volume $\lambda < R^2$. Note that the blown up symplectic manifold is *smaller* than the original one. The opposite operation, that is, to replace a sphere with -1 normal bundle (or self intersection) with a large enough ball is the *symplectic blowing down*. This is the process generalised in the symplectic fibre sum used in [10].

5.3. The symplectic fibre sum

In this §, I describe a construction, recently discovered by Gompf [10], although it seems to have been known for a very long time, at least by Gromov (!) as one[13] can see on page 343 of [13].

Suppose that we are given two symplectic manifolds W_1 and W_2 which both contain a symplectic embedded copy of the same surface Σ. We want to try to remove neighbourhoods of Σ_1 and Σ_2 in W_1, W_2 and to glue the complements together to get a symplectic manifold. Of course, we need the boundaries to be diffeomorphic, in other words, the normal bundles (which are symplectic 2-plane or complex line bundles) have to be either isomorphic or anti-isomorphic. For the same reasons as before, the right choice is that they are *anti*-isomorphic. It is more or less obvious. For instance, at the almost complex level we know how to glue together disc bundles of L and L^{-1} over Σ to get a complex manifold, namely $\mathbf{P}(L \oplus \mathbf{1})$, in which the normal bundle to the zero section is L and that to the section at infinity is L^{-1}. The same

[13]Thanks to E. Giroux for pointing out to me this remark in [13].

gluing map allows us to get an almost complex structure on $W = (W_1 - D(L)) \cup_{S(L)} (W_2 - D(L))$ which is homotopic to the given ones on $W_1 - D(L)$ and $W_2 - D(L)$ (this is the same argument as the use of the sphere in the connected sum problem).

So we are certain to be able to construct an almost complex manifold. What should we do to construct a symplectic one?

As in the blowing up and down process, the symplectic forms should agree near the parts glued together. This is what the *symplectic tubular neighbourhood theorem* allow us to assume[14]: there is a normal form for the symplectic form in a small enough neighbourhood of the submanifold.

In other words, if you understand one example, you understand everything. For instance, if $L \to \Sigma$ is a complex line bundle over the surface Σ (the normal bundle of Σ_1 in W_1), there is a symplectic form on a small enough disc bundle of L which can be used as a model. This is most conveniently defined on a compact version of L, that is on $\mathbf{P}(L \oplus \mathbf{1})$. We already have one if $\Sigma = S^2$, according to 2.3. The next exercise gives an easy and elegant general construction which I learnt from P. Iglesias.

5.3.1. Exercise. — Assume L is trivial and construct a symplectic form on $\mathbf{P}(L \oplus \mathbf{1})$. Assume now L non trivial with Euler class k, and endowed with a Hermitian metric. Consider the unit sphere bundle $S(L)$ with its S^1-action by rotations in the fibres.

1. Show that there exists a connection form α on $S(L)$ such that $d\alpha = \pi^\star \eta$ for some *volume form* η on Σ (in other words, one can assume that the curvature of α never vanishes) with $\int_\Sigma \eta = k$ (see chapter IV).

2. Consider now the S^1-action on $\mathbf{P}^1(\mathbf{C})$ by $t \cdot [a, b] = [a, tb]$, put
$$H([a, b]) = \mu + \frac{|b|^2}{|a|^2 + |b|^2}$$
(H is a Hamiltonian for the action if $\mathbf{P}^1(\mathbf{C})$ is endowed with the standard symplectic form ω_0). Investigate the kernel of the 2-form
$$\widetilde{\omega} = d(H\alpha) + \omega_0$$
on $S(L) \times \mathbf{P}^1(\mathbf{C})$. Assume $\mu > 0$ and show that $\widetilde{\omega}$ defines a symplectic form ω on the quotient $S(L) \times_{S^1} \mathbf{P}^1(\mathbf{C})$ by the diagonal action.

3. Show that $S(L) \times_{S^1} \mathbf{P}^1(\mathbf{C})$ is actually diffeomorphic to $\mathbf{P}(L \oplus \mathbf{1})$ and compute the volumes of the surfaces sitting inside (fibres and sections) as in 2.3.

[14] In the present case (symplectic submanifold) as well as in all cases where the symplectic form, restricted to the submanifold has constant rank (e.g. symplectic, isotropic, co-isotropic submanifolds) this is the so called Darboux-Weinstein theorem [29], which can be obtained as the classical Darboux theorem (in chapter I) with the help of the path method of Moser. A more general result is due to Givental (see e.g. [6]) and asserts that if the restrictions of the forms induced on the submanifolds are isomorphic, then they are isomorphic on neighbourhoods.

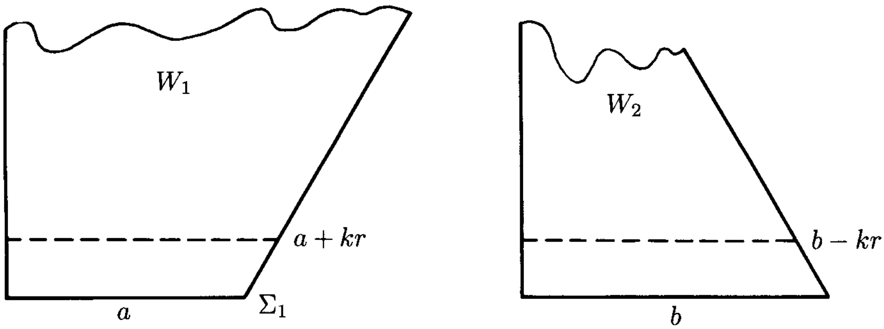

Figure 4

Now look at figure 4 which shows neighbourhoods of Σ_1 in W_1 and Σ_2 in W_2. Call $a = \int_{\Sigma_1} \omega_1$, $b = \int_{\Sigma_2} \omega_2$, and k the Euler class of the normal bundle to Σ_1.

After rescaling one of the symplectic forms, there exists an $r > 0$ such that $a + kr = b - kr$ and we can glue the complementaries W_i' of the neighbourhoods of Σ_i defined by r to get a closed symplectic manifold W (figure 5).

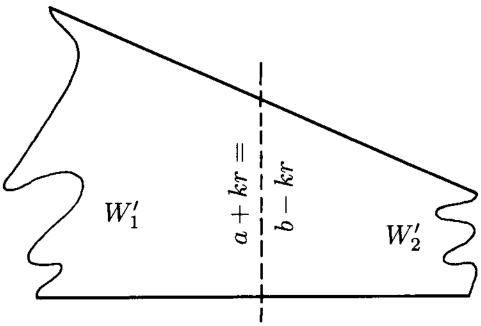

Figure 5

Remarks.

1. Although the diffeomorphism type of W is well defined, the symplectic form might depend on r (not least because of the rescaling).

2. Change the dimensions to get obvious generalisations.

3. Exercise 5.1.4 together with the adjacent remarks about volumes is supposed to explain why one cannot easily replace symplectic by Lagrangian submanifolds: a neighbourhood of a Lagrangian submanifold is isomorphic to a disc bundle of its cotangent bundle, whose boundary has contact type.

5.3.2. Exercise. — Suppose W contains a symplectic sphere Σ with self intersection -1. Using $\mathbf{P}^1(\mathbf{C}) \subset \mathbf{P}^2(\mathbf{C})$, show that one can view the symplectic blowing down of Σ as a symplectic fibre sum.

This construction is used by Gompf in two ways. First of all, using some sophisticated examples (construction of homotopy $K3$-surfaces without complex structure), he gets 4-dimensional simply connected symplectic manifolds which are non Kähler. Second, he constructs, for any finitely presented group G, a 4-dimensional symplectic manifold W with $\pi_1(W) = G$. This result, although new and spectacular (the situation for complex surfaces is quite different) is nevertheless very elementary. It uses only the symplectic fibre sum in the easiest case (submanifolds with *trivial* normal bundles) and the existence of a symplectic manifold V which contains a symplectic 2-torus T with trivial normal bundle and simply connected complement. The construction for $G = \mathbf{Z}$ (no symplectic 4-manifold with $\pi_1 = \mathbf{Z}$ was known before [10]) is given in the following exercise.

5.3.3. Exercise. — Consider a torus (genus 1 surface) $F = \mathbf{R}^2/\mathbf{Z}^2$, with the two circles α ($x_2 = 0$) and β ($x_1 = 0$) so that $\pi_1(F)$ is generated by α and β and that $\pi_1(F)/\langle\beta\rangle = G = \mathbf{Z}$. Consider another torus T^2 with one circle γ ($x_2 = 0$) sitting inside. Set $W = F \times T^2$, endow it with a product symplectic form ω_0.

1. Show that there exists a closed 1-form ρ on F such that $\rho|_\beta$ never vanishes. Let σ be a closed 1-form on T^2 having the same property with respect to γ. Put $\omega_t = \omega_0 + t\,(\rho \wedge \sigma)$. Show that, for $t > 0$ small enough, ω_t is a symplectic form[15] and $\beta \times \gamma$ a symplectic subtorus of (W, ω_t) with trivial normal bundle. Perform the surgery with the manifold V (with the properties stated above) along the tori, to give W_1.

2. Show that for t small enough, $\{z\} \times T^2$ is a symplectic subtorus in W (with trivial normal bundle). Choose z far enough from β, in such a way that $\{z\} \times T^2$ is still a submanifold of W_1. Perform the symplectic fibre sum along it with another copy of V, to give W_2. Show that $\pi_1(W_2) = \mathbf{Z}$.

There do exist complex (Kähler) surfaces with the stated properties, for instance the "rational elliptic surfaces" V obtained by blowing up nine points in general position in $\mathbf{P}^2(\mathbf{C})$, as will be seen from the next exercise.

5.3.4. Exercise. — Choose nine points $p_1, \ldots, p_9$ in $\mathbf{P}^2(\mathbf{C})$.

[15] At this point, there is a small technicality for a more general group G: if G has a presentation with k generators, one can try to begin with a genus k surface F. One then has to kill half of the generators of $\pi_1(F)$, but also all the relations. In order to have an analogue of ω_t, one needs a ρ. For obvious homological reasons, it might be necessary, in order to get one, to increase the genus... and thus the number of elements to kill.

1. Show that, for a generic choice of the points p_i, the set of all cubic curves through these points is a pencil (a complex projective line in the space of all cubic curves). Hint: recall that, in general, ten points define a plane cubic.

2. Assume that the nine points are chosen in this way. Let V be the complex surface obtained by blowing up $\mathbf{P}^2$ at the nine points (so that V is a Kähler manifold). Show that the "map" $\mathbf{P}^2 \to \mathbf{P}^1$ which, to each point p associates the unique cubic of the pencil through p is actually a well defined map $f : V \to \mathbf{P}^1$, almost all of its fibres being smooth cubics.

3. Recall that a smooth plane cubic is diffeomorphic to a torus. Check that almost all the fibres of f are symplectic tori with trivial normal bundles. Show that the fundamental group of the complement of a smooth cubic in $\mathbf{P}^2$ is generated by a small loop in the complex normal line at some point. Deduce that V is simply connected.

Appendix: The canonical almost complex structure on the manifold of 1-jets of pseudo-holomorphic mappings between two almost complex manifolds, by Paul Gauduchon

A.1. Minimal connections on an almost complex manifold

Let (M, J) be an almost complex manifold of (real) dimension $n = 2m$. We denote by N the *Nijenhuis tensor*, or *complex torsion*, defined by:

(A.1.1) $$N(X,Y) = [JX, JY] - J[JX,Y] - J[X,JY] - [X,Y],$$

for all X, Y in T_xM, for all $x \in M$. Recall from § 1.3 that, according to the Newlander-Nirenberg theorem [27], N vanishes identically if and only if the almost complex structure J is integrable.

Let D be a J-linear connection on M, i.e. a $\mathbf{R}$-linear connection preserving the almost complex structure J (for basic facts about connections, see chapter IV). The torsion T of D splits into a J-invariant part, denoted by T^+, and a J-skew-invariant part, denoted by T^-, with respect to the involutive action of J defined by: $T \mapsto T(J., J.)$. The component T^- in turn splits into two components, denoted respectively by T^{--} and T^{-+}, satisfying the following identities:

(A.1.2) $$T^{--}(JX,Y) = T^{--}(X,JY) = -J(T^{--}(X,Y))$$

and

(A.1.3) $$T^{-+}(JX,Y) = T^{-+}(X,JY) = J(T^{--}(X,Y)).$$

Then, the following hold (see, e.g. [22]):

- The component T^{--} of the torsion of D is independent of D and is equal to $\frac{1}{4}N$;
- there exists a non-empty (affine) space of J-linear connections on M, of which the torsion is reduced to the component $T^{--} = \frac{1}{4}N$.

Connections of this type will be called *minimal.*

Two minimal J-linear connections D and D' are related by: $D' = D + A$, where A is a 1-form on M with values in the vector bundle of J-linear endomorphisms of the tangent bundle TM; moreover, since D and D' have the same torsion, A is *symmetric*, i.e. satisfies the identity:

(A.1.4) $$A_X Y = A_Y X, \quad \forall X, Y \in T_x M, \quad \forall x \in M.$$

Then, the symmetry of A and the J-linearity of A_X, for any X, imply:

(A.1.5) $$A_{JX} = J \circ A_X, \quad \forall X \in T_x M, \quad \forall x \in M.$$

A.2. The canonical almost complex structure on $J^1(M_1, M_2)$.

Let (M_1, J_1) and (M_2, J_2) be two almost complex manifolds, of (real) dimension n_1 and n_2 respectively. A mapping $f : M_1 \longrightarrow M_2$ is *pseudo-holomorphic* if $df \circ J_1 = J_2 \circ df$ (df denoting the differential of f).

A.2.1. Exercise. — Denote by N_i the Nijenhuis tensor of (M_i, J_i). If f is a pseudo-holomorphic mapping, check that

$$N_2\left(df(X), df(Y)\right) = df\left(N_1(X, Y)\right).$$

Let $J^1(M_1, M_2)$ denote the manifold of 1-jets of pseudo-holomorphic mappings from (M_1, J_1) into (M_2, J_2), identified with the total space of the (complex) vector bundle E over the product $M_1 \times M_2$, whose fibre $E_{(x_1,x_2)}$ at the point (x_1, x_2) of $M_1 \times M_2$ is the (complex) vector space of **C**-linear homomorphsims from $T_{x_1}M_1$ to $T_{x_2}M_2$. Chose a J_1-linear connection D^1 on M_1 and a J_2-linear connection D^2 on M_2, both minimal.

These connections induce a **C**-linear connection, denoted by ∇, acting on the sections of the vector bundle E, defined, for any section a of E, by:

(A.2.2) $$(\nabla_{(X_1,X_2)}\mathrm{a})(Y_1) = \partial_{X_1}(\mathrm{a}(Y_1)) + D^2_{X_2}(\mathrm{a}(Y_1)) - \mathrm{a}(D^1_{X_1}Y_1),$$

for all Y_1 in $T_{x_1}M_1$,with the following significance: in the left hand side, Y_1 denotes any extension (independent of the variable x_2) of the vector Y_1 on the factor $M_1 \times \{x_2\}$; in the first term on the right hand side, the variable x_2 is "frozen", so that $\mathrm{a}(Y_1)$ takes its values in the fixed vector space $T_{x_2}M_2$, and ∂_{X_1} denotes the ordinary derivative in the direction of X_1; in the second term of the right hand side, the variable x_1 is "frozen", so that $a(Y_1)$ is considered as a vector field on M_2.

The linear connection ∇ determines (and is determined by) a *horizontal distribution*, denoted by H^∇, on the total space of E, by which we make the following identification, for any vector U tangent at ξ to (the total space of) E:

(A.2.3) $$U = \left(v^\nabla(U), (X_1, X_2)\right), \quad \forall U \in T_\xi E, \xi \in E_{(x_1,x_2)},$$

where:

- $X = (X_1, X_2)$ is the natural projection of U into $T_{(x_1,x_2)}(M_1 \times M_2)$;
- $v^\nabla(U)$ is the *principal part* of U w.r.t. ∇, where v^∇ denotes the (vertical) projection of $T_\xi E$ onto the vertical subspace of $T_\xi E$, identified with the fibre $E_{(x_1,x_2)}$, along the horizontal subspace H^∇_ξ.

For a different choice of (minimal) connections $D'^1 = D^1 + A^1$ and $D'^2 = D^2 + A^2$ on M_1 and M_2, the induced connection ∇' is related to ∇ by: $\nabla' = \nabla + B$, where B is the 1-form on M_1 with values in the vector bundle End E of $\mathbf{C}$-linear endomorphisms of E, equal to:

(A.2.4) $$B_{(X_1,X_2)}\xi = A^2_{X_2} \circ \xi - \xi \circ A^1_{X_1},$$

for all ξ in $E_{(x_1,x_2)} = \mathrm{Hom}_{\mathbf{C}}(T_{x_1}M_1, T_{x_2}M_2)$. It follows readily from (A.2.4) that the principal parts $v^{\nabla'}(U)$ and $v^\nabla(U)$ of a same vector U w.r.t. ∇' and ∇ are related by:

(A.2.5) $$v^{\nabla'}(U) = v^\nabla(U) + A^2_{X_2} \circ \xi - \xi \circ A^1_{X_1}, \quad \forall U \in T_\xi E,$$

where $X = (X_1, X_2)$ is the natural projection of U into $T_{(x_1,x_2)}(M_1 \times M_2)$.

With the help of the connection ∇, we construct an almost complex structure $\mathcal{J}$ on the manifold $J^1(M_1, M_2) = E$, via the identification (A.2.3), by putting:

(A.2.6) $$\mathcal{J}U = (J_2 \circ v^\nabla(U), (J_1X_1, J_2X_2)) \quad \forall U \in T_\xi E, \quad \forall \xi \in E.$$

PROPOSITION A.2.7. — *The almost-complex structure $\mathcal{J}$ is independent of the choice of the (minimal) connections D^1 and D^2.*

Proof. — Let $\mathcal{J}'$ be the almost complex structure on $E = J^1(M_1, M_2)$ built from the connection ∇'. By (A.2.6), $\mathcal{J}'U$ is represented, via the identification (A.2.3) w.r.t. ∇', by the pair $(J_2 \circ v^{\nabla'}(U), (J_1X_1, J_2X_2))$. By (A.2.5), the principal part $v^{\nabla'}(\mathcal{J}'U)$ of $\mathcal{J}'U$ with respect to the initial connection ∇ is equal to

$$J_2 \circ v^{\nabla'}(U) - A^2_{J_2X_2} \circ \xi + \xi \circ A^1_{J_1X_1},$$

equal, by (A.2.5), to

$$J_2 \circ v^\nabla(U) + J_2 \circ (A^2_{X_2} \circ \xi - \xi \circ A^1_{X_1}) - A^2_{J_2X_2} \circ \xi + \xi \circ A^1_{J_1X_1}.$$

By (A.1.5), the latter expression is equal to $J_2 \circ v^\nabla(U)$, i.e. the principal part of $\mathcal{J}U$ w.r.t. ∇. □

Remark. — It follows from Proposition A.2.7 that the almost complex structure $\mathcal{J}$ defined by (A.2.6) on $J^1(M_1 \times M_2)$ is canonical, i.e. only depends on J_1 and J_2.

A.2.8. Exercise. — Check that $\mathcal{J}$ is integrable if and only if J_1 and J_2 are integrable (Hint: compute the Nijenhuis tensor $\mathcal{N}$ of $\mathcal{J}$ by using *distinguished*, vertical and horizontal (w.r.t. ∇), vector fields on E, i.e., respectively, sections of E, considered as vertical vector fields, constant on fibres, and horizontal lifts of vector fields on the basis $M_1 \times M_2$).

Each pseudo-holomorphic map f from (M_1, J_1) into (M_2, J_2) canonically lifts, via its 1-jet, to a map, denoted by $\tilde{f}$, from M_1 into $J^1(M_1, M_2)$. Via the identification $J^1(M_1, M_2) = E, \tilde{f}$ is expressed by:

$$\tilde{f}(x) = df_{|T_xM_1}, \quad \forall x \in M_1. \tag{A.2.9}$$

PROPOSITION A.2.10. — *For any pseudo-holomorphic map f from (M_1, J_1) into (M_2, J_2), the canonical lift $\tilde{f}$ is a pseudo-holomorphic map from (M_1, J_1) into $(J^1(M_1, M_2), \mathcal{J})$.*

Proof. — Chose minimal connections D^1 and D^2 on M_1 and M_2 respectively, so that the almost complex structure $\mathcal{J}$ of $J^1(M_1, M_2) = E$ is expressed by (A.2.6) w.r. to the induced connection.

For each point x of M_1 and each vector X of T_xM_1, the vector $d\tilde{f}(X)$, in $T_{\tilde{f}(x)}E$, is represented, via the identification (A.2.3), by the pair:

$$d\tilde{f}(X) = (\nabla_X df, (X, df(X))), \tag{A.2.11}$$

where, for convenience, we still denote by ∇ the $\mathbf{C}$-linear connection induced by D^1 and D^2 on the (complex) vector bundle, over $M_1, T^\star M_1 \otimes_{\mathbf{C}} f^\star TM_2$ (of which df is a section).

Since the torsion of D^1, resp. D^2, is equal to the Nijenhuis tensor N_1, resp. N_2, of J_1, resp. J_2, the "Hessian" ∇df is symmetric. Indeed, we have:

$$\begin{aligned}(\nabla_X df)(Y) - (\nabla_Y df)(X) &= N_2(df(X), df(Y)) - df(N_1(X,Y)) \\ &= 0,\end{aligned}$$

for the differential df of any pseudo-holomorphic mapping f exchanges the Nijenhuis tensors (exercise A.2.1). It then follows from the $\mathbf{C}$-linearity of df and the connections D^1 and D^2, that ∇df satisfies the relation:

$$(\nabla_{J_1X} df)(Y) = J_2((\nabla_X df)(Y)), \quad \forall X, Y \in T_xM_1, \quad \forall x \in M_1. \tag{A.2.12}$$

By (A.2.11) and (A.2.6), we infer:

$$d\tilde{f}(J_1X) = (J_2 \circ \nabla_X df, (J_1X, J_2 df(X)) = \mathcal{J}(d\tilde{f}(X)), \quad \forall X \in T_xM_1, \quad \forall x \in M_1,$$

i.e.

$$d\tilde{f} \circ J_1 = \mathcal{J} \circ d\tilde{f}. \tag{A.2.13}$$

□

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Part 2

Riemannian geometry and linear connections

Chapter III

Some relevant Riemannian geometry

Jacques Lafontaine

with an appendix by M.-P. Muller

The first two sections of this chapter are an introduction to Riemannian geometry. It is not self-contained, and precise references are provided when necessary. However, we chosed to give some proofs which have a metric flavour.

Sections 3 and 4 and the appendix deal with results which are crucial in the study of pseudo-holomorphic curves, namely the Wirtinger and isoperimetric inequalities.

In section 5, following arguments kindly communicated by J. Duval and D. Bennequin, we show how these Riemannian techniques can be used to prove compactness properties for pseudo-holomorphic curves.

1. Riemannian manifolds as metric spaces

1.1. Introduction

Let U be an open set of $\mathbf{R}^n$, and assign to each $m \in U$ a positive definite quadratic form g_m whose coefficients are smooth functions. For each—say piecewise smooth—curve $c : [a, b] \mapsto U$ the *length* of c is defined by the integral

$$\ell(c) = \text{length}(c) = \int_a^b \sqrt{g_{c(t)}(\dot{c}(t), \dot{c}(t))}dt.$$

Then, for any pair (x, y) of points in U, define $\text{dist}_g(x, y)$ to be the infimum of the lengths of all the curves from x to y in U. It can be easily checked that dist_g is indeed a distance on U, which induces the usual topology, and that conversely dist_g determines g.

DEFINITION 1.1.1. — *A Riemannian metric on a manifold M is a metric such that M can be covered by open sets which are isometric to open sets of $\mathbf{R}^n$, equipped with such a metric.*

More classically, a Riemannian metric is a field $m \mapsto g_m$ of symmetric positive definite forms... The definition above is of course related to the data of this field in local coordinates with respect to some atlas of the manifold.

Remark. — Strictly speaking, one should distinguish the infinitesimal data g, that is the Riemannian tensor, and the distance dist_g. In fact, when speaking of Riemannian metrics, one refers to both, and this is not confusing.

The following properties justify the importance of Riemannian metrics. They are just trivial consequences of elementary properties of positive definite quadratic forms.

a) any convex combination of Riemannian metrics is still a Riemannian metric.

b) any submanifold of a Riemannian manifold (M, g) inherits a natural Riemannian metric.

Either a) (partitions of unity) or b) (elementary embedding theorems in Euclidean space) can be used to prove the existence of Riemannian metrics on any smooth manifold[1]. The numerical invariants which are provided by a Riemannian metric may give "quantitative versions" of topological properties. Namely, a Riemannian manifold which is complete for dist_g has a finite diameter if and only if M is compact (1.3.5). A more subtle numerical invariant is the *injectivity radius*, which guarantees, for any compact (M, g), the existence of an $r > 0$ such that any open ball $B(x, r)$ is diffeomorphic to $\mathbf{R}^n$ (cf. 1.3.10).

In contrast with these global invariants, there are local invariants, which are given by g viewed as a tensor. The most elementary one is the canonical measure. It is obtained by assigning to each tangent space T_mM a suitable normalisation of the Lebesgue measure: we decide that the cube generated by an orthonormal basis has measure one. In other words, if with respect to some local coordinates

$$g = \sum_{i,j} g_{ij} dx^i dx^j,$$

then for the same coordinates

$$v_g = \sqrt{\det(g_{ij})} dx^1 \dots dx^n.$$

Therefore, we can define the (possibly infinite) volume of (M, g), namely $\mathrm{vol}(M, g) = \int_M v_g$, and by the remark b) above, the volume of any submanifold of M as well.

1.2. The Levi-Civita connection: why and how

For a better understanding of the metric dist_g, it is very useful to find curves whose length is equal to the distance between their extremities. It is clear than any curve contained in such a curve will satisfy the same property, which is indeed a local property. It turns out (cf. [17] ch.2 or [13] 2.97 for instance) that minimising the *energy* is much more convenient. Indeed, it is an easy consequence of 1.3.3 below that a curve is energy minimising if and only if it is length minimising and parametrised proportionnally to arc-length.

[1] Of course, all manifolds are paracompact.

DEFINITION 1.2.1. — *The energy of a piecewise smooth curve $c : [a, b] \mapsto M$ is just the integral*

$$\frac{1}{2}\int_a^b g(\dot c, \dot c)dt.$$

With the usual notations of classical mechanics, it is "well known" that the curves which are extremal for the energy satisfy the *Lagrange equations*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q^i} - \frac{\partial L}{\partial q^i} = 0$$

for the Lagrangian $L(q, \dot q) = \frac{1}{2}\sum_{ij} g_{ij}(q)\dot q^i \dot q^j$. Therefore, we get the second order system

$$\sum_{j=1}^{n} g_{ij}(q)\ddot q^i - \frac{1}{2}\sum_{j,k=1}^{n} \partial_i g_{jk}\dot q^j \dot q^k.$$

Inversing the matrix (g_{ij}), we see that this system is equivalent to a system of the form $\ddot q = F(q, \dot q)$ so that classical existence and uniqueness theorems can be used.

If we want a better insight into what is happening, a more intrinsic approach is suitable. Namely, a one parameter family of curves $c_t(u)$ being given, we want to compute the derivative of the energy $E(c_t)$, say for $t = 0$, in terms of the vector field

$$X(u) = \frac{d}{dt}c_t(u)_{|t=0}$$

along $c_0 = c$. To do that, we must compare tangent vectors at different points of the manifold. Linear connections do the job.

DEFINITION 1.2.2. — *A linear connection on a manifold M is a* **R**-*bilinear map D from $\mathcal{X}(M) \times \mathcal{X}(M)$ into $\mathcal{X}(M)$ such that for any smooth function f,*
i) $D_{fX}Y = fD_XY$
ii) $D_X fY = fD_XY + (X \cdot f)Y.$

Remark. — We just extend the usual properties of the directional derivative in $\mathbf{R}^n$. Usually, D is supposed *torsion-free*: we require that
iii) $D_XY - D_YX = D_{[X,Y]}$.

Remark. — More generally, one defines connections on vector bundles (see chapter IV). Denoting by $\Gamma(E)$ the vector space of sections of a vector bundle $E \mapsto M$, a connection is then an **R**-bilinear map from $\mathcal{X}(M) \times \Gamma(E)$ into $\Gamma(E)$ satisfying i) and ii) above. Of course, iii) does not make sense in that case.

Coming back to the tangent bundle, a basic result is then the following.

PROPOSITION 1.2.3. — *Given a Riemannian manifold (M, g), there exists a unique torsion-free connection D such that*

$$X \cdot g(Y, Z) = g(D_XY, Z) + g(Y, D_XZ).$$

Remark. — We just mimic the formula for the derivative of a scalar product in Euclidean space.

Proof. — It is enough to determine D on coordinate vector fields ∂_i. We must have

$$\partial_i g_{jk} = g(D_{\partial_i}\partial_j, \partial_k) + g(\partial_j, D_{\partial_i}\partial_k).$$

After circular permutation, summing the two first relation and substracting the third we get

$$2g(D_{\partial_i}\partial_j, \partial_k) = \partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk}.$$

The result follows, since g is non-degenerate (the positivity of g plays no role here). □

Now, we need to refine a little bit the definition of a connection, since we want to work with vector fields along curves. Denote by $\mathcal{X}(c, M)$ the space of vector fields along the curve c (recall that they are maps $X : I \mapsto TM$ such that $X(u) \in T_{c(u)}M$ for any u). The following result is straightforward.

PROPOSITION 1.2.4. — *Given a connection D, there exists a unique linear map D^c from $\mathcal{X}(c, M)$ into itself such that*
i) $D^c(fX) = \dot{f}X + fD^cX$ for any X in $\mathcal{X}(c, M)$ and any $f \in C^\infty(I)$.
ii) If $X(s) = Y_{c(s)}$, where Y is a vector field on M, then $D^cX = D_{\dot{c}}Y$.

Remark. — Our notation is correct, since by 1.2.2 i) the value of D_XY at $p \in M$ only depends on X_p.

Proof. — Just notice that, restricting the interval if necessary, any $X \in \mathcal{X}(c, M)$ can be written as

$$X(s) = \sum_i f^i(s)V_i(s),$$

where the V_i are coordinate vector fields. □

DEFINITION 1.2.5. — *D^c is the covariant derivative along c.*

Clearly, if X and Y are vector fields along c, and if D is the Levi-Civita connection,

$$\frac{d}{ds}g(X(s), Y(s)) = g(D^cX(s), Y(s)) + g(X(s), D^cY(s)).$$

Now, we can write the derivative of energy in the following way:

THEOREM 1.2.6 (first variation formula). — *Let $(t, s) \mapsto c_t(s)$ be a family of smooth curves, where $t \in]-\varepsilon, \varepsilon[$ and $s \in [a, b]$. Let $X(s) = \frac{d}{dt}c_t(s)_{|t=0}$, and $c_0 = c$. Then*

$$\frac{d}{dt}E(c_t)_{|t=0} = [g(X(s), \dot{c}(s))]_a^b - \int_a^b g((D^c\dot{c})(s), X(s))ds.$$

Proof. — Interchanging derivatives, we first get

$$\frac{d}{dt}E(c_t) = \int_a^b g((D^c X)(s), \dot{c}(s))ds.$$

Then integrate by parts, using the preceding remark (we invite the reader to fill the gap in this argument, that is to define and use a covariant derivative for vectors along the (possibly singular) surface $(t, s) \mapsto c_t(s)$). □

DEFINITION 1.2.7. — *A geodesic of a Riemannian metric (M, g) is a curve c such that $D^c\dot{c} = 0$.*

In local coordinates, we get the differential system

$$\ddot{c}^i + \sum_{j,k} \Gamma^i_{jk}(c)\dot{c}^j\dot{c}^k = 0,$$

the coefficients Γ^i_{jk} being defined by

$$D_{\partial_j}\partial_k = \sum_i \Gamma^i_{jk}\partial_i.$$

Indeed, we recover the Lagrange equations.

The following facts are both important and straightforward:

a) for any $m \in M$ and $u \in T_mM$, there is a geodesic $c :]-\varepsilon, \varepsilon[\to M$ such that $c(0) = m$ and $\dot{c}(0) = u$, and c is unique in the sense that two such geodesics coincide on the intersection of their intervals of definition;

b) if a curve $s \mapsto c(s)$ is a geodesic, so is the curve $s \mapsto c(\lambda s)$ for any constant λ;

c) since $\frac{d}{ds}g(\dot{c}, \dot{c}) = 2g(\dot{c}, D^c\dot{c}) = 0$, $g(\dot{c}, \dot{c})$ is *constant* along any geodesic. Therefore, after a suitable scaling of the parameter, we can supposed that geodesics are parameterised by arc-length.

DEFINITION 1.2.8. — *The exponential map at $m \in M$, denoted by $\exp_m v$, where $v \in T_mM$, is given by the position at $s = 1$ of the unique geodesic c such that $c(0) = m$ and $\dot{c}(0) = v$.*

For the moment, $\exp_m$ is only defined on a (star-shaped) neighbourhood of 0 in T_mM. The remark b) above shows that $\exp_m tv$ is just the position at time t of the geodesic starting from m with initial speed v. Since we already know, from general properties of differential equations, that $\exp_m$ is smooth, we get the following result.

PROPOSITION 1.2.9. — *i) The differential of $\exp_m$ at $0 \in T_mM$ is the identity of T_mM.*
ii) The differential of the map $v_x \mapsto (x, \exp_x v_x)$ from TM to $M \times M$ has maximal rank at 0_x.

Proof. — We have already proved the first part. For ii), taking local coordinates around m and the corresponding coordinates for TM, we see that the differential at 0_x is given by the matrix

$$\begin{pmatrix} I & 0 \\ I & I \end{pmatrix}.$$

□

Therefore, using the inverse function theorem, we see that $\exp_m$ provides local coordinates. They are called *normal coordinates*. They are particularly well adapted to the metric, as we shall see now.

1.3. Minimising properties of geodesics

PROPOSITION 1.3.1 (Gauss lemma). — *The geodesics issuing from m meet the images by $\exp_m$ of the spheres $S(0,r)$ orthogonally.*

Proof. — Take two unit orthogonal vectors u and v in T_mM, and apply the first variation formula to $c(t,\theta) = \exp_m(t(u\cos\theta + v\sin\theta))$ for $\theta = 0$. □

1.3.2. Scholium. — For $ru \in T_mM$ $(r \neq 0)$, if we identify as usual $T_{ru}T_mM$ with T_mM, then $T_{ru}\exp_m$ is an isometry when restricted to $\mathbf{R}u$, and preserves the orthogonal splitting

$$T_mM = \mathbf{R}u \oplus (\mathbf{R}u)^\perp.$$

In dimension 2, taking polar coordinates in T_mM, we have

$$\exp_m^\star g = dr^2 + f^2(r,\theta)d\theta^2,$$

with $f(0,\theta) = 0$ and $\partial_r f(0,\theta) = 1$.

COROLLARY 1.3.3. — *For any $m \in M$, there is a strictly positive r such that any geodesic issuing from m of length smaller than r realises the distance between its extremities. In other words,*

$$\forall u \in T_mM \text{ with } \|u\| = 1, \forall t \in [0,r], \quad \mathrm{dist}_g(m, \exp_m tu) = t.$$

Proof. — By the inverse function theorem, there is an open set U containing $0 \in T_mM$ such that $\exp_m$ is a diffeomorphism of U onto $\exp_m(U)$. Then take a closed ball $B(0,r) \subset U$. Let $x = \exp_m ru$. Using polar coordinates in T_mM, any curve $c : [0,1] \mapsto \exp_m(B(0,r))$ joining m and x can be written as

$$c(t) = \exp_m(s(t)v(t)) \quad \text{with } s(0) = 0,\ s(1) = r,\ v(1) = u \text{ and } \|v(t)\| = 1.$$

Then, using the scholium above, we get

$$\|\dot{c}(t)\|^2_{c(t)} = |s'(t)|^2 + \|T_{s(t)v(t)}\exp_m \cdot s(t)v'(t)\|^2_{c(t)}.$$

Now it is clear that $\ell(c) \geq r$, and that equality occurs if and only if v' vanishes identically. But this means that $c(t) = \exp_m tu$.

We supposed that $c([0,1]) \subset \exp_m(B(0,r))$. If this is not the case, we proceed just in the same way, considering the first point where c hits the boundary of the ball. □

Remarks. — i) The same argument, using 1.2.9 ii), proves that there exists a $\delta > 0$ such that any two points x and y whose distance is smaller than δ can be joined by a unique geodesic of length $\operatorname{dist}_g(x,y)$.

ii) Since the converse is clear (use the first variation formula), 1.3.3 allows a purely metric characterisation of geodesics.

Another consequence is the following "intuitive" property, that we leave to the reader.

1.3.4. Exercise. — For any $p, q \in M$, and for δ small enough, there exists p_0 such that $\operatorname{dist}_g(p,q) = \operatorname{dist}_g(p,p_0) + \operatorname{dist}_g(p_0,q)$ and $\operatorname{dist}_g(p,p_0) = \delta$.

Using this property, one can prove that geodesics can be extended indefinitely. This is the Hopf-Rinow theorem (see [13] ch.2, or [17] ch.2 for the proof, and also [14] for a purely metric view-point).

THEOREM 1.3.5. — *For a Riemannian manifold (M,g) the following properties are equivalent.*

i) Any closed and bounded subset is compact.

ii) The metric dist_g *is complete.*

iii) There is an $m \in M$ such that $\exp_m$ *is defined everywhere on T_mM.*

iv) For every $m \in M$, $\exp_m$ *is defined everywhere on T_mM. Moreover, if one of these properties is satisfied, any two points of M can be joined by a minimising geodesic.*

DEFINITION 1.3.6. — *A complete Riemannian manifold (M,g) is a manifold which is a complete metric space for* dist_g.

Now it is clear that a compact Riemannian manifold is complete, and that conversely, a complete Riemannian manifold is compact if and only if its diameter is finite. *From now on, all the Riemannian manifolds are assumed to be complete.* A remarkable consequence of completeness is the following.

THEOREM 1.3.7. — *Let (M,g) be a complete Riemannian manifold, $m \in M$ and $u \in T_mM$. Let*

$$t_0 = \sup\{t > 0, s \to \exp_m su \text{ is minimising on } [0,t]\}.$$

Then, if $t_0 < \infty$, one of the following properties occurs.

i) There are (at least) two distinct minimising geodesics from m to $\exp_m t_0u$

ii) $\exp_m$ *is singular at t_0u.*

Proof. — By assumption, there is a sequence t_n, with $t_n > t_0$, such that $\mathrm{dist}_g(m, \exp_m t_n u) < t_n$. But $\exp_m t_n u$ can be joined to m by a minimising geodesic, so that

$$\exp_m t_n u = \exp_m s_n u_n, \quad \text{with } s_n = \mathrm{dist}_g(m, \exp_m t_n u).$$

Using the compactness of the unit sphere in $T_m M$, and taking a subsequence if necessary, we can suppose that u_n converges to some $v \in U_m M$. Clearly, s_n converges to t_0. Now, if $v \neq u$, situation i) occurs. And if $v = u$, we see that $\exp_m$ is not injective in any neighbourhood of $t_0 u$. Therefore $t_0 u$ is a singular point of $\exp_m$. □

The following definition is useful to precise case ii).

DEFINITION 1.3.8. — *If $t_0 u \in T_m M$ is a critical point of $\exp_m$, the point $p = \exp_m t_0 u$ is said to be conjugate to m along the geodesic $s \mapsto \exp_m su$.*

Using the same kind of arguments (see [9] p. 272 or [8] p. 94 for details) one proves the following.

THEOREM 1.3.9. — *Let (M, g) be a complete Riemannian manifold, UM be its unit tangent bundle, let $\rho(u_m)$ be the lowest upper bound of the $r > 0$ such that the geodesic $t \mapsto \exp_m(tu_m)$ is minimising on $[0, r]$. Then ρ is a continuous map from UM to $\overline{\mathbf{R}}_+$.*

Remark. — In particular, if M is compact, $\inf \rho(u_m)$ is stricly positive and attained.

DEFINITION 1.3.10. — *The injectivity radius* $\mathrm{inj}(M, g)$ *of (M, g) is the infimum of $\rho(u_m)$.*

In contrast with the compact case, for a noncompact manifold, the injectivity radius can be infinite (e.g. Euclidean space!) or zero (cf. 4.1.1d).

As illustrated in 1.3.7, controlling the injectivity radius involves both local and global properties of the Riemannian metric. This may be quite delicate (cf. [8], ch. 5). The story, a little bit of which will be told in 2.4.3, begins with the following refinement of 1.3.7.

LEMMA 1.3.11. — *Let $p \in M$ such that* $\mathrm{dist}_g(m, p) = \rho(m)$. *Then either p is conjugate to m along some minimising geodesic, or there are two geodesics c_1 and c_2 from m to p such that $\dot{c}_1(p) = -\dot{c}_2(p)$.*

Proof. — Using 1.3.7, we may assume there are two minimising geodesics c_1 and c_2 from m to p such that p is not conjugate to m along either. If $\dot{c}_1(p) \neq -\dot{c}_2(p)$, there exists $v \in T_p M$ such that

$$g(v, -\dot{c}_1(p)) > 0 \quad \text{and} \quad g(v, -\dot{c}_2(p)) > 0.$$

Let $\sigma : [0, \varepsilon[\to M$ be a curve such that $\sigma(0) = p$ and $\dot{\sigma}(0) = v$.

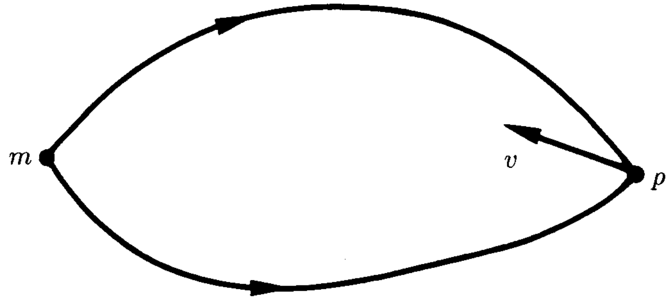

Figure 6

Since p is not conjugate to m along c_1 or c_2, there exist for small t one-parameter families c_1^t and c_2^t from m to $\sigma(t)$ such that $c_1^0 = c_1$ and $c_2^0 = c_2$. Then, by the first variation formula, one has $\ell(c_i^t) < \operatorname{dist}_g(m,p)$ for t small enough. Suppose for instance that

$$\ell(c_1^t) \leq \ell(c_2^t) < \ell(c_1) = \ell(c_2).$$

Using 1.3.7 again we see that the geodesic c_2^t is minimising at most up to $\sigma(t)$, a contradiction. □

1.4. Some basic examples

1.4.1. Flat tori. — The geodesics of Euclidean space are just the straight lines, provided with an affine parameter. A more interesting example is provided by flat tori, i.e. quotients of $\mathbf{R}^n$ by a lattice Λ: the length of a curve in $\mathbf{R}^n/\Lambda$ is just the length of one of its lifts in $\mathbf{R}^n$. Then the geodesics of $\mathbf{R}^n/\Lambda$ are the projections of the straight lines of $\mathbf{R}^n$ by the covering map $p : \mathbf{R}^n \mapsto \mathbf{R}^n/\Lambda$. Be careful: the metric structure of such a torus strongly depends on the lattice Λ: for instance, the lengths of closed geodesics are just the norms of the vectors of the lattice.

1.4.2. Submanifolds. — Let M be a submanifold of a Riemannian manifold $(\tilde{M},\tilde{g})$, equipped with the induced metric g, and let $\tilde{D}$ and D the corresponding Levi-Civita connections. Then for any $m \in M$, and any vector fields X and Y on N, $(D_XY)_m$ is the orthogonal projection of $(\tilde{D}_{\tilde{X}}\tilde{Y})_m$ on T_mM, where $\tilde{X}$ and $\tilde{Y}$ are local prolongations de X and Y to a neighbourhood of m in $\tilde{M}$. This is a direct consequence of the characterisation of the Levi-Civita connection in proposition 1.2.3. Taking $\tilde{M} = \mathbf{R}^n$, we see that $D_{\dot{c}}^c$ is the orthogonal projection of $\ddot{c}$ on $T_{c(t)}M$. It follows easily that the geodesics of the round sphere $S^n \subset \mathbf{R}^{n+1}$ are the great circles.

1.4.3. The standard sphere. — There is a more geometric way to get this result: since the image of a geodesic by an isometry is a geodesic, using the uniqueness property of geodesics we see that a curve which is the fixed point set of an isometry is a geodesic. But the great-circles are the intersections with the sphere of the fixed point sets of orthogonal symmetries with respect to 2-planes through the origin. If $x \in S^n$ and if $u \in T_x S^n$ is a unit vector, the geodesic starting from x with initial speed u is just

$$s \mapsto (\cos s)x + (\sin s)u,$$

so that

$$\exp_x^\star g = ds^2 + (\sin^2 s)g_{S^{n-1}}.$$

In particular, for $n = 2$,

$$\exp_x^\star g = ds^2 + \sin^2 s d\theta^2.$$

1.4.4. Hyperbolic space. — With minor modifications, the above applies to *hyperbolic space*, defined as follows: consider $\mathbf{R}^{n+1}$ with the quadratic form

$$\langle x, x\rangle = -x_0^2 + x_1^2 + \cdots x_n^2,$$

and set

$$H^n = \{x \in \mathbf{R}^{n+1},\ \langle x, x\rangle = -1 \quad \text{and} \quad x_0 \geq 1\}.$$

Clearly, $T_x H^n = \mathbf{R}x^\perp$ (the orthogonality is meant for the form $\langle\ ,\ \rangle$, not the Euclidean one of course), and the restriction of $\langle\ ,\ \rangle$ to $T_x H^n$ is positive definite. The intersections of H^n with 2-planes through the origin (which are half hyperbolas for our Euclidean eyes) provide the geodesics of this Riemannian metric. Indeed, both arguments we saw for the sphere work for hyperbolic space, replacing *Euclidean* symmetries or projections by their *Lorentzian* analogues. The geodesic starting from x with initial speed u is now

$$s \mapsto (\cosh s)x + (\sinh s)u,$$

so that

$$\exp_x^\star g = ds^2 + (\sinh^2 s)g_{S^{n-1}}.$$

In particular, for $n = 2$, $\exp_x^\star g = ds^2 + \sinh^2 s d\theta^2$. In constrast with the case of the sphere, we see that $\exp_x$ is a global diffeomorphism, and that $\mathrm{dist}_g(x, \exp_m su) = |s|$ for any s. Topologically, the situation is the same as for $\mathbf{R}^n$. But metrically, it is completely different: since

$$\exp_x^\star v_g = ds \wedge \left(\sinh^{n-1} s\right) \omega_S^{n-1},$$

where ω_S^{n-1} is the standard volume form of the unit sphere, we have for the volume of the ball $B(x, r)$ the asymptotic formula

$$\mathrm{vol}(B(x, r)) \simeq c_n \exp(n-1)r,$$

and the $(n-1)$-dimensional volume of the boundary, that is

$$S_g^{n-1}(x, r) = \{y \in H^n,\ \mathrm{dist}_g(x, y) = r\}$$

is also $O(\exp(n-1)r)$.

1.4.5. Exercise. — Show that the n-form $dx^1 \wedge \cdots \wedge dx^n$ has no bounded primitive on $\mathbf{R}^n$.

2. The geodesic flow and its linearisation

2.1. The geodesic spray

It is well known that a second order differential equation can be considered as a first order one: it is sufficient to consider the first derivatives as unknown functions. If we do this for the equation of geodesics, we get a differential equation on the tangent bundle TM. More directly, let (M, g) be a complete Riemannian manifold, and consider the transform

$$\Phi_t(m,u) = \left(\exp_m tu, \frac{d}{dt}\exp_m tu\right)$$

of TM into itself: we assign to (m, u) the position and the velocity at time t of the geodesic whose position and velocity at time 0 are m and $u \in T_mM$. Clearly,

$$\Phi_{t+t'} = \Phi_t \circ \Phi_{t'},$$

so that Φ_t is a one parameter group of diffeomorphisms of TM, and also of the unit tangent bundle UM (since $g(\dot{c},\dot{c})$ is constant along a geodesic).

DEFINITION 2.1.1. — *Φ_t is the geodesic flow of (M, g); its infinitesimal generator is the geodesic spray.*

Taking local coordinates (q^i) on M, the vector fields $\partial/\partial q^i$ define a local trivialisation of TM. Denote by $(q^i, \dot{q}^i)$ the corresponding local coordinates on TM. With respect to theses coordinates, the geodesic spray Ξ is just

$$\Xi(q,\dot{q}) = \sum_{i=1}^n \dot{q}^i \frac{\partial}{\partial q^i} - \sum_{i=1}^n \Gamma^i_{jk}(q)\dot{q}^j\dot{q}^k \frac{\partial}{\partial \dot{q}^i}.$$

Fortunately, a more geometric description of Ξ is available, but for this we will need to make a slight detour.

A Riemannian metric (and more generally a pseudo-Riemannian one) provides a vector bundle isomorphism between TM and $T^\star M$. Usually, the isomorphism from TM to $T^\star M$ is denoted by $\flat$, and the inverse by $\sharp$. (Indeed, this notation reminds us of what happens to indices in local coordinates). Therefore, we can transport the metric and the energy to $T^\star M$. Set

$$H(q,p) = \frac{1}{2} g_q(p^\sharp, p^\sharp).$$

In local coordinates, we have

$$H = \frac{1}{2}\sum_{i,j} g^{ij} p_i p_j.$$

Now, the vector field Ξ (or, more precisely its image by $\sharp$) is just the Hamiltonian of H for the standard symplectic structure of T^*M. To see this, let us write down explicitly Hamilton's equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}.$$

The first just says that

$$\dot{q}^i = \sum_{j=1}^{n} g^{ij} p_i,$$

i.e. that $\dot{q} = p^\sharp$, and the second gives the equation of geodesics we saw in 1.2.7. Let us see now a less down-to-earth argument.

2.2. The geometry of the tangent bundle

Consider TM as a vector bundle, and let $\pi : TM \to M$ be the natural projection. Then π is a submersion, and for any $u_m \in T_mM \subset TM$, the kernel of $T_{u_m}\pi$ is just the tangent space at u_m to the fibre $\pi^{-1}(m) = T_mM$. Denote by V_uTM this kernel. It is a n-dimensional subspace of T_uTM, and is naturally isomorphic to T_mM.

DEFINITION 2.2.1. — *V_uTM is called the vertical tangent space at u. The union of all the V_uTM is naturally a vector bundle of rank* $\dim M$ *over TM, called the vertical bundle.*

Now, once a linear connection on TM is given, one can define intrinsically an n-dimensional distribution H_uTM which is everywhere transverse to V_uTM. In particular, the restriction of T_up to H_uM is a linear isomorphism. To see this, pick a vector field X in a neighbourhood of m such that $X_m = u_m$. If we view X as a local section of π, it is clear that $\mathrm{Im}\,(T_mX(T_mM))$ is an n-dimensional subspace of T_uTM which is transverse to V_uTM. This space only depends on the first jet of X at m. Now, we choose X such that $(D_uX)_m = 0$, and take

$$H_uTM = \mathrm{Im}\,(T_mX(T_mM))$$

for such an X.

The next notion will be discussed in chapter IV.

DEFINITION 2.2.2. — *The n-plane distribution $u \mapsto H_uTM$ is the horizontal distribution associated with the connection D.*

An alternative description of H_uTM is the following. A curve in TM is just a vector field along a curve in M; such a curve, say $t \mapsto (c(t), X(t))$ is horizontal if and only if its tangent vector is everywhere horizontal. This amounts to saying that $D^cX = 0$.

Now, let us remember the Riemannian metric g. Using the decomposition of T_uTM above, one defines on TM a Riemannian metric, by prescribing the following properties.

i) The spaces H_uTM and V_uTM are orthogonal.

ii) The natural isomorphism $T_mM \simeq V_uTM$ and the projection $T_up : H_uTM \mapsto T_mM$ are isometries.

An alternative description of this metric is given by the lengths of curves in TM: if $t \mapsto (c(t), X(t))$ is such a curve, where $t \in [a,b]$, its length is just

$$\int_a^b \sqrt{g((\dot{c}(t), \dot{c}(t)) + g((\dot{D}^c X(t), \dot{D}^c X(t))} dt.$$

Usually, it is more convenient and tractable to consider the unit tangent bundle UM, equipped with the induced metric, instead of the whole of TM. We invite the reader to check the following examples.

2.2.3. Examples. — If M is Euclidean space $\mathbf{R}^n$, then UM is $\mathbf{R}^n \times S^{n-1}$ with the standard product metric. The geodesic flow is given by $\Phi_t(m,u) = (m + tu, u)$.

If M is the standard sphere S^2, then US^2 is $SO(3)$ equipped with a bi-invariant metric; and the geodesic flow Φ_t is given by right multiplication by the matrix

$$\begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If we lift this action to the universal cover, we get the *Hopf action*

$$\tilde{\Phi}_t(z,z') = (e^{it}z, e^{it}z') \quad \text{on} \quad S^3_r = \{(z,z') \in \mathbf{C}^2,\ |z|^2 + |z'|^2 = r^2\}.$$

2.2.4. Exercise. — Check that if S^2 is the sphere of radius 1, and if US^2 is equipped with its natural metric, then its double cover is the sphere S^3 of radius 2.

Now, pulling back the natural symplectic structure of T^*M to TM by the isomorphism $\sharp$, we get a symplectic form Ω on TM. The above decomposition of $T(TM)$ enables us to give a neat description of Ω.

THEOREM 2.2.5. — *Let $u \in TM$ and $X, Y \in T_u(TM)$. Then*

$$2\Omega_u(X,Y) = g(X^h, Y^v) - g(Y^h, X^v),$$

where (X^h, X^v) and (Y^h, Y^v) are the decompositions of X and Y into their horizontal and vertical components.

Remark. — For computing these scalar products, we have identified $H_u(TM)$ and $V_u(TM)$ with $T_{\pi(u)}M$ via the procedure explained above.

Proof. — It suffices to check the equality in suitable coordinates. Take exponential (normal) coordinates $q = (q^i)$ based at $\pi(u) = m$ in a neighbourhood U of m, the naturally associated coordinates $(q, \dot{q})$ for $\pi^{-1}(U)$ and (q, p) for $\pi^{\star-1}(U)$. Then $q^i(m) = 0$, $g_{ij}(0) = \delta_{ij}$ and $\Gamma_{ij}^k(0) = 0$. Moreover, the first derivatives of g_{ij} also vanish at $q(m)$. In these coordinates, the isomorphism $\flat$ is given by

$$(q^i, \dot{q}^i) \longmapsto (q^i, \sum_{k=1}^n g_{ki}\dot{q}^k).$$

Therefore, although $(q, \dot{q})$ are *not* Darboux coordinates for Ω, for points whose coordinates are $(0, \dot{q}^i)$ we have

$$\Omega = \sum_{i=1}^n d\dot{q}^i \wedge dq^i \qquad (\text{since } \partial_k g_{ij}(0) = 0).$$

On the other hand, since $\Gamma_{ij}^k(0) = 0$, the cordinates vector fields $\partial/\partial q^i$ on TM are horizontal at u (as for the vector fields $\partial/\partial\dot{q}^i$, they are of course always vertical). The result follows. □

COROLLARY 2.2.6. — *The geodesic spray is the Hamiltonian of the energy on* TM *with respect to the symplectic form* Ω.

Proof. — Check this using the same coordinates. □

In particular, we recover the fact that the energy is constant along geodesics.

Now, let us come back to TM. It is endowed with a symplectic form Ω, a Riemannian metric G, together with the Lagrangian plane fields $u \mapsto H_u(TM)$ and $u \mapsto V_u(TM)$; both are isometric to T_mM. Therefore we get as a free gift an almost complex structure J on TM which is calibrated by Ω and such that $G(X, Y) = \Omega(JX, Y)$. But this almost complex structure *is not integrable* unless (M, g) is locally isometric to Euclidean space. Its integrability turns out to be equivalent to the integrability of the horizontal plane field. We leave the proof to the reader as an instructive (but unpleasant) exercise. This non-integrability can be read on the curvature tensor as we shall see in exercise 2.3.2.

2.3. Jacobi fields and curvature

DEFINITION 2.3.1. — *The curvature tensor of a Riemannian metric* (M, g) *is the field of endomorphisms of* TM *given by*

$$R(X, Y) = [D_X, D_Y] - D_{[X,Y]}.$$

In other words, take three tangent vectors x, y, z at $m \in M$, and local vector fields X, Y, Z whose values at m are x, y, z. Then the value at m of the vector field

$$R(X, Y)Z = D_X(D_Y Z) - D_Y(D_X Z) - D_{[X,Y]}Z$$

only depends on x, y, z (check that the map

$$(X, Y, Z) \mapsto R(X, Y)Z$$

is $\mathcal{C}^\infty(M)$-linear).

2.3.2. Exercise. — Let ξ and η be two vector fields on M, and $H\xi$ and $H\eta$ their horizontal lifts. Show that the vertical component of $[\xi, \eta]$ at $u \in TM$ is $R(\xi, \eta)u$ (we have used once more the canonical isomorphism of $V_u(TM)$ with $T_{\pi(u)}M$).

Remarks. — a) The curvature of a connection on a vector bundle is defined in exactly the same way: replace z by a vector of the fibre over m, and Z by a local section s such that $s(m) = z$.

b) Both sign conventions coexist in the literature.

It is convenient to set $R(x, y, z, t) = g(R(x, y)z, t))$. Then the following algebraic properties hold.

PROPOSITION 2.3.3. — *For any $x, y, z, t \in T_mM$ one has*
i) $R(x, y, z, t) = -R(y, x, z, t)$ and $R(x, y, z, t) = -R(x, y, t, z)$.
ii) $R(x, y)z + R(y, z)x + R(z, x)y = 0$ ("first Bianchi identity")
iii) $R(x, y, z, t) = R(z, t, x, y)$.

Proof. — i) and ii) can be checked directly. One can then deduce iii) by elementary but tricky (cf. [17], ch.2 for a nice presentation) algebraic manipulations (a better argument will be given in 2.2.5 below). □

There are many ways to interpret the curvature tensor. All of them amount to looking at second order invariants of the metric, since we already know there are no first order invariants. Here we shall focus on the linearisation of the geodesic flow. Consider a one parameter family of geodesics, that is a map $c : I \times]-\varepsilon, \varepsilon[\to c(s, t)$ such that any curve $c_t : s \mapsto c(s, t)$ is a geodesic. Let $J(s)$ be the vector field along $s \mapsto c(s, 0)$ given by

$$J(s) = \frac{d}{dt} c(s, t)_{|t=0}.$$

DEFINITION 2.3.4. — *J is a Jacobi field along c.*

2.3.5. Example. — Consider the round sphere S^n, take two unit orthogonal vectors $u, v \in T_mS^n$, and set

$$c(s, t) = (\cos s)m + \sin s((\cos t)u + (\sin t)v).$$

Let $V(s)$ be the vector field along $c_0(s) = c(s, 0)$ which is parallel to v. Then $J(s) = (\sin s)V(s)$ is a Jacobi field along the geodesic c_0. With the same notation,

$J(s) = (\sinh s)V(s)$ is a Jacobi field along the geodesic $c_0(s) = (\cosh s)m + (\sinh s)u$ of the hyperbolic space H^n.

PROPOSITION 2.3.6. — *The Jacobi vector fields along a geodesic c satisfy the linear differential equation*

$$\ddot{J}(s) + R_{c(s)}(J(s), \dot{c}(s))\dot{c}(s) = 0.$$

For vector fields along c, we have set $D^c X = \dot{X}$ and $D^c(D^c X) = \ddot{X}$.

Proof (sketch). — We work with vector fields along the geodesic variation $c(s,t)$. Then, using covariant derivation *along this* c we have

$$D_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} = 0, \qquad \text{and consequently} \qquad D_{\frac{\partial}{\partial t}}(D_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s}) = 0.$$

The desired equation follows by permutation of derivatives (the curvature was made for that). □

Remark. — The converse is true: any solution of the differential equation above is the first derivative of some geodesic variation (cf. [13], 3.45).

2.3.7. Example (continued). — In 1.4.3 we have $\ddot{J}(s) = (-\sin s)V(s)$ for the sphere, and $\ddot{J}(s) = (\sinh s)V(s)$ for hyperbolic space. Comparing with the differential equation, we find that for any pair of orthogonal unit tangent vectors we have

$$R(v,u)u = v \text{ on } S^n \text{ and } R(v,u)u = -v \text{ on } H^n,$$

so that

$$R(x,y,z,t) = \pm(g(x,t)g(y,z) - g(x,z)g(y,t)).$$

Coming back to the general case, we see that $\dot{c}(s)$ and $s\dot{c}(s)$ are Jacobi fields. They correspond to the geodesic variations given by $(s,t) \mapsto c(t+s)$ (translation of c along itself) and $(s,t) \mapsto c(ts)$. On the other hand, for any Jacobi field, the second derivative of the function $s \mapsto g(J(s), \dot{c}(s))$ clearly vanishes. Using this remark and basic properties of linear equations, we get the following.

PROPOSITION 2.3.8. — *The vector space of Jacobi fields along a geodesic c has dimension $2n$; the subspace of Jacobi fields which are normal to c has dimension $2n-2$. A Jacobi field is normal to c if and only if for some s_o one has $g(J(s_o), \dot{c}(s_o)) = 0$ and $g(\dot{J}(s_o), \dot{c}(s_o)) = 0$.* □

Let us now calculate the differential of the exponential map.

PROPOSITION 2.3.9. — *Let $m \in M$ and $u, v \in T_m M$. Let c be the geodesic $s \mapsto \exp_m sv$ and J the Jacobi field along c such that $J(0) = 0$ and $\dot{J}(0) = u$. Then*

$$T_{rv} \exp_m \cdot u = \frac{J(r)}{r}.$$

Proof (sketch). — Use the geodesic variation $(s,t) \mapsto \exp_m(s(v+tu))$. □

Remark. — In particular, p is conjugate to m along the geodesic $s \mapsto \exp_s u$ if and only if there exists a Jacobi field along the geodesic which vanishes at both extremities. Then m is conjugate to p along the reversed geodesic.

COROLLARY 2.3.10. — *A Riemannian manifold with vanishing curvature tensor is locally isometric to Euclidean space.*

Proof. — In this case, $\exp_m$ is a local isometry from T_mM equipped with the Euclidean metric g_m into M. □

Now, Proposition 2.3.9 can be refined: instead of differentiating $\exp_m$ for given m, one can differentiate the geodesic flow Φ_t.

THEOREM 2.3.11. — *Let X be a tangent vector to TM at the point u_m, and let Y and Z be its horizontal and vertical components. Then, using the canonical isomorphisms with T_mM, one has*

$$T_u\Phi_t \cdot X = (J(t), \dot{J}(t))$$

where J is the Jacobi field along the geodesic $\pi(\Phi_t u)$ such that $J(0) = Y$ and $\dot{J}(0) = Z$.

For the proof, see [15], p. 264 or [5], ch. 1. □

This property has a nice symplectic interpretation. We already know that Φ_t is a Hamiltonian flow, and therefore preserves the symplectic form Ω. Now, take two Jacobi fields J_1 and J_2 along a geodesic c. Combining theorems 2.2.5 and 2.3.11, we see that

$$g(J_1(t), \dot{J}_2(t)) - g(J_2(t), \dot{J}_1(t))$$

is *constant* along c. We get in this way a symplectic structure on the vector space of Jacobi fields along a geodesic.

Remarks. — a) Taking the derivative of the expression above gives

$$g(J_1(t), \ddot{J}_2(t)) = g(J_2(t), \ddot{J}_1(t)),$$

or, in other words,

$$g(J_1(t), R(J_2(t), \dot{c}(t)) \cdot \dot{c}(t)) = g(J_2(t), R(J_1(t), \dot{c}(t)) \cdot \dot{c}(t)).$$

Therefore

$$R(J_1(t), \dot{c}(t), \dot{c}(t), J_2(t)) = R(J_2(t), \dot{c}(t), \dot{c}(t), J_1(t)).$$

This gives the right reason for the symmetry of the curvature tensor we saw in 2.3.3 iii).

b) Clearly the 2-space generated by $\dot{c}(t)$ and $t\dot{c}(t)$ is symplectic, and its symplectic orthogonal is just the space of Jacobi fields which are orthogonal to c. At least infinitesimally, we have got a symplectic reduction of (TM, Ω), the reduced space being the space of geodesics.

c) Unfortunately, this reduction is very seldom a bona fide symplectic manifold but it is in two important cases. Firstly when the $\mathbf{R}$-action of the geodesic flow factors through an S^1-action which is free, that is when all of the geodesics are closed with the same length (see [5]). Secondly for simply connected manifolds with non-positive curvature.

2.3.12. Exercise. — Show that the space of geodesics of the Euclidean space $\mathbf{R}^{n+1}$ is diffeomorphic to the cotangent bundle $T^\star S^n$ of the n-sphere, equipped with $\pm$ the standard symplectic structure.

2.4. Back to the metric structure

There are numerous results relating curvature properties with topological properties of the underlying manifold. We shall be concerned rather with *metric properties* which can be deduced from curvature assumptions. Let us begin with the 2-dimensional case, which will be crucial for us.

In this case, the space of normal Jacobi fields is of course one-dimensional. Let (u, v) be an orthogonal basis of $T_{c(0)}M$, with $u = \dot{c}(0)$, and let $V(s)$ be the vector field along c such that $\dot{V} \equiv 0$ and $V(0) = v$. Then the normal Jacobi fields take the form $y(s)V(s)$, and clearly the fonction y satisfies a second order scalar differential equation, say

$$y''(s) + k(s)y(s) = 0.$$

Recalling that normal coordinates in dimension 2 are given by

$$\exp^\star_{c(0)} g = ds^2 + f^2(s,\theta)d\theta^2,$$

we infer, using 2.3.9, that

$$k(s) = -\frac{1}{f} \cdot \frac{\partial^2 f(s,\theta)}{\partial s^2}.$$

Let us relate this to the general setting of the curvature tensor. In dimension 2, since $\Lambda^2 T_m M$ is one-dimensional, the space of quadrilinear forms which satisfy the symmetry properties of 2.3.3 is also one-dimensional, and one checks that

$$R(x, y, z, t) = K(g(x,t)g(y,z) - g(x,z)g(y,t))$$

or, equivalently, that $R(x, y)z = K(g(y, z)x - g(x, z)y)$. In other words, the information contained in this complicated object R is reduced to a single scalar function K. The equation of normal Jacobi fields becomes

$$\ddot{J}(s) + K(c(s))J(s) = 0$$

and $K(c(s))$ is just the fonction k.

DEFINITION 2.4.1. — *K is the Gaussian curvature of (M, g).*

Now, we can use classical properties of differential equations to get geometric information from assumptions about K.

Firstly, if K is negative everywhere, any solution of the Jacobi fields equation is convex when it is positive, and concave when it is negative, and therefore vanishes once at most. Using 2.3.9 we see that the exponential map has maximal rank everywhere. Moreover, since $\exp_m^\star g$ is then a complete Riemannian metric on $T_m M$, one deduces that $\exp_m$ is *a covering map* (Cartan-Hadamard theorem).

On the other hand, if K is positive, the Sturm-Liouville comparison theorem can be applied. Let us recall it for completeness.

THEOREM 2.4.2. — *Let $y''(s) + h(s)y(s)$ be a second order scalar linear differential equation. Suppose that for any s one has $a^2 \leq h(s) \leq b^2$. Let s_0 be the first positive zero of a solution y such that $y(0) = 0$ and $y'(0) \neq 0$. Then*

$$\frac{\pi}{b} \leq s_0 \leq \frac{\pi}{a}.$$

Using 1.3.7, we see that *if $K \geq a^2$, the length of any minimising geodesic is smaller than π/a.* In particular, if (M, g) is complete, its diameter is smaller than π/a (Bonnet theorem). Curvature bounds also give volume (or rather area estimates), by using the same idea, and also injectivity radius estimates.

THEOREM 2.4.3 (Klingenberg's lemma). — *Let (M, g) be a compact Riemannian surface such that $K \leq b^2$. Then*

$$\operatorname{inj}(M, g) \geq \min\left\{\frac{\pi}{b}, \frac{l}{2}\right\},$$

where l is the length of the shortest closed geodesic in M.

Proof. — Take p, q such that $\operatorname{dist}_g(p, q) = \operatorname{inj}(M, g)$. Suppose q is not conjugate to p along any minimal geodesic. Then 1.3.11 says there exists two minimising geodesics c_1 and c_2 from p to q such that

$$c_1(s_0) = c_2(s_0) = q \quad \text{and} \quad \dot{c}_1(s_0) = -\dot{c}_2(s_0).$$

Now exchange p and q. Using the remark following 2.3.9, we see that p is not conjugate to p either. Then, using 1.3.11 again, we see that $c_1(0) = -c_2(0)$, so that c_1 and c_2 fit together to give a smooth closed geodesic.

On the other hand, if q is conjugate to p, then the Sturm-Liouville theorem says that $\operatorname{dist}_g(p, q) \geq \dfrac{\pi}{b}$. □

To see what happens in higher dimensions, one introduces the following

DEFINITION 2.4.4. — *The sectional curvature $\sigma(P)$ of a 2-plane $P \subset T_m M$ is $R(x, y, y, x)$ where (x, y) is an orthonormal basis of P (clearly, this number does not depend on the choice of the basis).*

2.4.5. Example. — For the standard sphere, $\sigma \equiv 1$, and for hyperbolic space, $\sigma \equiv -1$.

Now, the Hadamard and Bonnet theorems are valid in higher dimensions if we replace the Gaussian curvature by the sectional curvature in the statement (see [13] or [9] for details). In higher dimensions, Bonnet's theorem is called Myers theorem (cf. for instance [13], 3.85; we do not give the best statement, but it does not matter here). Klingenberg's lemma holds in the same way, but the Sturm-Liouville technique must be replaced by more delicate Jacobi field estimates, initiated by Rauch (see [8], ch.I, or [9]).

The influence of curvature on the metric can also be expressed in a very crude but useful way. Namely, using 2.3.9, together with Jacobi field estimates, one can control the norm of the differential of $\exp_m$ using curvature estimates. On the other hand, even the simple-minded example of the locally Euclidean torus $\mathbf{R}^2/\varepsilon\mathbf{Z} \oplus \mathbf{Z}$, with small ε, shows that curvature alone does not always give information about the size of balls on which $\exp_m$ is a diffeomorphism.

DEFINITION 2.4.6. — *A Riemannian manifold (M, g) has bounded geometry if $\mathrm{inj}(M, g) > 0$ and if there is a constant $A > 0$ such that $|\sigma(P)| < A$, for any $m \in M$ and any 2-plane $P \subset T_m M$.*

Compact manifolds have of course bounded geometry; so have universal covers of compact Riemannian manifolds, equipped with the lifted Riemannian metric, and tangent spaces to compact Riemannian manifolds, equipped with their natural Riemannian metric.

The main property of manifolds with bounded geometry is the following "folk" result:

PROPOSITION 2.4.7. — *There is a universal function $f(r, A)$ such that any ball of radius say $\mathrm{inj}(M, g)/2$ is diffeomorphic to the Euclidean ball of the same radius, the diffeomorphism being realised by a Lipschitz map whose dilation $\mathcal{D}$ satisfies*

$$|\log \mathcal{D}| \leq f(\mathrm{inj}(M, g), \sup|\text{sectional curvature}|)\,.$$

In dimension 2 this is now straightforward. In higher dimensions it follows again from the Jacobi field estimates we referred to above. □

Bounded geometry implies for instance a quantitative version of the Poincaré lemma. To see this, introduce for differential forms on a compact domain D of (M, g) the L^∞-norm

$$\|\alpha\|_\infty = \sup_{x \in D} |\alpha_x|,$$

where $|\alpha_x|$ is just the natural Euclidean norm in $\Lambda^p T_x M$, $p = \deg(\alpha)$.

PROPOSITION 2.4.8. — *If (M, g) has bounded geometry, there exist positive numbers δ and C such that, for any Riemannian ball Δ of radius smaller than δ, any closed form $\alpha \in \Omega^p(\Delta)$ admits a primitive ω such that*

$$\|\omega\|_\infty \leq C\|\alpha\|_\infty$$

Taking $\delta = \operatorname{inj}(M, g)/2$, this is a trivial consequence of the preceeding result, combined with the usual proof of the Poincaré lemma, where an explicit primitive is provided. □

3. Minimal manifolds

3.1. The second fundamental form

Let $i : M \to \overline{M}$ be an immersion of M into $\overline{M}$. If $\overline{M}$ carries a Riemannian metric g, we can equip M with the metric $i^\star g$. If X and Y are two vector fields on $\overline{M}$ which are tangent to M, for $m \in M$ we have the natural decomposition

$$\overline{D}_X Y = \left(\overline{D}_X Y\right)^T + \left(\overline{D}_X Y\right)^N$$

of $\overline{D}_X Y$ into its tangential and normal component. Now $\left(\overline{D}_X Y\right)^T$ defines a metric connection on M, and since $[X_{|M}, Y_{|M}] = [X, Y]_{|M}$, this connection is symmetric: using 1.4.2, we recover the Levi-Civita connection of M for the induced metric. As for $\left(\overline{D}_X Y\right)^N$, the basic properties of connections show that it is $\mathcal{C}^\infty(M)$-linear with respect to X and Y. Therefore, we have a symmetric 2-form on TM with values in the normal bundle $N(M)$.

DEFINITION 3.1.1. — *The symmetric 2-form $h(X, Y) = \left(\overline{D}_X Y\right)^N$ is the second fundamental form of M.*

Remark. — To be completely correct, we should proceed as follows: take $X, Y \in \Gamma(TM)$, extend them (locally) to $\overline{M}$, and check that $\left(\overline{D}_X Y\right)^T$ and $\left(\overline{D}_X Y\right)^N$ do not depend on the extensions. Moreover, this definition applies to immersed manifolds.

The second fundamental form measures the "bending" of M in $\overline{M}$ in the following way: pick a unit section ν of $N(M)$. For small t, we have, at least locally, the embedding $m \mapsto \exp_m t\nu$ of M into $\overline{M}$. If g_t is the metric induced by this embedding, then

$$\langle h, \nu\rangle = \frac{1}{2}\frac{d}{dt} g_{t|_{t=0}}.$$

(This is the best way to see why the second fundamental form of a round sphere in Euclidean space is proportional to the metric).

This remark makes it natural to consider the trace of h, which is a section of $N(M)$. Indeed, since the derivative of a determinant is basically a trace, one should expect some stationariness property for the volume of a submanifold such that $\mathrm{tr}(h) = 0$.

3.2. Area variation formulas for minimal surfaces

DEFINITION 3.2.1. — *The mean curvature vector H of a submanifold $M \subset \overline{M}$ is the trace of the second fundamental form. When $H \equiv 0$, M is said to be minimal.*

Then one has the following

PROPOSITION 3.2.2 (First variation formula). — *Let $M \subset \overline{M}$ be a submanifold, X a compactly supported vector field along M and $\varphi_t : M \to \overline{M}$ a one parameter family of immersions such that $X_x = \frac{d}{dt}\varphi_t(x)|_{t=0}$. Then*

$$\frac{d}{dt}\mathrm{vol}(\varphi_t(M))|_{t=0} = -\int_M \langle H, X\rangle v_g\,.$$

Proof. — Here and in 3.2.4, we follow the elegant set-up due to B. Lawson, cf. [16] or [20]. For any vector field Y, recall that

$$L_X Y = \overline{D}_X Y - \overline{D}_Y X.$$

Since L_X and $\overline{D}_X$ have natural prolongations as derivation of the whole tensor algebra, so has the map $Y \to \overline{D}_Y X$. Denote by $\mathcal{A}^X$ this derivation.

LEMMA 3.2.3. — *Let $e_1, \ldots, e_n$ be a local orthonormal frame of M, and set $\xi = e_1 \wedge e_2 \wedge \cdots \wedge e_n$. Then*

$$\frac{d}{dt}\mathrm{vol}(\varphi_t(M))|_{t=0} = \int_M \langle \mathcal{A}^X\xi, \xi\rangle v_g\,.$$

Remark. — Observe that the second member is invariantly defined.

Proof. — This is straightforward, since

$$\mathrm{vol}(\varphi_t(M)) = \int_M \|\varphi_{t\star}\xi\| v_g = \int_M \varphi_t^\star v_g$$

and

$$\begin{aligned}\frac{d}{dt}\|\varphi_{t\star}\xi\|^2 &= L_X\overline{g}(\xi,\xi) = ((\overline{D}_X - \mathcal{A}^X)\overline{g})(\xi,\xi)\\ &= -(\mathcal{A}^X g)(\xi,\xi) = 2g(\mathcal{A}^X\xi,\xi).\end{aligned}$$

□

End of the proof of 3.2.2. — With the notations of the lemma, we have

$$\begin{aligned}\langle \mathcal{A}^X\xi,\xi\rangle &= \sum_{i=1}^n \langle \mathcal{A}^X e_i, e_i\rangle = \sum_{i=1}^n \langle \overline{D}_{e_i}X, e_i\rangle \\ &= \sum_{i=1}^n \langle \overline{D}_{e_i}X^N, e_i\rangle + \sum_{i=1}^n \langle \overline{D}_{e_i}X^T, e_i\rangle \\ &= -\sum_{i=1}^n \langle h(e_i,e_i), X\rangle + \sum_{i=1}^n \langle D_{e_i}X^T, e_i\rangle \\ &= -\langle H, X\rangle + \operatorname{div}(X^T).\end{aligned}$$

Recall that the *divergence* of a vector field on a Riemannian manifold is just the trace of its covariant derivative. We have

$$(\operatorname{div}X^T)v_g = L_{X^T}v_g = d(i_{X_T}v_g),$$

so that in the preceding formula the term coming from X^T integrates to zero. □

Lemma 3.2.3 is also crucial for the following basic properties of minimal manifolds of Euclidean space.

THEOREM 3.2.4 (Monotonicity lemma). — *Let $M^n \subset \mathbf{R}^N$ be a minimal submanifold. Then for any $p \in M$, we have*

$$\operatorname{vol}(B(p,r)\cap M) \geq \omega_n r^n$$

where ω_n is the volume of the n-dimensional unit Euclidean ball.

Proof. — Suppose $p = 0$, and take the vector field $X = f(\|x\|)x$. Then

$$\mathcal{A}^X e_i = \overline{D}_{e_i}X = f'(\|x\|)\frac{\langle x, e_i\rangle}{\|x\|}x + f(\|x\|)e_i,$$

so that

$$\sum_{i=1}^n \langle \mathcal{A}^X e_i, e_i\rangle = f'(\|x\|)\frac{\left\|x^T\right\|^2}{\|x\|} + nf(\|x\|).$$

If X is compactly supported in M and M is minimal, then

$$n\int_M f(\|x\|)v_g = -\int_M f'(\|x\|)\frac{\left\|x^T\right\|^2}{\|x\|}v_g.$$

Taking f piecewise linear, with $f(t) = 1$ if $0 \leq t \leq r$ and $f(t) = 0$ if $t \geq r + \varepsilon$ we get, setting $M_r = M \cap B(p,r)$,

$$\int_{M_{r+\varepsilon}} f(\|x\|)v_g \leq \frac{r}{\varepsilon}\int_{M_{r+\varepsilon}\setminus M_r} v_g.$$

In the limit, setting $V(r) = \operatorname{vol}(B(p,r)\cap M)$, we get $nV(r) \leq rV'(r)$. Therefore, the function $r \mapsto V(r)/r^n$ is non-decreasing. On the other hand, comparing M in the neighbourhood of p to its tangent plane, we get

$$\lim_{r\to 0}\frac{V(r)}{r^n} = \omega_n.$$

Remarks. — i) In particular, there are no compact minimal surfaces in $\mathbf{R}^N$ (there are indeed many ways to prove this result, this is just one of them).

ii) This result means that, in spite of their volume minimising property, minimal varieties are rather "big". This is not really surprising. For instance, it follows from the Gauss equation that minimal surfaces in $\mathbf{R}^N$ have *negative* Gaussian curvature, compare with 5.1.1.

3.3. Minimal varieties in Kähler manifolds

The following property is both trivial and basic.

PROPOSITION 3.3.1. — *Let M be a complex submanifold of a Kähler manifold $\overline{M}$. Then M is minimal.*

Proof. — By assumption, the tangent bundle is complex. Then so is $N(M)$, since J is isometric. Moreover, we know that $\overline{D} \circ J = J \circ \overline{D}$. Combining with 1.4.2, we see that

$$h(Ju, v) = h(u, Jv) = J(h(u, v)) \quad \text{for any } u, v \in TM,$$

so that the second fundamental form h is J-antiinvariant, and has vanishing trace. □

In fact, much more is true.

THEOREM 3.3.2 (Wirtinger inequality). — *Let M be a complex curve in a compact Kähler manifold $\overline{M}$. Then, for any (real!) 2-dimensional manifold M' homologous to M, one has*

$$\text{area}(M') \geq \text{area}(M),$$

and equality holds if and only if M' is complex.

Proof. — Let M' be any oriented submanifold of $\overline{M}$. For any $m \in M'$, pick a direct orthonormal basis e, f. Then

$$0 \leq \omega(e, f) = \overline{g}(Je, f) \leq \|Je\| \, \|f\| \, .$$

Moreover, equality holds if and only if the vector space $T_m M'$ is complex. In other words, for the induced metric g we have

$$0 \leq \omega_{|M'} \leq v_g .$$

Now, if M' is homologous to M, then

$$\text{vol}(M) = \int_M \omega = \int_{M'} \omega \leq \text{vol}(M')$$

with equality if and only if M' is complex. □

3.3.3. Exercise. — Prove the same result for complex submanifolds of any dimension.

Remark. — The very same argument applies when J is an almost complex structure calibrated by a symplectic form ω : in that case, J-holomorphic curves are still homologically non trivial and minimise area in their homology class (even though they may not be minimal). And of course, we have some coarse estimate of the area if J is only tamed, as was already explained in the introduction.

4. Two-dimensional Riemannian manifolds

4.1. Riemannian and complex geometry

Given a 2-dimensional oriented Riemannian manifold (M, g), let J_m be the rotation of angle $\frac{\pi}{2}$ of T_mM. Then J is clearly an almost complex structure, which is complex for dimensional reasons (see the discussion and references in chapter II), and M becomes a Riemann surface. Moreover, replacing g by $e^u g$, for any (smooth) function u does not change the complex structure. This fact has two important consequences.

Locally. Integrability just says that for any $m \in M$ there are always local coordinates (x, y) in a suitable neighbourhood $U \ni m$ such that $g_{|U} = f(dx^2 + dy^2)$ (these coordinates are traditionally called *conformal* or *isothermal*).

Globally. The uniformisation theorem (cf. for instance [10]) says that, as a Riemann surface, the universal cover $\widetilde{M}$ of M is the projective line $\mathbf{CP}^1$, the complex line, or the unit disc D.

From the point of view of Riemannian geometry, in the first two cases the complex structures come from the standard round metric on S^2 and the flat metric on $\mathbf{R}^2$ respectively.

As for the third case, the so-called Poincaré models make it clear that H^2 (cf. 1.4.4) gives us D with its complex structure. To see this, take ordinary polar coordinates (ρ, θ) on $\mathbf{R}^2$. It suffices to find a non-vanishing function f such that

$$f^2(\rho)(d\rho^2 + \rho^2 d\theta^2) = dr^2 + \sinh^2 r d\theta^2.$$

An elementary computation gives

$$f(\rho) = \frac{2}{1-\rho^2} \quad \text{that is} \quad g = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}.$$

Pulling back the metric to the half-plane $\mathrm{Im}(z) > 0$ by the transform $z \mapsto \dfrac{z-i}{z+i}$, one gets the Poincaré half-plane model, with $g = \dfrac{dx^2 + dy^2}{y^2}$.

4.1.1. Exercise. — a) Show that the geodesics of the half-plane model are half-circles whose centre is on the real axis or half-lines orthogonal to the axis (use the fact that

$$z \longrightarrow \frac{az+b}{cz+d}, \quad \text{with } a,b,c,d \in \mathbf{R},\ ad-bc=1$$

are isometries).

b) Let Γ_λ be the group generated by the isometry $z \mapsto \lambda z$ ($\lambda > 1$). Show that the quotient P/Γ_λ is isometric to the cylinder $\mathbf{R}/(\log\lambda)\mathbf{Z}$ equipped with the metric

$$\cosh^2 l ds^2 + dl^2.$$

c) Show that this Riemannian manifold is conformal to a unique annulus $A_a = \{z \in \mathbf{C}, a < |z| < 1\}$, with $a > 0$.

d) Taking the group generated by $(x,y) \mapsto (x+c,y)$, show that one obtains a Riemannian manifold which is conformal to the punctured unit disc, and isometric to $S^1 \times \mathbf{R}$ equipped with the metric $e^{-2s}d\theta^2 + ds^2$ (this is a typical example of metric with vanishing injectivity radius).

This exercise covers the so called "exceptional" Riemann surfaces (cf. [10], IV.6), tori excepted.

Coming to the general case, the following property will be important in chapters VII and VIII.

PROPOSITION 4.1.2. — *If M is a surface with non-abelian fundamental group, $\widetilde{M}$ is the unit disc.*

Proof. — The complex manifold $\widetilde{M}$ is either $\mathbf{C}$ or D. But the first case is excluded, since any discrete goup of complex automorphisms of $\mathbf{C}$ acting properly is abelian. □

A basic example of this situation is the sphere with $n \geq 3$ punctures.

Remark. — On the other hand, it is possible to prove that a compact manifold carries a metric with constant negative curvature if and only if its Euler characteristic is negative. The "only if" part is of course the Gauss-Bonnet theorem, and the if part comes from the celebrated "pants decomposition", cf. [19] or [13], III.L.

We shall see now how the difference between $\mathbf{C}$ and D can be read on metric properties of the corresponding Riemannian manifolds, namely $\mathbf{R}^2$ and H^2.

4.2. The isoperimetric inequality

The set-up of the isoperimetric problem is roughly as follows. Given a Riemannian manifold (M^n, g), find a (if possible optimal!) numerical function f such that, for any compact disc D with rectifiable boundary ∂D, one has $\operatorname{vol}(\partial D) \geq f(\operatorname{vol}(D))$.

DEFINITION 4.2.1. — *Such an f is an isoperimetric profile of (M^n, g).*

4.2.2. Example. — Among domains in the Euclidean plane with fixed area, the minimal length is achieved for the circle. In other words, $f(t) = \sqrt{4\pi t}$ is the optimal isoperimetric profile.

Intuitively, this can be justified by the following naive approach. Let $s \mapsto c(s)$ be an arc-length parameterisation of ∂D. Consider a 1-parameter variation of c of the type

$$c_t(s) = c(s) + tf(s)\nu(s),$$

where $\nu(s)$ is the oriended normal vector. For the domain D_t enclosed by c_t we have

$$\frac{d}{dt}\text{length}(\partial D_t)_{|t=0} = -\int_0^L \kappa(s)f(s)ds$$

and

$$\frac{d}{dt}\text{area}(D_t)_{|t=0} = -\int_0^L f(s)ds.$$

Here, L is the length of ∂D and $\kappa(s)$ the (extrinsic) curvature of c, compare with 3.2.2. Now, if D has minimal boundary length for fixed area, then κ is constant, and c is a circle.

There is a quite serious gap in this argument: we implicitly supposed the existence and regularity of a minimising D. Correct proofs were obtained in the 19th century only, see [18] for a very nice discussion. However, this variational approach gives some idea of what is going on. It can be applied directly to the sphere and the hyperbolic plane, and gives some evidence of why domains with minimal boundary length for fixed area should still be bounded by curves with constant curvature: they are also geodesic discs in these cases. For the proof, see [7], I.2.

Denote by S_k^2 and H_k^2 respectively the sphere and the hyperbolic plane of constant curvature k and $-k$ ($k > 0$), and by Δ_r a geodesic disc of radius r. Using 1.4.3 and 1.4.4, we have the following data:

manifold	length$(\partial\Delta_r)$	area(Δ_r)
$\mathbf{R}^2$	$2\pi r$	πr^2
S_k^2	$2\pi\dfrac{\sin\sqrt{k}r}{\sqrt{k}}$	$\dfrac{2\pi}{k}(1-\cos\sqrt{k}r)$
H_k^2	$2\pi\dfrac{\sinh\sqrt{k}r}{\sqrt{k}}$	$\dfrac{2\pi}{k}(\cosh\sqrt{k}r - 1)$

(the second line is valid if $r \leq \pi/\sqrt{k}$; otherwise, we have length$(\partial\Delta_r) = 0$ and area$(\Delta_r) = 4\pi/k$). In all the cases, the isoperimetric profile is

$$f(t) = \sqrt{4\pi t - (\text{curvature})t^2},$$

or more acurately $\sup(0, \sqrt{4\pi t - (\text{curvature})t^2})$. Now, this property can be generalised when the Gaussian curvature is bounded below.

THEOREM 4.2.3 (Bol, Fiala). — *Let (M,g) be a 2-dimensional Riemannian manifold whose curvature is smaller than K. Then the function*

$$f(t) = \sup(0, \sqrt{4\pi t - Kt^2})$$

is an isoperimetric profile of (M,g).

The proof, elementary but tricky, consists in introducing the domain $D_t \subset D$ whose boundary is $\{x \in D, \mathrm{dist}(x, \partial D) \leq t\}$, until the area of D_t vanishes, and finding estimates for the (right) derivatives of the area of D_t and the length of ∂D_t. It will be given in the appendix to this chapter (see also [7] and their references).

Therefore, there is a sharp constrast between Euclidean space and a simply connected manifold with strictly negative curvature, constant or not. In the latter case, for big enough domains, we have an isoperimetric inequality of the type

$$\mathrm{area}(D) \leq C\mathrm{length}(\partial D) \quad \text{instead of} \quad \mathrm{area}(D) \leq C'\mathrm{length}^2(\partial D)\cdot$$

This fact is strongly related to the non-existence of conformal maps of $\mathbf{R}^2$ into a space of negative curvature (see [14], ch.6) and chapters VII and VIII).

5. An application to pseudo-holomorphic curves

As suggested in the introductory chapter of this book and explained more fully in chapter VIII, pseudo-holomorphic curves should enjoy some sort of compactness property. This fact is strongly related to the existence of uniform bounds for metric invariants attached to them. At this point, one should compare this kind of phenomenon with the compactness theory for Riemannian manifolds developed by J. Cheeger, M. Gromov and Peters, cf. [14] and [21]. Their theory is both more general and coarser, but there are some common ideas: in particular, a control of the injectivity radius prevents degeneracies.

In this section we prove that the set of J-holomorphic curves of degree one in $\mathbf{CP}^2$ is compact for the compact open topology. A very terse outline is given in [3]. We thank D. Bennequin and J. Duval for having provided many of the details, in particular in [11].

5.1. Curvature and injectivity radius estimates

PROPOSITION 5.1.1. — *Let (V,ω) be a symplectic manifold with a tamed almost complex structure J. Equip V with a Hermitian metric g. There is an upper bound $A(J,g)$ for the Gaussian curvature of J-holomorphic curves in V.*

Proof. — Let Σ be a J-holomorphic curve. Denote by $\overline{D}$ and D the Levi-Civita connections of (V,g) and $(\Sigma, g_{|\Sigma})$ respectively, and recall that for tangent vector fields to Σ

$$\overline{D}_X Y = D_X Y + h(X,Y)\,,$$

where h is the second fundamental form (cf. 3.1.1). The following property shows that pseudo-holomorphic curves are not very far from being minimal.

LEMMA 5.1.2. — *The norm of the mean curvature vector H of a pseudo-holomorphic curve satisfies the bound*

$$\|H\| \leq 2 \left\|\overline{D}J\right\|.$$

Since JY is also tangent to Σ, we have

$$\overline{D}_X JY = D_X JY + h(X, Y).$$

Now, since Σ is 2-dimensional, it is Kähler, so that $D_X JY = JD_X Y$. On the other hand, from the very definition of the covariant derivative,

$$\overline{D}_X JY = J\left(\overline{D}_X Y\right) + \left(\overline{D}_X J\right)(Y).$$

Combining these relations, we obtain

$$Jh(X, Y) = h(X, JY) + \left(\overline{D}_X J\right)(Y),$$

and we infer that

$$|h(X, X) + h(JX, JX)| \leq 2\left|\overline{D}J\right|$$

if $|X| = 1$, proving the lemma. □

To finish the proof of 5.1.1, just recall how the Gauss equation (cf. [13], 5.5) relates the curvature tensors $\overline{R}$ and R of V and Σ respectively. We have

$$R(u, Ju, Ju, u) = \overline{R}(u, Ju, Ju, u) + g(h(u, u), h(Ju, Ju)) - g(h(u, Ju), h(u, Ju))$$

The result follows, if V is compact or has bounded geometry. □

As for the control of the injectivity radius, it can be obtained in the following setting.

THEOREM 5.1.3. — *Let J be an almost complex structure on $\mathbf{CP}^n$ tamed by the standard symplectic form, and let g be a Hermitian metric. Then there is an a priori lower bound for the injectivity radius of pseudo-holomorphic curves which are homologous to the projective line.*

Proof. — Using 5.1.1, it is enough, using Klingenberg's lemma (cf. 2.4.3), to find a lower bound for the length of closed simple geodesics of the J-holomorphic curve Σ. Let γ be such a geodesic. It divides Σ into two pieces D_1 and D_2, which can be considered as 2-chains in V with boundary γ. On the other hand, if length$(\gamma) \leq \frac{1}{2}\operatorname{inj}(V, g)$, then γ bounds a disc Δ, which is contained in a ball of radius $\frac{1}{2}\operatorname{inj}(V, g)$. With a suitable orientation for Δ we have the equality

$$[\Sigma] = [D_1 + \Delta] + [D_2 - \Delta]$$

between *integral homology classes.* Therefore, using our homology assumption on Σ, we infer that the integral of ω on one of the two chains on the right member is non positive. Suppose for instance that $\int_{D_1+\Delta}\omega \leq 0$. We must have

$$\left|\int_\Delta \omega\right| \geq \int_{D_1} \omega$$

On one hand, let k be the Gaussian curvature of Σ, and A an upper bound obtained in 5.1.1. Then

$$\int_{D_1} \omega = \frac{1}{A}\int_{D_1} A\omega \geq \frac{1}{A}\int_{D_1} k\omega = \frac{2\pi}{A}.$$

(The last equality comes from Gauss-Bonnet theorem, since D_1 is a disc with geodesic boundary).

On the other hand, we have seen in 2.4.8 that in the small ball containing Δ, the form ω admits a primitive α, with

$$\|\alpha\|_\infty \leq C\,\|\omega\|_\infty,$$

where C depends only on the metric. Then

$$\int_{D_1} \omega = \int_\gamma \alpha \leq C\,\mathrm{length}(\gamma).$$

Therefore, we must have $C\,\mathrm{length}(\gamma) \geq 2\pi/A$, or $\mathrm{length}(\gamma) \geq 2\pi/AC$ as claimed. □

Remarks. — a) The assumption about the homology class is of course crucial: the statement is already false for complex curves of degree 2 in the standard projective plane, as can be see with the conics $XT = \varepsilon T^2$ in homogenous coordinates.

b) The relevant notion in that context is that of *simple homology class*, see chapter VIII or [23].

5.2. An equicontinuity theorem

Combining 5.1.1 and 5.1.3, we get the following technical lemma.

LEMMA 5.2.1. — *Let (S,g) be a Riemannian surface with Gaussian curvature smaller than K and injectivity radius larger than r. Then any domain D such that* $\mathrm{length}(\partial D) \leq r$ *and* $\mathrm{area}(D) < 2\pi/K$ *has diameter smaller than* $3r$.

Proof. — Let $x, y \in D$ be such that $\mathrm{dist}(x,y) = \mathrm{diam}(D)$, and let $x_1, y_1 \in \partial D$ realise the distance of x and y to the boundary. Then

$$\mathrm{dist}(x,y) \leq \mathrm{dist}(x,x_1) + \mathrm{length}(\partial D) + \mathrm{dist}(y,y_1).$$

It follows that if $\mathrm{diam}(D) > 3r$, the distance of x or y to the boundary is bigger than r, and D contains a Riemannian ball of radius r, say $B(x,r)$. Using 1.3.2, write the metric in normal coordinates at x, namely

$$ds^2 + f^2(s,\theta)d\theta^2.$$

Then the Sturm-Liouville therorem says that

$$f(s,\theta) \geq \frac{\sin\sqrt{K}s}{\sqrt{K}}.$$

Now

$$\operatorname{area}(D) \geq \operatorname{area}(B(x,r)) = \int_0^{2\pi}\int_0^r f(s,\theta)ds d\theta \geq \frac{2\pi}{K}(1-\cos\sqrt{K}r).$$

On the other hand, we know from 4.2.3 that

$$\operatorname{length}^2(\partial D) \geq 4\pi \operatorname{area}(D) - K\operatorname{area}^2(D).$$

If $\operatorname{length}(\partial D) \leq r$ and $\operatorname{area}(D) \leq \frac{2\pi}{K}$, we get

$$\operatorname{area}(D) \leq 2\pi/K\left(1-\sqrt{1-\frac{Kr^2}{4\pi^2}}\right),$$

a contradiction as soon as $r \leq \pi/K$. □

COROLLARY 5.2.2. — *Under the same assumptions, for any $\delta > 0$ there exists $\eta > 0$ such that, for a closed curve $c \subset f(S^2)$ of length smaller than η, one of the components of $f(S^2) \setminus c$ has a diameter smaller than δ.*

Proof. — The same homological argument as in 5.1.3 says that one of the discs which bound the loop must have small area. □

THEOREM 5.2.3. — *Let J be a tame almost complex structure on $\mathbf{CP}^n$. Let a_1, a_2, a_3 be three distinct points in $S^2 = \mathbf{CP}^1$, and d a positive number. Then the set of J-holomorphic embeddings of degree 1 of S^2 into $\mathbf{CP}^n$ such that $\operatorname{dist}(a_i, a_j) \geq d$ for all i, j is equicontinuous.*

Proof. — Equip $\mathbf{CP}^n$ with a J-Hermitian metric g. Denote by f any J-holomorphic map satisfying our assumptions. We shall prove that there exists a $\rho > 0$ such that, for any disc $D_\rho \subset S^2$ of radius ρ for the standard metric,

$$\operatorname{diam}(f(D_\rho)) \leq \frac{\operatorname{inj}(g)}{2}.$$

Then it will be possible to apply the Schwarz lemma.

By 5.1.1 and 5.1.3 we already control the curvature and the injectivity radius of $f(S^2)$. If we can control the area of $f(D_\rho)$ and the length of $\partial f(D_\rho)$, we will be able to use 5.2.1. Take $\ell > 0$, and set

$$\rho = \inf\{r, 0 < r < \pi \text{ and } \operatorname{length}(\partial f(D_r)) \geq \ell\}.$$

We look for a lower bound on ρ. Take concentric discs in S^2, and work in normal coordinates. Since f maps S^2 into $f(S^2)$ conformally,

$$f^\star g = F^2(r,\theta)(dr^2 + \sin^2 r d\theta^2).$$

Let us evaluate the area of the annulus $f(D_c \setminus D_\rho)$. On the one hand,

$$\begin{aligned}\text{area}(f(D_c \setminus D_\rho)) &= \int_\rho^c \int_0^{2\pi} F^2(r,\theta) \sin r dr d\theta \\ &\le \int_\rho^c \frac{\sin r}{2\pi} \left(\int_0^{2\pi} F(r,\theta) d\theta \right)^2 dr \quad \text{(Cauchy-Schwarz).}\end{aligned}$$

On the other hand,

$$\text{length}\,(\partial f(D_r)) = \sin r \int_0^{2\pi} F(r,\theta) d\theta.$$

But the area of the annulus is of course smaller than the area of $f(S^2)$, which is a priori bounded (cf. the introductory chapter). Thus we get

$$\text{cons.} \ge \int_\rho^c \frac{\text{length}^2\,(\partial f(D_r))}{2\pi \sin r} dr \ge \frac{\ell^2}{2\pi} \int_\rho^c \frac{dr}{\sin r}$$

and this gives a lower bound for ρ.

Now, 5.2.2 says that either $f(D_\rho)$ or its complement has controlled diameter. Choose c smaller than the mutual distances of the points a_i. Then D_ρ contains at most one point a_i, its complement at least two of them, so that diam $(f(S^2 \setminus D_\rho)) \ge d$. Then if ℓ is small enough, we are sure that the "small" disc is D_ρ. □

There is still some work to be done to go from this (C^0!) equicontinuity result to compactness: given a convergent sequence f_n of pseudo-holomorphic embeddings, we must prove that the limit has the same property. First use the fact that the Schwarz lemma also gives control of higher order derivatives. In particular, the limit curve will be immersed. It will be embedded in $\mathbf{CP}^2$ by the adjunction formula, cf. chapter VI.

Appendix: Bol's isoperimetric inequality, by M.-P. Muller

In the plane equipped with a Riemannian metric whose Gaussian curvature k is bounded above by a constant K, consider a 1-connected domain with area a, whose frontier is a closed rectifiable curve of length l. In the case where the Gaussian curvature k is constant, the inequality $l^2 \ge 4\pi a - Ka^2$ is well-known [4], [22]. In the general case we are concerned with, the result is due to G. Bol [6] (see also [2] if $k \le 0$, and [12] if $0 \le k \le K$). In this appendix, we give a proof of Bol's inequality.

The curve can be approximated by a $\mathcal{C}^\infty$ regular simple closed curve c_0 whose length is close to 1 and which bounds a domain D_0 whose area is close to a. Moreover, this curve c_0 can be assumed to be in general position for the double points of the parallel curves (wavefronts) and the singularities of the caustic (locus of the singular points of the parallel curves) associated to c_0 (see [1]). Notice that a real analytic approximation of the Riemannian metric ($\mathcal{C}^2$ topology in a compact containing the domain) and of the curve could also be chosen to exclude non generic situations.

In D_0 bounded by the curve c_0, let D_t ($t > 0$) be the set of points whose distance to c_0 is greater than or equal to t. We denote by c_t its (oriented) boundary. Considering $\varepsilon > 0$ smaller than the injectivity radius of the Riemannian metric, the boundaries of the balls with radius ε and centre a point of $c_{t-\varepsilon}$ are regular curves, and c_t is their envelope. As a consequence, at an angular point of c_t, the angle between the oriented normals is positive. In the generic situation the number of angular points is finite. They appear on c_t, for $t > t_0$, in the following cases:

1. when c_{t_0} has a double point. Generically, there is at most one double point for a given value t_0, and the curve has at this point a contact of order 1. Moreover, the total number of double points is finite, because in the generic situation there are finitely many geodesic curves which join two points of c_0 and are orthogonal to c_0 at these points (see figure 7),

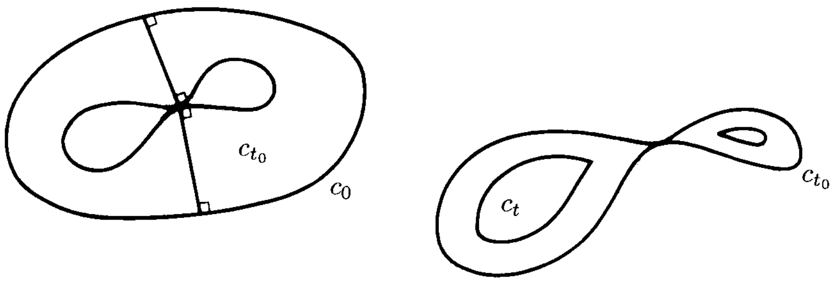

Figure 7

2. when the geodesic curvature of c_{t_0} is infinite at a point, i.e. when the wavefront reaches the caustic (thus at one of its singular points), see figure 8.

Let $l(t)$ be the length of c_t. Generically, it is a continuous function of t. As long as c_t is regular, $l(t)$ has a derivative and

$$l'(t) = -\int_{c_t} C$$

where C is the geodesic curvature. If there are angular points on the curve c_t, it is still possible to obtain the (right) derivative of l at t. Indeed, for $\Delta t > 0$, the contribution Δl of an angular point with angle α to the global length variation $l(t + \Delta t) - l(t)$ has an equivalent $\Delta l \sim -2\Delta t \tan(\alpha/2)$ (see figure 9).

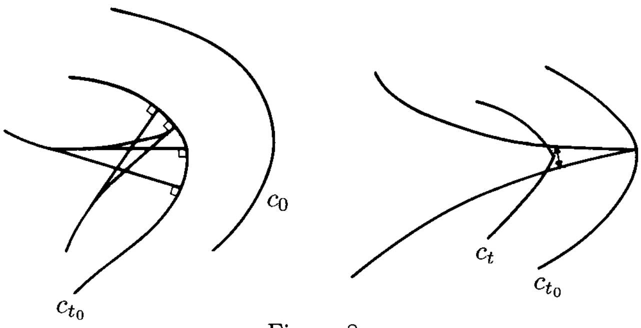

Figure 8

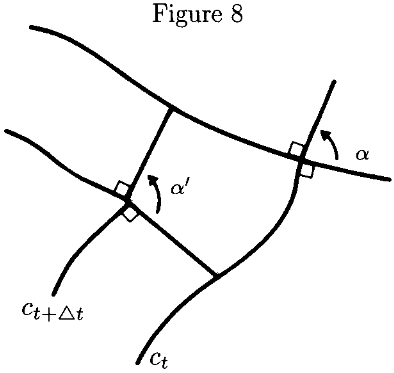

Figure 9

If c_t has a double point, it has to be considered as the union of two loops, each one having an angular point with angle π, for the preceeding evaluation. Notice that it is not possible to estimate Δl for $\Delta t > 0$, essentially because it is impossible to go back and rebuild $c_{t'}$, for $t' < t$, solely from knowledge of c_t.

Let us summarise the preceding remarks: the function l is (generically) continuous, has a right derivative l' (taking values in $[-\infty, +\infty[$) at every value of t, and

$$l'(t) = -\int_{c_t} C - 2\sum_i \tan\frac{\alpha_i}{2}$$

where C is the geodesic curvature and the $\alpha_i > 0$ are the angles at the angular points of c_t.

By the Gauss-Bonnet theorem, the preceeding formula can be rewritten

$$l'(t) = -2\pi p + \int_{D_t} k - 2\sum_i \left(\tan\frac{\alpha_i}{2} - \frac{\alpha_i}{2}\right)$$

where p is the number of connected components of D_t. On the other hand, the function $a(t)$, area of D_t, is differentiable and $a'(t) = -l(t)$.

Let us consider now the "default" function f, defined by $f = l^2 - 4\pi a + Ka^2$ where K is a constant such that $k \leq K$ on D_0. Its right derivative is

$$f' = 2l(l' + 2\pi - Ka).$$

Notice that in the expression of $l'(t)$, the contribution of the angular points is negative because

$$\tan \frac{\alpha_i}{2} \geq \frac{\alpha_i}{2}$$

and $p \geq 1$. Thus $l' \leq -2\pi + Ka$, and so $-\infty \leq D' \leq 0$. A continuous function which has a negative right derivative is decreasing. Consider the value T for which D_T is non empty and $a(T) = 0$ (generically, D_T is a single point: the connected components of D_t, whose number is bounded by the the total number of double points on all the curves, shrink to a point and disappear one after the other). We have

$$f(0) \geq f(T) = l^2(T) \geq 0$$

and hence the result. □

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Chapitre IV

Connexions linéaires, classes de Chern, théorème de Riemann-Roch

Paul Gauduchon

Ce chapitre se divise en trois parties.

La première, §§ 1.1-1.5, est un abrégé de la théorie générale des connexions linéaires sur un fibré vectoriel complexe. L'accent est mis sur les différentes manifestations de la courbure d'une connexion et leur équivalence.

La deuxième, §§ 2.1-2.7, expose les bases de la théorie des classes de Chern d'un fibré vectoriel complexe, sous l'angle topologique (théorie de l'obstruction) et géométrique (théorème de Chern-Weil).

La troisième partie, §§ 3.1-3.2, présente la formule générale de Riemann-Roch-Hirzebruch, incluant une démonstration complète, extraite de [Gun1], [Gun2] de la formule de Riemann-Roch sur une surface de Riemann compacte.

L'ensemble est aussi auto-suffisant que possible et il est recommandé au lecteur de suivre l'ordre indiqué des paragraphes. Les références se limitent aux ouvrages que nous avons consultés pour l'élaboration de ces notes.

1. Connexions linéaires

1.1. Connexions linéaires sur un fibré vectoriel complexe

Nous considérons la situation générique suivante : $E \xrightarrow{\pi} M$ où M est une variété différentiable (réelle) de dimension n, E un fibré vectoriel complexe (localement trivial) de rang r, considéré comme une variété (réelle) de dimension $n + 2r$, π la projection de E sur M. Sauf mention contraire explicite, nous supposons M connexe. Nous notons TM, $T^\star M$ les fibrés tangent et cotangent de M, $\Gamma(\mathbf{C})$ l'espace des fonctions à valeurs complexes (différentiables) définies sur M, $\Gamma(E)$ l'espace des sections (différentiables) de E.

Les produits tensoriels $\otimes$ sont relatifs au corps $\mathbf{C}$ des nombres complexes, de sorte que $T^\star M \otimes E$ note le produit tensoriel sur $\mathbf{C}$ du complexifié de $T^\star M$ avec E. Convention similaire pour $\Lambda^\star M \otimes E$, $T^\star M \otimes E$, $\Lambda^\star M \otimes \operatorname{End} E$ etc..., où $\Lambda^\star M = \Lambda^\star(T^\star M)$ note le fibré des formes extérieures (réelles) de M, $\operatorname{End} E$ le fibré des endomorphismes $\mathbf{C}$-linéaires de E.

Une *connexion linéaire* sur E est un opérateur différentiel ∇ d'ordre 1, opérant sur les sections de E, à valeurs dans le fibré vectoriel $T^\star M \otimes E$, de symbole principal égal à l'identité, i.e. satisfaisant l'*identité de Leibniz* :

$$\nabla(f\xi) = df \otimes \xi + f\nabla\xi, \quad \forall f \in \Gamma(\mathbf{C}) \ , \ \forall \xi \in \Gamma(E). \tag{1.1.1}$$

Pour toute section ξ de E, la *dérivée covariante* $\nabla\xi$ est ainsi une 1-forme sur M à valeurs dans E. Pour tout vecteur X dans l'espace tangent $T_xM, x \in M$, la valeur de $\nabla\xi$ en X, notée $\nabla_X\xi$, qui est un élément de la fibre E_x de E en x, est la *dérivée covariante de* ξ *suivant* X. La section ξ est ∇-*parallèle en* x si $\nabla_X\xi$ est nul pour tout élément X de T_xM, ∇-parallèle, si ξ est annulée par ∇.

Il résulte de (1.1.1) que la différence de deux connexions linéaires sur E est un opérateur d'ordre 0 de E dans $T^\star M \otimes E$, i.e. une 1-forme sur M à valeurs dans le fibré vectoriel $\operatorname{End} E$. Ainsi, l'ensemble des connexions linéaires sur E est muni d'une structure naturelle d'*espace affine*, modelé sur l'espace vectoriel (complexe) $\Gamma(T^\star M \otimes \operatorname{End} E)$, noté $\mathcal{A}(E)$.

Le fibré produit $M \times V$, $V = \mathbf{C}^r$, possède une *connexion produit (= triviale)*, notée d, définie par

$$d_X\xi = X \cdot \xi, \quad \forall X \in T_xM, \quad \forall X \in M, \quad \forall \xi \in \Gamma(M \times V)$$

où $X \cdot \xi$ note la dérivée ordinaire suivant X de ξ, considérée comme une fonction définie sur M à valeurs dans l'espace vectoriel V.

Une *trivialisation* (locale) φ de E au-dessus d'un ouvert $\mathcal{U}$ de M, i.e. un isomorphisme de fibrés vectoriels complexes :

$$\varphi : E_{|\mathcal{U}} \to \mathcal{U} \times V$$

de la restriction de E à $\mathcal{U}$ sur le fibré produit $\mathcal{U} \times V$, détermine une *connexion triviale*, notée $d^{(\varphi)}$, sur $E_{|\mathcal{U}}$. La 1-*forme de connexion de* ∇ *relative* à φ est la différence

$$A^{(\varphi)} = \nabla - d^{(\varphi)}$$

vue comme une 1-forme sur $\mathcal{U}$ à valeurs dans l'espace vectoriel $\operatorname{End} V$ des endomorphismes ($\mathbf{C}$-linéaires) de V via la trivialisation φ.

Si $\xi^{(\varphi)} : \mathcal{U} \to V$ est l'expression de ξ relativement à φ, l'expression $(\nabla_X\xi)^{(\varphi)}$ de $\nabla_X\xi$ relative à φ s'écrit

$$(\nabla_X\xi)^{(\varphi)} = X \cdot \xi^{(\varphi)} + A_X^{(\varphi)}(\xi^{(\varphi)}). \tag{1.1.2}$$

Soient φ_1, φ_2, deux trivialisations locales de E au-dessus des ouverts $\mathcal{U}_1$ et $\mathcal{U}_2$ respectivement, d'intersection non-vide $\mathcal{U}_{12}$. Soit φ_{12} la *fonction de transition* sur $\mathcal{U}_{12}$, à valeurs dans $\operatorname{Aut} V$, définie par

$$(\varphi_2 \circ \varphi_1^{-1})(x, v) = (x, \varphi_{12}(x)(v)), \quad \forall x \in \mathcal{U}_{12}, \quad \forall v \in V.$$

Les expressions $\xi^{(\varphi_1)}$ et $\xi^{(\varphi_2)}$ d'une même section ξ de E relativement à φ_1 et φ_2 sont liées par

$$\xi^{(\varphi_2)} = \varphi_{12}(\xi^{(\varphi_1)}), \tag{1.1.3}$$

tandis que les 1-formes de connexion de la même connexion linéaire ∇ relatives à φ_1 et φ_2 sont liées par

$$A^{(\varphi_2)} = \varphi_{12} \circ A^{(\varphi_1)} \circ \varphi_{12}^{-1} - d\varphi_{12} \circ \varphi_{12}^{-1} \tag{1.1.4}$$

comme il résulte simplement de (1.1.2) et (1.1.3).

Remarque. — La relation (1.1.4) concerne les expressions d'une *même* connexion linéaire ∇ relativement à des trivialisations (locales) différentes de E. Nous obtenons une relation d'apparence similaire en faisant agir le *groupe de jauge* $G(E)$ sur l'espace affine $A(E)$, où $G(E)$ est le groupe $\Gamma(\mathrm{Aut}\ E)$ des sections du fibré Aut E des automorphismes de fibré vectoriel complexe de E, de la façon suivante :

$$\gamma \cdot \nabla = \gamma \circ \nabla \circ \gamma^{-1} \ , \quad \forall \gamma \in \mathcal{G}(E) \ , \ \forall \nabla \in \mathcal{A}(E), \tag{1.1.5}$$

soit encore

$$\gamma \cdot \nabla = \nabla - \nabla\gamma \circ \gamma^{-1}, \tag{1.1.6}$$

où $\nabla\gamma$ note la dérivée covariante de γ, vue comme une section du fibré vectoriel $\mathrm{End}(E)$ (voir le § 1.2.4). Les relations (1.1.4) et (1.1.6) peuvent être déduites l'une de l'autre en considérant que la connexion ∇ induit, via les isomorphismes φ_1 et φ_2, deux connexions *distinctes* sur le fibré produit $\mathcal{U}_{12} \times V$, dérivées l'une de l'autre par l'action de φ_{12}, vu comme un élément du groupe de jauge de $\mathcal{U}_{12} \times V$.

Une connexion linéaire ∇ sur E est *localement triviale* si E peut être trivialisé localement par des sections ∇-parallèles de E, i.e. ∇ est localement isomorphe à la connexion naturelle $d^{(\varphi)}$ associée à une trivialisation locale φ de E (= la 1-forme de connexion $A^{(\varphi)}$ est nulle). Il résulte de (1.1.4) que E admet une connexion localement triviale si et seulement si il existe un recouvrement ouvert $\{\mathcal{U}_i\}$ de M et une trivialisation φ_i de E au-dessus de chaque ouvert $\mathcal{U}_i$ tels que les fonctions de transition φ_{ij} correspondantes, définies pour chaque paire (i,j) telle que l'intersection $\mathcal{U}_i \cap \mathcal{U}_j$ est non-vide, sont *constantes*.

De façon générale, si le fibré E est défini au moyen du recouvrement $\{\mathcal{U}_i\}$ de M et de ses fonctions de transitions φ_{ij}, la donnée d'une connexion linéaire sur E équivaut à la donnée d'une famille $\{A_i\}$ de 1-formes à valeurs dans l'espace vectoriel End V, liées entre elles par

$$A_j = \varphi_{ij} \circ A_i \circ \varphi_{ij}^{-1} - d\varphi_{ij} \circ \varphi_{ij}^{-1}$$

sur $\mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j$ dès lors que cet ouvert est non-vide.

1.2. Propriétés fonctorielles des connexions linéaires

1.2.1. — Soit $\psi : N \to M$ une application (différentiable) d'une variété N de dimension quelconque dans M. Le *fibré induit* $\psi^\star E$ de E par ψ est le fibré vectoriel complexe au-dessus de N obtenu en effectuant le produit fibré $N \times_\psi E$ de N par E au-dessus de ψ. Les éléments de $\psi^\star E$ sont donc les paires (y, u) de $N \times E$ telles que $\psi(y) = \pi(u)$ et la projection de $\psi^\star E$ sur N est définie par : $(y, u) \mapsto y$. En d'autres termes, $\psi^\star E$ est le fibré vectoriel complexe sur N dont la fibre en y est l'espace vectoriel $E_{\psi(y)}$.

Toute connexion linéaire ∇ sur E *induit* une connexion linéaire ∇^ψ sur $\psi^\star E$ définie comme suit : si ξ est une section de E, on pose

$$\nabla^\psi_Y \tilde{\xi} = (y, \nabla_{\psi_\star(Y)}\xi), \quad \forall Y \in T_yN, \quad \forall y \in N, \tag{1.2.2}$$

où $\tilde{\xi}$ note la section *induite* de $\psi^\star E$, définie à partir de ξ par

$$\tilde{\xi}(y) = (y, \xi(\psi(y))), \quad \forall y \in N. \tag{1.2.3}$$

La connexion linéaire ∇^ψ est entièrement déterminée par (1.2.2) car toute section de $\psi^\star E$ est localement engendrée, sur $\Gamma_N(\mathbf{C})$, par les sections de la forme (1.2.3).

Remarque. — La connexion induite ∇^ψ est plus aisément définie en termes de distribution horizontale, introduite au § 1.3. Pour tout élément (y, u) de $\psi^\star E$, un vecteur tangent en (y, u) à $\psi^\star E$ est identifié à une paire (Y, U) dans $T_yN \times T_uE$ telle que $\psi_\star(Y) = \pi_\star(U)$.

Le relèvement horizontal $\tilde{Y}_{(y,u)}$ de Y en (y, u) relatif à ∇^ψ est alors *défini* par

$$\tilde{Y}_{(y,u)} = (Y, \widetilde{(\psi_\star(Y))}_u)$$

où $\widetilde{(\psi_\star(Y))}_u$ note le relèvement horizontal du vecteur $\psi_\star(Y)$ en u relativement à ∇.

Le lecteur vérifiera sans peine que les deux définitions de la connexion induite ∇^ψ coïncident.

1.2.4. — Deux connexions linéaires ∇^1 et ∇^2 sur les fibrés vectoriels E^1 et E^2 respectivement (au-dessus d'une même variété M) déterminent une connexion linéaire ∇ sur le produit tensoriel $E^1 \otimes E^2$, définie par

$$\nabla(\xi^1 \otimes \xi^2) = \nabla^1\xi^1 \otimes \xi^2 + \xi^1 \otimes \nabla^2\xi^2, \quad \forall \xi^1 \in \Gamma(E^1), \quad \forall \xi^2 \in \Gamma(E^2). \tag{1.2.5}$$

De même, toute connexion linéaire ∇ sur E détermine une connexion linéaire, encore notée ∇, sur le dual (complexe) $E^\star$ de E, définie par

$$(\nabla\sigma)(\xi) = d(\sigma(\xi)) - \sigma(\nabla\xi), \quad \forall \sigma \in \Gamma(E^\star)\ ,\ \forall \xi \in \Gamma(E). \tag{1.2.6}$$

Plus généralement, ∇ détermine, via (1.2.5) et (1.2.6), une connexion linéaire, encore notée ∇, sur tout fibré vectoriel (complexe) construit fonctoriellement à partir de E, tels que $E^\star$, $\operatorname{End} E = E^\star \otimes E, \Lambda^\star E$ etc... En particulier, si α est une section de $\operatorname{End} E$, on a, pour tout ξ dans $\Gamma(E)$, $(\nabla\alpha)(\xi) = \nabla(\alpha(\xi)) - \alpha(\nabla\xi)$, c'est à dire

$$\nabla\alpha = [\nabla, \alpha]. \tag{1.2.7}$$

1.2.8. — Toute connexion linéaire ∇ sur E *s'étend* en une *différentielle extérieure* (relative à ∇) notée d^∇, ou simplement ∇, définie sur les formes sur M à valeurs dans E, i.e. sur les sections de $\Lambda^\star M \otimes E$, par

$$(1.2.9)\quad \begin{aligned}(d^\nabla\psi)(X_0,X_1,\dots,X_p) &= \sum_{i=0}^{p}(-1)^i\nabla_{X_i}(\psi(X_0,\dots,\hat{X}_i,\dots,X_p)) \\ &+ \sum_{0\leq i<j\leq p}(-1)^{i+j}\psi([X_i,X_j],\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_p),\end{aligned}$$

pour toute p-forme ψ à valeurs dans E (où $X_0,\dots,X_p$ sont $(p+1)$ vecteurs quelconques au point considéré de M), $0\leq p\leq n$.

Si $\Psi=\psi\otimes u$ est une p-forme décomposée sur M à valeurs dans E, où ψ est une p-forme scalaire (complexe) et u une section de E, on a

$$(1.2.10)\qquad d^\nabla\Psi = d\psi\otimes u+\sum_{i=1}^{n}(e_i^\star\wedge\psi)\otimes\nabla_{e_i}u,$$

où $\{e_i\}, i=1,\dots,n$, est un repère quelconque de T_xM au point x considéré et $\{e_i^\star\}$ le repère dual (algébrique) de $T_x^\star M$. On observe que si ψ est une 0-forme à valeurs dans E, i.e. une section de E, $d^\nabla\Psi$ défini par (1.2.9) ou (1.2.10) coïncide avec la dérivée covariante $\nabla\Psi$.

Si Ψ_1 est une p-forme à valeurs dans E^1, Ψ_2 une q-forme à valeurs dans E^2, le produit extérieur formel $\Psi_1\wedge\Psi_2$ est une $(p+q)$-forme à valeurs dans le produit tensoriel $E^1\otimes E^2$, muni de la connexion linéaire ∇ induite par ∇^1 et ∇^2. On déduit aisément de (1.2.10) l'identité suivante :

$$(1.2.11)\qquad d^\nabla(\Psi_1\wedge\Psi_2)=d^{\nabla^1}\Psi_1\wedge\Psi_2+(-1)^p\Psi_1\wedge d^{\nabla^2}\Psi_2.$$

L'identité (1.2.11) vaut, en particulier, quand $\Psi_1=\omega$ est une p-forme scalaire, i.e. quand E^1 est le fibré produit $M\times\mathbf{C}$ et $\nabla^1=d$ la connexion naturelle, dont l'extension au sens de (1.2.9)–(1.2.10) coïncide avec la différentielle extérieure ordinaire opérant sur les formes scalaires. Dans ce cas, l'identité (1.2.11) se réduit à l'*identité de Leibniz*

$$d^\nabla(\omega\wedge\Psi)=d\omega\wedge\Psi+(-1)^p\omega\wedge d^\nabla\Psi,$$

pour toute E-forme Ψ (qui coïncide avec l'identité de Leibniz (1.1.1) quand $\omega=f$ est une 0-forme, i.e. une fonction scalaire sur M).

Contrairement au cas scalaire, le carré $d^\nabla\circ d^\nabla$ n'est pas nul en général. Toutefois, c'est un opérateur d'ordre 0, explicité par

$$(1.2.12)\qquad (d^\nabla\circ d^\nabla)=-R^\nabla\wedge\Psi,\quad \forall\Psi\in\Gamma(\Lambda^\star M\otimes E),$$

où R^∇ est la *courbure* de ∇, vue comme une 2-forme sur M à valeurs dans le fibré vectoriel $\operatorname{End} E$, et où le "produit extérieur" $R^\nabla \wedge \Psi$ est le composé du produit extérieur formel, à valeurs dans $\operatorname{End}(E) \otimes E$, et de la contraction naturelle de $\operatorname{End}(E) \otimes E$ sur E (qui, à $\alpha \otimes u$, associe $\alpha(u)$).

La relation (1.2.12) constitue une définition possible de la courbure de ∇, voir le § 1.4. Nous en déduisons immédiatement l'*identité de Bianchi* suivante, satisfaite par toute connexion linéaire ∇,

$$d^\nabla R^\nabla = 0, \tag{1.2.13}$$

i.e., en tant que 2-forme à valeurs dans $\operatorname{End}(E)$, la courbure R^∇ est *fermée* relativement à ∇ (opérant sur les sections de $\operatorname{End}(E)$ par (1.2.7)). On a, en effet, pour toute section ξ de E, la suite d'égalités évidentes :

$$(d^\nabla R^\nabla)(\xi) = d^\nabla(R^\nabla(\xi)) - R^\nabla(\nabla\xi) = -d^\nabla \circ (d^\nabla \circ d^\nabla)\xi + (d^\nabla \circ d^\nabla) \circ d^\nabla \xi = 0.$$

1.3. Distribution horizontale d'une connexion

Pour tout élément u de E, nous notons $T_u E$ l'espace tangent en u au fibré E (vu comme une variété de dimension $n + 2r$), $T_u^V E$ le sous-espace tangent *vertical*, dont les éléments sont tangents à la fibre $E_x, x = \pi(u)$. Comme E_x est un espace vectoriel, nous avons l'identification canonique :

$$T_u^V E = E_x \ , \quad \forall u \in E_x, \quad \forall x \in M. \tag{1.3.1}$$

En particulier, pour toute section ξ de E et tout élément X de $T_x M$, la dérivée covariante $\nabla_X \xi$ de ξ suivant X peut être — et sera — considérée comme un élément de $T_u^V E$, i.e. un élément vertical de $T_u E$. Dans $T_u E$, nous avons également le vecteur $\xi_\star X$, image de X par la différentielle de ξ (vue comme une application de la variété M dans la variété E), dont la projection dans $T_x M$, via $\pi_\star$, est égale à X puisque ξ est un inverse à droite de π. Soit $\tilde{X}_u$ la différence, dans $T_u E$, des deux vecteurs $\xi_\star X$ et $\nabla_X \xi$:

$$\tilde{X}_u = \xi_\star X - \nabla_X \xi. \tag{1.3.2}$$

Il résulte aisément de l'identité de Leibniz (1.1.1) que $\tilde{X}_u$ *ne dépend pas de la section* ξ, pourvu que $\xi(x) = u$, i.e. $\tilde{X}_u$ *ne dépend que de* X *et de* u. Comme $\xi_\star X$, le vecteur $\tilde{X}_u$ se projette en X dans $T_x M$ puisque $\nabla_X \xi$ est vertical. Le vecteur $\tilde{X}_u$ est le *relèvement horizontal*, relatif à ∇, de X en u. Les éléments x et u étant fixés, respectivement dans M et E_x, l'application

$$X \mapsto \tilde{X}_u \ , \ X \in T_x M$$

est un homomorphisme de $T_x M$ dans $T_u E$, pour lequel $\pi_\star$ est un inverse à gauche et dont l'image, notée H_u^∇, est transverse à $T_u^V E$. Le sous-espace H_u^∇ est le *sous-espace horizontal* de $T_u E$, relatif à ∇, et l'ensemble des H_u^∇ constitue le *sous-fibré* *horizontal*, ou la *distribution horizontale*, sur E déterminé(e) par la connexion ∇. Nous obtenons ainsi une décomposition en somme directe de TE :

$$TE = T^V E \oplus H^\nabla \tag{1.3.3}$$

en sous-fibrés vertical (indépendant de ∇) et horizontal (dépendant de ∇). La projection de TE sur la composante verticale $T^V E$ déterminée par (1.3.3) sera notée v^∇.

La connexion ∇ est entièrement déterminée en retour par la distribution H^∇ au moyen de la relation suivante, immédiatement déduite de (1.3.2),

$$\nabla_X \xi = v^\nabla(\xi_\star X), \quad \forall X \in T_x M, \quad \forall \xi \in \Gamma(E). \tag{1.3.4}$$

On observe que pour tout élément u de la fibre E_x, l'espace horizontal H_u coïncide avec l'image $\xi_\star(T_x M)$ par la différentielle de *toute* section ξ, ∇-parallèle en x et telle que $\xi(x) = u$. En particulier, si $u = 0_x$ est l'origine de E_x, l'espace horizontal $H^\nabla_{0_x}$ coïncide avec l'espace tangent en 0_x à la variété M elle-même, identifiée à la section nulle de E. Par définition même de H^∇, une section ξ de E est ∇-parallèle si et seulement si l'image $\xi(M)$ de M par ξ est une variété intégrale de H^∇. Inversement, puisque H^∇ est transverse aux fibres de E, toute variété intégrale de H^∇ est localement l'image d'une section ∇-parallèle de E. Il en résulte que

PROPOSITION 1.3.5. — *La distribution horizontale H^∇ est intégrable si et seulement si la connexion ∇ est localement triviale.*

Le défaut d'intégrabilité de la distribution horizontale H^∇ est exactement mesuré par la courbure R^∇ de ∇, voir le § 1.4.

Soit u un élément fixé de la fibre E_x. Toute courbe (différentiable) $c : [0,1] \to M$ issue de x se relève en une unique courbe *horizontale* (= tangente à la distribution horizontale H^∇) $\tilde{c}_u : [0,1] \to E$ issue de u. L'élément $\tilde{c}_u(1)$ est le *transporté parallèle de* u le long de la courbe c (relativement à ∇). Il est clair, que $\tilde{c}_u(1)$ ne dépend pas du paramétrage choisi de c. Le transporté parallèle $\tilde{c}_u(1)$ est la solution de l'équation différentielle suivante :

$$\nabla^c_{d/dt}\xi = 0, \quad \xi(0) = u,$$

où ξ est une section de $c^\star E$, ∇^c la connexion induite, d/dt le champ de vecteur canonique de $[0,1]$. Il en résulte que le *transport parallèle* (relatif à ∇)

$$u \longmapsto \tilde{c}_u(1), \quad \forall u \in E_x,$$

est un *isomorphisme* ($\mathbf{C}$-linéaire) de la fibre E_x sur la fibre $E_{c(1)}$. Tout champ de vecteurs X sur M se relève en un champ de vecteurs horizontal $\tilde{X}$ sur E, tel que $\tilde{X}_u$ est le relèvement horizontal du vecteur X_x en u. Tout flot local φ_t^X de X se relève en un flot local $\varphi_t^{\tilde{X}}$ de X, tel que $\varphi_t^{\tilde{X}}(u)$ soit le transporté parallèle de u le long de la courbe $\varphi_t^X(x)$.

Nous pouvons dès lors donner une description alternative de la dérivée covariante $\nabla_X \xi$ d'une section ξ de E :

$$\nabla_X \xi = [\tilde{X}, \xi], \tag{1.3.6}$$

où $\nabla_X \xi$ et ξ sont considérés comme des *champs de vecteurs* verticaux sur E, *constants le long des fibres*, via l'identification (1.3.1).

1.4. La courbure d'une connexion linéaire

La courbure R^∇ d'une connexion linéaire ∇ a été définie au § 1.2.8 par (1.2.12), soit donc par

$$d^\nabla \circ \nabla \xi = -R^\nabla(u), \ \forall u \in E_x, \forall x \in M, \tag{1.4.1}$$

où ξ est une section quelconque de E prenant la valeur u au point x considéré. Plus usuellement, la courbure d'une connexion linéaire est définie comme le défaut de commutation de la dérivée covariante associée, i.e. par

$$R^\nabla_{X,Y} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y], \quad \forall X, Y \in T_x M, \quad \forall x \in M, \tag{1.4.2}$$

soit donc, avec les mêmes notations que pour (1.4.1),

$$R^\nabla_{X,Y} u = \nabla_Y(\nabla_X \xi) - \nabla_X(\nabla_Y \xi) - \nabla_{[Y,X]}\xi.$$

Il est très facile de vérifier que les deux définitions (1.4.1) et (1.4.2) de la courbure R^∇ coïncident. Dans les deux cas, R^∇ est interprétée comme une 2-forme sur M à valeurs dans le fibré vectoriel End(E). Comme $\nabla_X \xi$, $R^\nabla_{X,Y} u$ peut être considéré comme un élément de $T^V_u E$, cf. le § 1.3, et, lorsque X et Y sont fixés dans $T_x M$, $R^\nabla_{X,Y}$ peut être considéré comme un *champ de vecteurs* (linéaire) défini sur la fibre E_x. On a alors la relation suivante :

PROPOSITION 1.4.3.

$$R^\nabla_{X,Y} = v^\nabla([\tilde{X}, \tilde{Y}]), \quad \forall X, Y \in T_x M, \quad \forall x \in M.$$

Ici, dans le membre de droite, X et Y notent des extensions (quelconques) des vecteurs X et Y au voisinage du point considéré, $\tilde{X}$ et $\tilde{Y}$ les relèvements horizontaux de X et Y sur E. Avec cette convention, la démonstration de 1.4.3 est la suivante. Fixons un élément u de la fibre E_x et considérons une section ξ de E telle que $\xi(x) = u$. La dérivée covariante $\nabla_X \xi$ de ξ suivant le *champ de vecteurs* X sera considérée comme un champ de vecteurs vertical sur E, *constant* le long des fibres (via l'identification (1.3.1)). Nous faisons de même pour $\nabla_Y \xi$. Comme les champs de vecteurs $\nabla_X \xi$ et $\nabla_Y \xi$ sont tangents aux fibres et constants le long des fibres de E, leur crochet $[\nabla_X \xi, \nabla_Y \xi]$ est nul sur E. On a, de façon tautologique, l'égalité :

$$\begin{aligned}
[\tilde{X}, \tilde{Y}] &= [\tilde{X} + \nabla_X \xi, \tilde{Y} + \nabla_Y \xi] - [\nabla_X \xi, \tilde{Y} + \nabla_Y \xi] \\
&\qquad - [\tilde{X} + \nabla_X \xi, \nabla_Y \xi] - [\nabla_X \xi, \nabla_Y \xi] \\
&= [\tilde{X} + \nabla_X \xi, \tilde{Y} + \nabla_Y \xi] - [\nabla_X \xi, \tilde{Y}] - [\tilde{X}, \nabla_Y \xi] \\
&= [\tilde{X} + \nabla_X \xi, \tilde{Y} + \nabla_Y \xi] + \nabla_Y(\nabla_X \xi) - \nabla_X(\nabla_Y \xi)
\end{aligned}$$

(par (1.3.6)). Par (1.3.2), les restrictions à la sous-variété $\xi(M)$ des champs de vecteurs $\tilde{X} + \nabla_X \xi$ et $\tilde{Y} + \nabla_Y \xi$ coïncident respectivement avec $\xi_\star X$ et $\xi_\star Y$. On a donc :

$$[\tilde{X} + \nabla_X \xi, \tilde{Y} + \nabla_Y \xi] = [\xi_\star X, \xi_\star Y] = \xi_\star [X, Y].$$

Nous obtenons finalement l'expression

$$[\tilde{X}, \tilde{Y}] = \xi_\star [X, Y] + \nabla_Y (\nabla_X \xi) - \nabla_X (\nabla_Y \xi), \tag{1.4.4}$$

où $\nabla_X(\nabla_Y \xi)$ et $\nabla_Y(\nabla_X \xi)$ sont verticaux sur E, tandis que la composante verticale $v^\nabla(\xi_\star[X,Y])$ est égale, par (1.3.4), à $\nabla_{[X,Y]}\xi$. La proposition 1.4.3 se déduit alors directement de (1.4.4). □

La relation 1.4.3 peut être considérée comme une *définition* de R^∇ quand la connexion ∇ est elle-même définie par la distribution H^∇. Contrairement aux deux premières définitions (1.4.1) et (1.4.2), la définition 1.4.3 a un sens pour toute connexion sur E, linéaire ou non, i.e. pour toute distribution de n-plans sur E (ou un espace fibré quelconque, vectoriel ou non), transverse aux fibres.

De 1.4.3, considéré au choix comme une définition ou un théorème, nous déduisons le fait fondamental suivant (qui est une tautologie ou un théorème suivant le point de vue adopté) :

PROPOSITION 1.4.5. — *La connexion ∇ est localement triviale si et seulement si sa courbure R^∇ est nulle (= ∇ est plate).*

En effet, l'annulation du second membre de 1.4.3 est exactement la condition d'intégrabilité de la distribution H^∇ donnée par le théorème de Frobenius.

Si ∇^0 et ∇^1 sont deux connexions linéaires sur E, i.e. deux éléments de $\mathcal{A}(E)$, liées par

$$\nabla^1 = \nabla^0 + \eta, \tag{1.4.6}$$

où η est une 1-forme à sur M à valeurs dans $\operatorname{End}(E)$, i.e. un élément de $\Omega^1 M \otimes \operatorname{End} E$, nous déduisons aisément de (1.4.1) la relation suivante entre les courbures correspondantes :

$$R^{\nabla^1} = R^{\nabla^0} - d^{\nabla^0}\eta - [\eta, \eta], \tag{1.4.7}$$

où $[\eta, \eta]$ est la 2-forme à valeurs dans $\operatorname{End}(E)$ définie par

$$[\eta, \eta](X, Y) = [\eta(X), \eta(Y)] \ , \ \forall X, Y \in T_x M \ , \forall x \in M.$$

En particulier, l'expression $(R^\nabla)^{(\varphi)}$ de R^∇ relative à une trivialisation locale φ de E se déduit de la 1-forme de connexion $A^{(\varphi)}$ par

$$(R^\nabla)^{(\varphi)} = -dA^{(\varphi)} - [A^{(\varphi)}, A^{(\varphi)}]. \tag{1.4.8}$$

Considérons *l'application courbure* $R : \nabla \to R^\nabla$ définie sur l'espace *affine* $\mathcal{A}(E)$ des connexions linéaires sur E dans l'espace *vectoriel* $\Omega^2 M \otimes \mathrm{End}(E)$ des 2-formes sur M à valeurs dans $\mathrm{End}(E)$. Si $\nabla^0 = \nabla$ et $\nabla^1 = \gamma.\nabla$ appartiennent à la même orbite du groupe de jauge $\mathcal{G}(E)$, on déduit aisément de (1.1.5) la relation :

$$R^{\gamma\cdot\nabla} = \gamma \circ R^\nabla \circ \gamma^{-1},$$

qui exprime l'*équivariance* de l'application courbure relativement à $\mathcal{A}(E)$. En linéarisant la relation (1.4.7), nous obtenons l'expression suivante de la différentielle $T_\nabla R$ de l'application courbure R en ∇ :

$$(1.4.9)\quad (T_\nabla R)\eta := \left(\frac{d}{dt}R^{\nabla+t\eta}\right)_{t=0} = -d^\nabla \eta,\ \forall \eta \in T_\nabla(E) = \Omega^1 M \otimes \mathrm{End}\, E.$$

Remarque. — Considérons, dans l'espace affine $\mathcal{A}(E)$, la droite

$$(1.4.10)\qquad \nabla^t = t\nabla^1 + (1-t)\nabla^0,\ \forall t \in \mathbf{R},$$

passant par ∇^0 et ∇^1, liées par (1.4.6). La fonction courbure, restreinte à cette droite, s'écrit, en posant $R^t = R^{\nabla^t}$,

$$R^t = tR^1 + (1-t)R^0 + t(1-t)[\eta, \eta],$$

que l'on déduit simplement de (1.4.7).

Avec les notations des §§ 1.2.1 et 1.2.4, la courbure satisfait les propriétés fonctorielles suivantes :

$$(1.4.11)\qquad R^{\nabla^\psi}_{Y_1,Y_2}(x,u) = (x, R^\nabla_{\psi_* Y_1, \psi_* Y_2} u)\ ,\ \forall x \in M\ ,\forall u \in E_x,$$

$$(1.4.12)\quad R^\nabla(u_1 \otimes u_2) = R^{\nabla^1}(u_1) \otimes u_2 + u_1 \otimes R^{\nabla^2}(u_2),\ \forall u_1 \in E^1_x, \forall u_2 \in E^2_x,$$

$$(1.4.13)\qquad (R^\nabla \zeta)(u) = -\zeta(R^\nabla u),\quad \forall \zeta \in E^*_x,\quad \forall u \in E_x,$$

$$R^\nabla \alpha = [R^\nabla, \alpha],\quad \forall \alpha \in \mathrm{End}_x E.$$

1.5. La connexion de Chern d'un fibré vectoriel holomorphe hermitien

Nous considérons dans ce paragraphe le cas particulier où la base M du fibré vectoriel E est une variété *complexe*, de dimension complexe $m = n/2$. Nous notons $T^{0,1}M$, resp. $T^{1,0}M$, le fibré des vecteurs complexes de type $(0,1)$, resp. $(1,0)$, sur M, $\Lambda^{p,q}M$, le fibré des formes (complexes) de type (p,q), $0 \leq p,q \leq m$.

Une *structure holomorphe* sur E détermine un *opérateur de Cauchy-Riemann* $\bar\partial$ opérant sur les sections de E à valeurs dans $\Lambda^{0,1}M \otimes E = \Gamma(\Lambda^{0,1}M \otimes E)$, vérifiant l'identité de type Leibniz suivante :

$$(1.5.1)\qquad \bar\partial(f.\xi) = \bar\partial f \otimes \xi + f\bar\partial\xi,\quad \forall f \in \Gamma(\mathbf{C}),\quad \forall \xi \in \Gamma(E),$$

où $\bar{\partial} f$ est la partie de type $(0,1)$ de df, tel que les sections (locales) holomorphes de E sont exactement celles qui sont annulées par $\bar{\partial}$.

Comme une connexion linéaire, l'opérateur $\bar{\partial}$ s'étend naturellement en une "différentielle extérieure" $d^{\bar{\partial}}$, de bidegré $(0,1)$, opérant sur les formes de type $(0,\star)$ à valeurs dans E, définie, pour toute forme décomposée $\Psi = \psi \otimes \xi$, où ψ est une $(0,p)$-forme scalaire et ξ une section de E, par

$$d^{\bar{\partial}} \Psi = \bar{\partial}\psi \otimes \xi + \sum_{\alpha=1}^{m} \bar{\varepsilon}_\alpha^\star \wedge \psi \otimes \bar{\partial}_{\bar{\varepsilon}_\alpha} \xi,$$

où $\{\bar{\varepsilon}_\alpha\}$ est un repère de $T_x^{0,1}M$ au point x considéré, $\{\bar{\varepsilon}_\alpha^\star\}$ le repère dual (algébrique) de $\Lambda_x^{0,1}M$, $\bar{\partial}\psi$ la partie de type $(0,p+1)$ de $d\psi$.

Comme E est localement engendré par ses sections holomorphes, on a

$$d^{\bar{\partial}} \circ d^{\bar{\partial}} = 0. \tag{1.5.2}$$

Inversement, tout opérateur $\bar{\partial}$ satisfaisant (1.5.1) et (1.5.2) détermine une structure holomorphe sur E, les sections holomorphes étant, *par définition*, les sections (locales) annulées par $\bar{\partial}$. La condition (1.5.2) implique que E peut être localement trivialisé par les éléments du noyau de $\bar{\partial}$ (théorème de Koszul-Malgrange).

Une connexion linéaire ∇ sur E est *compatible* avec une structure holomorphe $\bar{\partial}$ si la partie $\nabla^{0,1}$ de type $(0,1)$ de ∇ coïncide avec $\bar{\partial}$, i.e. les connexions linéaires compatibles avec une structure holomorphe sur E sont celles qui "complètent" l'opérateur de Cauchy-Riemann $\bar{\partial}$ en une connexion linéaire sur E.

Lorsque E est munie d'une *structure hermitienne* (fibrée) h, toute structure holomorphe $\bar{\partial}$ sur E peut être ainsi "complétée" en une connexion linéaire ∇ définie par

$$\nabla = \bar{\partial} + \sigma^{-1} \circ \bar{\partial} \circ \sigma, \tag{1.5.3}$$

où $\sigma : E \to E^\star$ note la dualité ($\mathbf{C}$-antilinéaire) de E sur le dual $E^\star$ induite par h, et où $\bar{\partial}$ note, dans le second terme, la structure holomorphe de $E^\star$ induite par celle de E. Il est facile de vérifier que la connexion ∇ est *hermitienne*, i.e. préserve h en ce sens que

$$h(\nabla\xi_1, \xi_2) + h(\xi_1, \nabla\xi_2) = d(h(\xi_1, \xi_2)), \quad \forall \xi_1, \xi_2 \in \Gamma(E),$$

et est l'*unique* connexion linéaire sur E qui soit à la fois hermitienne et compatible avec la structure holomorphe $\bar{\partial}$. Cette connexion est la *connexion de Chern* associée à la structure hermitienne h et la structure holomorphe $\bar{\partial}$.

La partie $\nabla^{1,0}$ de type $(1,0)$ de ∇ est égale à $\sigma^{-1} \circ \bar{\partial} \circ \sigma$. Il résulte de (1.5.2) que le "carré extérieur" de $\nabla^{1,0}$, comme celui de $\nabla^{0,1} = \bar{\partial}$, est nul, i.e. que la courbure R^∇ de la connexion de Chern ∇ *est de type* $(1,1)$. Inversement, *toute connexion linéaire hermitienne* ∇ *sur le fibré hermitien* (E,h), *dont la courbure* R^∇ *est de type* $(1,1)$, *détermine une structure holomorphe sur* E, dont l'opérateur de Cauchy-Riemann est égal à la partie $\nabla^{0,1}$ de type $(0,1)$ de ∇. En effet, le fait que R^∇ soit de type

$(1,1)$ implique que le "carré extérieur" de $\bar\partial = \nabla^{0,1}$, qui est égal, au signe près, à la composante de type $(0,2)$ de R^∇, est nulle, i.e. que $\bar\partial$ satisfait (1.5.2).

Cette observation est d'un intérêt particulier quand M est une *surface de Riemann* (i.e. $m = 1$), puisque, dans ce cas, toute 2-forme est de type $(1,1)$.

Le reste du paragraphe est consacrée à la description de la connexion de Chern ∇ attachée à un fibré vectoriel hermitien holomorphe $(E, h, \bar\partial)$, définie par (1.5.3). Considérons une trivialisation locale *holomorphe* $\varphi : E_{|\mathcal{U}} \to \mathcal{U} \times V$ de E au-dessus d'un ouvert $\mathcal{U}$ de M, et fixons une base $\{e_\lambda\}$, $1 \leq \lambda \leq r$, de l'espace vectoriel V. La structure hermitienne h est alors représentée sur $\mathcal{U}$ au moyen d'une matrice de fonctions $(h^{(\varphi)}_{\lambda\mu})$, où $h^{(\varphi)}_{\lambda\mu}$ est définie par

$$h^{(\varphi)}_{\lambda\mu}(x) = h(\varphi^{-1}(x, e_\lambda), \varphi^{-1}(x, e_\mu)), \quad 1 \leq \lambda, \mu \leq r, \quad \forall x \in \mathcal{U}.$$

La 1-forme de connexion $A^{(\varphi)}$ de ∇ relative à φ est également exprimée par une matrice de 1-formes, de type $(1,0)$ puisque ∇ est compatible avec la structure conforme. On déduit aisément de (1.5.3) l'expression suivante de $A^{(\varphi)}$ en fonction de $h^{(\varphi)}$ et de la matrice inverse $(h^{(\varphi)})^{-1}$:

$$A^{(\varphi)} = \partial h^{(\varphi)} \circ (h^{(\varphi)})^{-1}. \tag{1.5.4}$$

Puisque $A^{(\varphi)}$ est de type $(1,0)$ et la courbure R^∇ de ∇ est de type $(1,1)$, la relation (1.4.8) se réduit dans le cas présent, à

$$(R^\nabla) = -\bar\partial(\partial h^{(\varphi)} \circ (h^{(\varphi)})^{-1}). \tag{1.5.5}$$

Lorsque E est de rang 1, la "matrice" $h^{(\varphi)}$ est une fonction positive définie sur l'ouvert $\mathcal{U}$ et (1.5.4), (1.5.5) s'écrivent respectivement $A^{(\varphi)} = \partial \log h^{(\varphi)}$ et $(R^\nabla)^{(\varphi)} = \partial\bar\partial \log h^{(\varphi)}$. Dans ce cas, le fibré $\mathrm{End}(E)$ est canoniquement identifié au fibré produit $M \times \mathbf{C}$ et la courbure de toute connexion linéaire sur E est une 2-forme *scalaire* (complexe), à valeurs dans $i\mathbf{R}$ si la connexion est hermitienne. Nous obtenons alors une 2-forme réelle en substituant à R^∇ la *forme de Chern* γ_1^∇ définie par

$$\gamma_1^\nabla = \frac{1}{2\pi i} R^\nabla$$

dont l'expression locale relativement à φ est donc

$$(\gamma_1^\nabla)^{(\varphi)} = \frac{-1}{2\pi} i\partial\bar\partial \log h^{(\varphi)}.$$

Nous obtenons une expression formellement similaire, mais de nature globale, en considérant une section globale *holomorphe*, non-identiquement nulle, ξ de E (s'il en existe). On a alors la relation suivante :

$$\gamma_1^\nabla = \frac{-1}{2\pi} i\partial\bar\partial \log |\xi|^2, \tag{1.5.6}$$

qui s'interprète ainsi : le second membre est bien défini sur l'ouvert (non-vide) où ξ ne s'annule pas et s'étend en une forme $\mathcal{C}^\infty$, égale à γ^∇, sur la variété M toute entière.

L'argument est le suivant : pour toute trivialisation locale holomorphe φ de E, le carré $|\xi|^2$ de la norme de ξ s'écrit $h^{(\varphi)}\xi^{(\varphi)}\bar{\xi}^{(\varphi)}$, où $\xi^{(\varphi)}$ est holomorphe. L'expression locale du second membre, sur l'ouvert où il est défini, coïncide donc avec l'expression locale de γ_1^∇, qui est globalement défini sur M. L'argument montre que l'on peut remplacer ξ, dans (1.5.6), par une "section" *méromorphe* quelconque (non-identiquement nulle) de E.

La 2-forme γ_1^∇ est clairement fermée et sa classe de cohomologie dans $H^2(M, \mathbf{R})$ est la *classe de Chern réelle* de E, voir le § 2.1.

Remarque. — Lorsque M est *compacte*, on déduit aisément de (1.5.6), par un argument fondé sur le théorème de Stokes (voir par exemple [Ha-Gr] Ch.3), que la classe de cohomologie $[\gamma_1^\nabla]$ de γ_1^∇ est le dual de Poincaré du diviseur de toute "section" méromorphe non-triviale de L, i.e. que l'on a

$$(1.5.7) \qquad \int_M \gamma_1^\nabla \wedge \psi = \int_{D_\xi} \psi, \qquad \text{pour toute } (n-2)\text{-forme fermée } \psi,$$

où D_ξ est le diviseur de ξ, pour toute section méromorphe non-identiquement nulle ξ de E.

Lorsque M est une surface de Riemann, le diviseur de ξ est constitué des zéros et des pôles de ξ, comptés avec leurs multiplicités, et (1.5.7) s'écrit alors

$$(1.5.8) \qquad \int_M \gamma_1^\nabla = \#\{\text{zéros de } \xi\} - \#\{\text{pôles de } \xi\}.$$

Il apparaît, voir le § 2.1, que les premiers membres de (1.5.7) et (1.5.8) sont indépendants, non seulement de la section ξ, mais encore des structures holomorphes de M et L.

1.5.9. Exemple fondamental. — Considérons le cas où M est l'*espace projectif complexe* $\mathbf{P}^m = \mathbf{P}(\mathbf{C}^{m+1})$ dont les éléments sont les droites complexes (vectorielles) de $\mathbf{C}^m$, et où $E = H$ est le *fibré tautologique*, ou *fibré de Hopf*, dont la fibre H_x en x est la droite complexe x elle-même, muni de sa structure holomorphe naturelle et de la structure hermitienne tautologique induite par la structure hermitienne naturelle de $\mathbf{C}^{m+1}$: pour tout élément u de H_x, $|u|^2$ est le carré de la norme (hermitienne) de u vu comme un élément de $\mathbf{C}^{m+1}$.

On observe que la variété H privée de la section nulle n'est autre que l'espace $\mathbf{C}^{m+1}$ privé de l'origine 0, i.e. H est la variété obtenue par *éclatement* de 0 dans $\mathbf{C}^{m+1}$. De ce fait, l'ensemble des éléments unitaires de H, i.e. le S^1-fibré principal Q^H des "repères unitaires" de H, coïncide avec la sphère S^{2m+1} des éléments unitaires de $\mathbf{C}^{m+1}$, i.e. la projection naturelle de Q^H sur M coïncide avec la *fibration de Hopf*

$$S^{2m+1} \longrightarrow \mathbf{P}^m,$$

l'action du groupe S^1, réalisé comme le groupe des nombres complexes de norme 1, étant explicitée, pour tout $e^{2\pi i\theta}$ dans S^1, par

$$(v^0, \ldots, v^m) \cdot e^{2\pi i\theta} = (v^0 e^{2\pi i\theta}, \ldots, v^m e^{2\pi i\theta}),$$

pour tout $v = (v^0, \ldots, v^m)$ dans $S^{2m+1} \subset \mathbf{C}^{m+1}$. Le fibré H n'a pas de section holomorphe globale non-triviale, tandis que l'espace $H^0(M, H^\star)$ des sections holomorphes (globales) du fibré *dual* $H^\star$ s'identifie naturellement à l'espace dual $(\mathbf{C}^{m+1})^\star$: à tout élément ζ de $(\mathbf{C}^{m+1})^\star$ est associée la section (holomorphe) $\tilde{\zeta}$ de $H^\star$ définie (tautologiquement) par

$$(\tilde{\zeta}(x))(u) = \zeta(u), \quad \forall x \in \mathbf{P}^m \quad \forall u \in H_x. \tag{1.5.10}$$

Pour la structure hermitienne induite sur $H^\star$, on a

$$|\tilde{\zeta}(x)|^2 = |\zeta(u)|^2, \quad \forall x \in \mathbf{P}^m,$$

où u est un élément quelconque de norme 1 de la fibre H_x. Choisissons pour ζ la forme linéaire $\zeta_0 : v = (v^0, \ldots, v^m) \mapsto v^0$, et considérons l'ouvert affine $\mathcal{U}_0 = \mathbf{P}^m - \mathbf{P}(\zeta_0^\perp)$, où $\zeta_0^\perp$ note le noyau de ζ_0 dans $\mathbf{C}^{m+1}$. Sur $\mathcal{U}_0$, nous pouvons choisir comme coordonnées (complexes) globales les fonctions $z^1 = v^1/v^0, \cdots, z^m = v^m/v^0$, et on a

$$|\tilde{\zeta}|^2 = (1 + \sum_{\alpha=1}^m |z^\alpha|^2)^{-1},$$

de sorte que la forme de Chern $\gamma_1^{H^\star}$ de $H^\star$ s'écrit :

$$\gamma_1^{H^\star} = \frac{1}{2\pi} i\partial\bar{\partial} \log(1 + \sum_{\alpha=1}^m |z^\alpha|^2).$$

Cette 2-forme est la forme de Kähler d'une structure (kählérienne) de Fubini-Study de $\mathbf{P}^m$ (celle dont le volume total est égal à $1/m!$).

Un calcul facile, effectué, par exemple, sur la droite projective $\mathbf{P}(\mathbf{C}^2)$ où $\mathbf{C}^2$ est le plan complexe de $\mathbf{C}^{m+1}$ défini par les équations $v^2 = \cdots = v^m = 0$, montre que l'intégrale de $\gamma_1^{H^\star}$ sur chaque droite projective de $\mathbf{P}^m$ est égale à 1. Ainsi, on obtient

PROPOSITION 1.5.11. — *La classe de de Rham $[\gamma_1^{H^\star}]$ de la forme de Chern du fibré de Hopf dual de $\mathbf{P}^m$ est le générateur positif de la cohomologie entière $H^\star(\mathbf{P}^m, \mathbf{Z})$ de $\mathbf{P}^m$ via le plongement canonique de $H^\star(\mathbf{P}^m, \mathbf{Z})$ dans $H^\star_{dR}(\mathbf{P}^m, \mathbf{R})$.*

En vertu de la remarque précédente, c'est aussi le dual de Poincaré de la classe d'homologie de tout hyperplan projectif complexe $\mathbf{P}^{m-1}$ de $\mathbf{P}^m$.

2. Classes de Chern

2.1. La classe de Chern d'un fibré en droites

Dans ce paragraphe, nous considérons le cas où la base M est une variété quelconque (réelle) de dimension n et le fibré E est un *fibré en droites* (complexes), i.e. un fibré vectoriel complexe de rang 1, noté génériquement L. L'ensemble $\mathrm{Vect}_1 M$ des classes d'isomorphisme de fibrés en droites sur M est munie d'une loi de groupe abélien, où la multiplication est induite par le produit tensoriel et l'inverse par le dual. Si $[L]$ note la classe d'isomorphisme de L, on a donc

$$(2.1.1) \qquad [L_1] \cdot [L_2] = [L_1 \otimes L_2] \ , \ [L]^{-1} = [L^*].$$

Comme nous l'avons déjà observé, le fibré $\operatorname{End} L$ est naturellement identifié au fibré produit $M \times \mathbf{C}$ et toute connexion linéaire ∇ sur L induit la connexion naturelle (c'est à dire triviale) sur $\operatorname{End} L = M \times \mathbf{C}$.

La courbure R^∇ est alors une 2-forme *scalaire* (complexe) et l'identité de Bianchi (1.2.13) s'écrit $dR^\nabla = 0$ tandis que la relation (1.4.7) se réduit à $R^{\nabla^1} = R^{\nabla^0} - d\eta$, où η est une 1-forme scalaire (complexe) et d est la différentielle extérieure ordinaire.

Ainsi, R^∇ est fermée et sa classe de cohomologie $[R^\nabla]$ dans l'espace de de Rham $H^2_{dR}(M, \mathbf{C})$ est indépendante de la connexion linéaire ∇. Comme nous l'avons fait au paragraphe précédent, nous substituons à la classe $[R^\nabla]$ la classe $\frac{1}{2\pi i}[R^\nabla]$, qui est réelle, i.e. appartient à $H^2_{dR}(M, \mathbf{R})$. En effet, tout fibré L admet une structure hermitienne h et une connexion hermitienne ∇, dont la courbure R^∇ est à valeurs dans $i\mathbf{R}$. Nous obtenons ainsi une application c_1^R de $\mathrm{Vect}_1 M$ dans $H^2_{dR}(M, \mathbf{R})$, qui associe à $[L]$ la classe $\frac{1}{2\pi i}[R^\nabla]$. Il résulte clairement de (1.4.12) et (1.4.13) que c_1^R est un homomorphisme de groupes abéliens. L'image de $[L]$ par c_1^R, notée simplement $c_1^R(L)$ est la *classe de Chern réelle* de $[L]$, ou, simplement, de L (nous ne distinguerons plus, dans la suite, le fibré L de sa classe d'isomorphisme $[L]$).

En général, l'homomorphisme c_1^R n'est pas injectif, mais nous allons voir qu'il se factorise par la *classe de Chern* (entière) c_1, qui est un *isomorphisme* de groupes abéliens de $\mathrm{Vect}_1 M$ sur le groupe de cohomologie entière $H^2(M, \mathbf{Z})$, i.e. que l'on a le diagramme commutatif suivant :

$$\begin{array}{ccc} \mathrm{Vect}_1 M & \xrightarrow{c_1} & H^2(M, \mathbf{Z}) \\ {\scriptstyle c_1^R}\downarrow & \swarrow {\scriptstyle j} & \\ H^2_{dR}(M, \mathbf{R}) & & \end{array}$$

où j est l'homomorphisme naturel de $H^2(M, \mathbf{Z})$ dans $H^2_{dR}(M, \mathbf{R})$, et où c_1 est un isomorphisme de groupes abéliens. Pour définir c_1, il convient d'utiliser le formalisme de la cohomologie de Čech à valeurs dans un faisceau de groupes abéliens, dont nous résumons brièvement les éléments essentiels, renvoyant, par exemple, à [God], [Gun1] ou [Hir] pour un exposé détaillé.

A tout recouvrement ouvert localement fini $\mathcal{V} = \{\mathcal{U}_i\}, i \in I$, de M est associé un complexe simplicial $K_\mathcal{V}$, le nerf de $\mathcal{V}$, dont les sommets sont les éléments de I et les p-simplexes les $(p+1)$-uples $(i_0, \ldots, i_p)$ d'éléments de I tels que l'ouvert $\mathcal{U}_{i_0, \cdots, i_p} = \mathcal{U}_{i_0} \cap \cdots \cap \mathcal{U}_{i_p}$ soit non-vide.

Soit $\mathcal{F}$ un faisceau de groupes abéliens défini sur M. Une *p-cochaîne* φ sur $K_{\mathcal{V}}$, à valeurs dans $\mathcal{F}$, associe à tout p-simplexe ordonné $(i_0, \ldots, i_p)$ de $K_{\mathcal{V}}$ une section $\varphi_{i_0,\cdots,i_p}$ de $\mathcal{F}$ sur l'ouvert $\mathcal{U}_{i_0}, \ldots, i_p$. Nous supposons en outre que φ est alternée. Nous définissons un opérateur *cobord* ∂ de degré 1 sur le groupe gradué $C^\star(K_{\mathcal{V}}, \mathcal{F})$ des cochaînes sur $K_{\mathcal{V}}$ à valeurs dans $\mathcal{F}$ par

$$(\partial\varphi)_{i_0,\ldots,i_{p+1}} = \Sigma_{k=0}^{p+1}(-1)^k \varphi_{i_0,\ldots,\hat{i}_k,\ldots,i_{p+1}}, \quad \forall \varphi \in C^p(K_{\mathcal{V}}, \mathcal{F}),$$

où, pour ne pas alourdir la notation, nous notons de même $\varphi_{i_0,\ldots,\hat{i}_k,\ldots,i_{p+1}}$ (indice i_k omis) et sa restriction à l'ouvert $\mathcal{U}_{i_0,\ldots,i_{p+1}}$.

Nous obtenons ainsi un complexe différentiel $(C^\star(K_{\mathcal{V}}, \mathcal{F}), \partial)$ dont le groupe (gradué) de cohomologie est noté $H^\star(K_{\mathcal{V}}, \mathcal{F})$. La limite inductive de ces groupes, lorsque $\mathcal{V}$ décrit l'ensemble des recouvrements ouverts (localement finis) de M, est le groupe (gradué) de *cohomologie de Čech de M à valeurs dans le faisceau $\mathcal{F}$*, noté $H^\star(M, \mathcal{F})$.

Lorsque $\mathcal{F}$ est l'un des faisceaux constants $M \times \mathbf{C}, M \times \mathbf{R}, M \times \mathbf{Z}$, le groupe de cohomologie de Čech correspondant sera noté simplement $H^\star(M, \mathbf{C})$, $H^\star(M, \mathbf{R})$, $H^\star(M, \mathbf{Z})$. Ce dernier est naturellement isomorphe au groupe de cohomologie singulière de M, et il existe de même des isomorphismes naturels de $H^\star(M, \mathbf{C})$ sur $H^\star_{dR}(M, \mathbf{C})$ et de $H^\star(M, \mathbf{R})$ sur $H^\star_{dR}(M, \mathbf{R})$ que nous expliciterons plus loin dans le cas de $H^2(M, \mathbf{R})$.

Nous notons C le faisceau des germes de fonctions (différentiables) sur M à valeurs dans $\mathbf{C}$, $C^\star$, le faisceau des germes de fonctions (différentiables) sur M à valeurs dans le groupe (multiplicatif) $\mathbf{C}^\star = \mathbf{C} - \{0\}$.

Tout fibré en droites L sur M détermine, via un recouvrement ouvert trivialisant (localement fini) $\mathcal{V} = \{\mathcal{U}_i\}, i \in I$, et les fonctions de transitions $\Phi = \{\Phi_{i,j}\}$, vues comme un 1-cocycle sur $K_{\mathcal{V}}$ à valeurs dans le faisceau $C^\star$, un élément de $H^1(K_{\mathcal{V}}, C^\star)$, puis de $H^1(M, C^\star)$. Nous obtenons de cette manière un *isomorphisme* (de groupes abéliens) $\mathrm{Vect}_1 M \to H^1(M, C^\star)$.

L'homorphisme c_1 de $\mathrm{Vect}_1 M = H^1(M, C^\star)$ dans $H^2(M, \mathbf{Z})$ est alors défini comme suit. Supposons le fibré L réalisé par le recouvrement $\mathcal{V}$ et le 1-cocycle φ sur $K_{\mathcal{V}}$ à valeurs dans $C^\star$. Supposons, en outre, que $\mathcal{V}$ a été choisi tel que les ouverts $\mathcal{U}_i$ ainsi que les intersections $\mathcal{U}_{i,j}$ soient contractiles. Nous pouvons alors définir une 1-cochaîne θ sur $K_{\mathcal{V}}$, à valeurs dans le faisceau C, par

$$\theta_{i,j} = \frac{-1}{2\pi i} \log \varphi_{i,j}, \quad \forall (i,j) \in K^1_{\mathcal{V}}.$$

Comme φ est un cocycle, dans $C^1(K_{\mathcal{V}}, C^\star)$, la 2-cochaîne $\partial\theta$ prend ses valeurs dans le sous-faisceau constant $M \times \mathbf{Z}$. La classe de Chern (entière) $c_1(L)$ de L est, par définition, l'élément de $H^2(M, \mathbf{Z})$ engendré par ce cocycle.

Nous construisons un inverse pour c_1, de $H^2(M, \mathbf{Z})$ dans $H^1(M, C^\star)$, de la façon suivante. Soit α un élément de $H^2(M, \mathbf{Z})$, représenté, relativement au recouvrement $\mathcal{V}$ (ou, au besoin, un raffinement de $\mathcal{V}$), par un 2-cocycle a sur $K_{\mathcal{V}}$ à valeurs dans le faisceau constant $M \times \mathbf{Z}$. Considéré comme un 2-cocycle à valeurs dans C, a est un cobord. De facon précise, soit $\{\rho_i\}, i \in I$, une partition de l'unité subordonnée à $\mathcal{V}$ et considérons la 1-cochaîne θ sur $K_{\mathcal{V}}$ à valeurs dans C définie par

$$\theta_{i,j} = \sum_k a_{i,j,k} \cdot \rho_k, \quad \forall (i,j) \in K^1_{\mathcal{V}} \tag{2.1.2}$$

où la somme est effectuée sur les k tels que (i,j,k) est un 2-simplexe de $K_{\mathcal{V}}$. On a alors $a = \partial\theta$.

Considérons la 1-cochaîne sur $K_{\mathcal{V}}$ à valeurs dans $C^\star$ définie par

$$\varphi_{i,j} = \exp(-2i\pi \cdot \theta_{i,j}), \quad \forall (i,j) \in K^1_{\mathcal{V}}.$$

Comme $a = \partial\theta$ est à valeurs entières, φ est un 1-cocycle et détermine ainsi un élément [L] de $H^1(M, C^\star) = \mathrm{Vect}_1 M$. Les deux opérations sont clairement inverses l'une de l'autre, ce qui établit que la classe de Chern (entière) c_1 constitue un isomorphisme de $\mathrm{Vect}_1 M$ sur $H^2(M, \mathbf{Z})$.

Il reste à montrer que pour tout fibré en droite L, $j(c_1(L))$ coïncide avec $\frac{1}{2\pi i} \cdot [R^\nabla]$ dans $H^2_{dR}(M, \mathbf{R})$.

L'homomorphisme naturel de $H^2(M, \mathbf{Z})$ dans $H^2(M, \mathbf{R})$ est exprimé, relativement au recouvrement $\mathcal{V}$, par l'application tautologique qui, à la classe d'un 2-cocycle a à valeur dans $M \times \mathbf{Z}$, associe a lui-même, vu comme un 2-cocycle à valeurs dans $M \times \mathbf{R}$. Nous devons faire suivre cette application de l'isomorphisme naturel de $H^2(M, \mathbf{R})$ dans $H^2_{dR}(M, \mathbf{R})$ qui, dans le cas considéré ici, est explicité comme suit. Considérons la 1-cochaîne θ définie par (2.1.2), où a est considéré comme un 2-cocycle à valeurs dans $M \times \mathbf{R}$, puis, pour tout i dans I, la 1-forme réelle b_i définie par $\mathcal{U}_i$ par

$$b_i = \sum_k d\theta_{i,k} \cdot \rho_k,$$

où la somme est effectuée sur les k tels que (i,k) est un 1-simplexe de $K_{\mathcal{V}}$. Considérons enfin, sur chaque ouvert $\mathcal{U}_i$, la 2-forme réelle γ définie par $\gamma_i = db_i$.

On vérifie aisément que γ_i et γ_j coïncident sur $\mathcal{U}_{i,j}$ pour tout 1-simplexe (i,j) de $K_{\mathcal{V}}$, i.e. chaque γ_i est la restriction à $\mathcal{U}_i$ d'une 2-forme réelle *fermée* γ, globalement définie sur M, dont la classe de cohomologie est, suivant la théorie générale, l'élément de $H^2_{dR}(M, \mathbf{R})$ identifié à la classe de a dans $H^2(M, \mathbf{R})$.

Par ailleurs, on a clairement, pour tout simplexe (i,j) de $K_{\mathcal{V}}$,

$$b_i - b_j = d\theta_{i,j} = \frac{-1}{2\pi i}\, d\left(\log \varphi_{i,j}\right),$$

qui montre, cf. (1.1.5), que $-2\pi i \cdot b_i$ est l'expression locale sur $\mathcal{U}_i$ d'une connexion linéaire ∇ sur L. Par (1.4.8), la 2-forme γ, dont la classe $[\gamma]$ dans $H^2_{dR}(M, \mathbf{R})$ est égale à $j(c_1(L))$, est égale à $\frac{1}{2\pi i}[R^\nabla]$. □

Remarque. — La classe de Chern c_1 que nous venons de construire est l'homomorphisme "cobord" déduit de la suite exacte de faisceaux (dite *suite exacte exponentielle*)

$$0 \longrightarrow M \times \mathbf{Z} \to C \xrightarrow{\exp 2\pi i} C^\star \longrightarrow 0, \tag{2.1.3}$$

où $\exp 2\pi i$ note le morphisme de C dans $C^\star$ qui à tout germe f de C associe le germe $\exp\, 2\pi i f$ de $C^\star$, dans la suite exacte "longue" induite :

$$0 \longrightarrow \Gamma(Z) \longrightarrow \Gamma(C) \longrightarrow \Gamma(C^\star) \longrightarrow H^1(M,\mathbf{Z}) \longrightarrow H^1(M,C)$$

$$\longrightarrow H^1(M,C^\star) \longrightarrow H^2(M,\mathbf{Z}) \longrightarrow H^2(M,C) \longrightarrow \cdots$$

Le fait que c_1 soit un isomorphisme est la conséquence de l'*acyclicité* du faisceau C, due à l'existence de partitions de l'unité différentiables qui implique l'annulation des groupes de cohomologie $H^p(M,C)$ pour tout p positif.

L'identification de $H^2(M,\mathbf{R})$ et de $H^2_{dR}(M,\mathbf{R})$, que nous avons explicitée dans le cas de $H^2(M,\mathbf{R})$, se déduit de l'*exactitude du complexe de de Rham*

$$0 \to M \times \mathbf{R} \xrightarrow{i} C^0 \xrightarrow{d} C^1 \xrightarrow{d} \cdots \xrightarrow{d} C^n \to 0,$$

où C^p est le faisceau des p-formes extérieures (différentiables), i l'inclusion naturelle de $M \times \mathbf{R}$ dans $C^0 = C$, de la différentielle extérieure, et de l'acyclicité des faisceaux C^p, cf. [Hir] Ch. I §2 12.

L'exposé que nous avons présenté suit de près [Wei] Ch. V.

La construction de la classe de Chern c_1 est clairement *fonctorielle* en ce sens que, pour toute application ψ de N dans M, cf. § 1.2.1, on a

$$c_1(\psi^\star L) = \psi^\star(c_1(L)), \quad [L] \in \operatorname{Vect}_1 M, \tag{2.1.4}$$

donc aussi $c_1^R(\psi^\star L) = \psi^\star(c_1^R(L))$. Cette dernière relation peut se déduire directement de (1.4.11) qui, dans le cas où $E = L$ est un fibré en droites, s'écrit

$$R^{\nabla^\psi} = \psi^\star(R^\nabla).$$

Par ailleurs, nous avons le fait important suivant :

PROPOSITION 2.1.5. — *Tout fibré en droites L sur M est isomorphe au fibré induit $\psi^\star H^\star$ du fibré de Hopf dual $H^\star$ de l'espace projectif complexe $\mathbf{P}^m$, $m \geq \frac{n}{2}$, par une application ψ de M dans $\mathbf{P}^m$, unique à homotopie près.*

Démonstration. — L'argument est le suivant. Choisissons une structure hermitienne h sur L et considérons le S^1-fibré principal Q des "repères unitaires" de L. Pour tout entier m, considérons la sphère S^{2m+1}, vue comme le S^1-fibré principal $Q^{H^\star}$ des "repères unitaires" de $H^\star$ (via la fibration de Hopf de S^{2m+1} sur $\mathbf{P}^m$ induite par l'action *conjuguée* de l'action (1.5.10)), voir le § 1.5.9. Considérons l'espace $Q(S^{2m+1}) = (Q \times S^{2m+1})/S^1$, quotient du produit $Q \times S^{2m+1}$ par l'action de S^1 opérant simultanément (à droite) sur Q et S^{2m+1}. Cet espace est fibré au-dessus de M et ses sections, s'il en existe, sont exactement les applications S^1-équivariantes de Q dans S^{2m+1}, i.e. les morphismes de S^1-fibrés principaux de Q dans S^{2m+1}. En particulier, à chaque section $\tilde{\psi}$ de $Q(S^{2m+1})$ est attachée une application ψ de M dans $\mathbf{P}^m$ telle que Q coïncide avec le fibré induit $\psi^\star Q^{H^\star}$, donc aussi L avec le fibré induit $\psi^\star H^\star$. En outre, deux telles sections $\tilde{\psi}_1$ et $\tilde{\psi}_2$ sont homotopes si et seulement si ψ_1 et ψ_2 sont homotopes.

Par ailleurs, tout espace fibré au-dessus de M possède des sections si la fibre F est $(n-1)$-connexe (c'est à dire telle que les groupes d'homotopie $\pi_j(F)$ sont triviaux pour $0 \leq j \leq n-1$), qui sont homotopes entre elles si F est n-connexe, cf. [Ste] Cor. 29.3. Nous concluons en observant que la fibre S^{2m+1} de $Q(S^{2m+1})$ est $2m$-connexe et que la classe d'homotopie $[\psi]$, construite à partir d'une section quelconque de $Q(S^{2m+1})$ quand $2m$ est supérieur ou égal à n, est clairement indépendante de la structure hermitienne h. □

L'application ψ déterminée, à homotopie près, par 2.1.5 est l'*application classifiante associée à L*. On observe que la dimension m n'importe pas puisque, pour tout $m' \geq m$, le plongement canonique de $\mathbf{P}^m$ dans $\mathbf{P}^{m'}$ induit le fibré de Hopf de $\mathbf{P}^{m'}$ sur le fibré de Hopf de $\mathbf{P}^m$.

Il résulte du § 1.5.9 que la classe de Chern réelle $c_1^R(H^\star)$ du fibré de Hopf dual $H^\star$ sur $\mathbf{P}^m$, qui coïncide avec la classe de Chern $c_1(H^\star)$ puisque $H^2(\mathbf{P}^m, \mathbf{Z})$ est sans torsion, est le générateur positif de $H^2(\mathbf{P}^m, \mathbf{Z})$.

Via la proposition 2.1.5 et la propriété de fonctorialité (2.1.4) de la classe de Chern, nous obtenons la *définition* équivalente de $c_1(L)$:

DÉFINITION 2.1.6. — *La classe de Chern $c_1(L)$ d'un fibré en droites L sur M est l'image inverse $\psi^\star\zeta$, par l'application classifiante ψ, du générateur positif ζ de $H^2(\mathbf{P}^m, \mathbf{Z})$ (pour tout $m \geq \frac{n}{2}$).*

Nous renvoyons le lecteur à [Ste] ou [Hus] pour un exposé détaillé de la théorie générale des espaces classifiants.

2.2. Identité de la classe d'Euler et de la classe de Chern d'un fibré en droites

Dans ce paragraphe, comme dans le précédent, nous considérons le cas où $E = L$ est un fibré en droites (complexes) au-dessus d'une variété (réelle) de dimension n. Toutefois, le fibré L sera considéré comme un fibré vectoriel *réel* de rang 2 dont chaque fibre L_x est munie d'une structure complexe i, ou, ce qui est équivalent d'une structure conforme et d'une orientation. Nous nous proposons d'établir l'identification suivante.

PROPOSITION 2.2.1. — *Dans $H^2(M, \mathbf{Z})$, pour tout fibré en droites L sur M, $c_1(L) = e(L)$, la classe d'Euler entière du fibré réel orienté L.*

Pour ce faire, il est suffisant d'établir l'identité en termes de cohomologie réelle de Rham i.e. sous la forme :

PROPOSITION 2.2.2. — *Dans $H^2(M, \mathbf{R})$, pour tout fibré en droites L sur M, $c_1^{\mathbf{R}}(L) = e^R(L)$, la classe d'Euler réelle du fibré réel orienté L, image de $e(L)$ dans $H^2_{dR}(M, \mathbf{R})$.*

En effet, les deux identités sont équivalentes lorsque L est le fibré de Hopf (dual) $H^\star$ sur $\mathbf{P}^m$, puisque la cohomologie entière de $\mathbf{P}^m$ est sans torsion, et induisent l'identité 2.2.1 pour tout L et tout M du fait de 2.1.5 et de la propriété de fonctorialité de $e(L)$ identique à (2.1.4).

Par définition, la classe d'Euler $e(L)$ de L est la restriction à M, identifiée à la section nulle $0(M)$ dans L, de la *classe de Thom* de L. En termes de cohomologie de de Rham, la classe de Thom de L appartient, cf. [Bo-Tu] Ch.1, à la cohomologie (de de Rham) à support *verticalement* compact $H^2_{dR,vc}(L, \mathbf{R})$, i.e. la cohomologie du complexe des formes différentielles sur L dont le support est compact dans chaque fibre, et est représentée par toute 2-forme réelle sur L fermée, à support compact dans chaque fibre et d'intégrale égale à 1 sur chacune d'elles.

Sur chaque fibre privée de l'origine $L_x - \{0_x\}$, la structure conforme et l'orientation de L_x déterminent une 1-*forme angulaire* $\omega = \frac{1}{2\pi} d\theta$ (où θ note l'angle orienté de deux vecteurs non-nuls de L_x), déterminée, en termes de la structure complexe i, par

$$\omega_u(u) = 0, \quad \omega_u(iu) = \frac{1}{2\pi}, \quad \forall u \in L_x - \{0_x\},$$

où $\{u, iu\}$ est considéré comme une base de $T_u(L_x) = L_x$. Toute connexion linéaire ∇ sur L détermine une 1-*forme* de *transgression* sur $L - 0(M)$, dont la restriction à chaque fibre $L_x - \{0_x\}$ coïncide avec la 1-forme angulaire ω. Cette 1-forme, notée ω^∇, est définie par

$$(\omega^\nabla)_{|T_u^V L} = \omega, \quad (\omega^\nabla)_{|H_u^\nabla} = 0, \quad \forall u \in L - 0(M),$$

où H_u^∇ est l'espace horizontal en u déterminé par ∇, voir le § 1.3. La 1-forme de transgression ω^∇ et la courbure R^∇ de ∇ sont liées par la relation (2.2.3), où γ_1^∇ est la *forme de Chern* de ∇ et $\mathrm{Ré}(\gamma_1^\nabla)$ la partie réelle de γ_1^∇, qui est fermée et représente $c_1^R(L)$ dans $H^2_{dR}(M, \mathbf{R})$:

$$d(\omega^\nabla) = -\pi^\star(\mathrm{Ré}(\gamma_1^\nabla)) \tag{2.2.3}$$

Pour montrer (2.2.3), nous utilisons les notations et conventions des § 1.3 et 1.4. Puisque la 1-forme angulaire $\omega = \frac{1}{2\pi}d\theta$ est fermée, la restriction de $d(\omega^\nabla)$ à chaque fibre $L_x - \{0_x\}$ est nulle. Pour tout élément $\tilde{X}$ de H_u^∇ et tout élément ξ de $T_u^V L$, on a

$$d(\omega^\nabla)(\tilde{X},\xi) = \tilde{X}\cdot\omega^\nabla(\xi) - \xi\cdot\omega^\nabla(\tilde{X}) - \omega^\nabla([\tilde{X},\xi]) = \tilde{X}\cdot\omega^\nabla(\xi) - \omega^\nabla([\tilde{X},\xi]),$$

car $\omega^\nabla(\tilde{X}) = 0$; donc $d(\omega^\nabla)(\tilde{X},\xi) = 0$ car le champ de 1-formes $\{\omega\}$, qui est déterminé par la seule structure complexe i, est de ce fait préservé par la connexion ($\mathbf{C}$-linéaire) ∇, cf. (1.3.6).

Enfin, si $\tilde{X}$ et $\tilde{Y}$ sont deux éléments de H_u^∇, on a

$$\begin{aligned} d(\omega^\nabla)(\tilde{X},\tilde{Y}) &= \tilde{X}\cdot\omega^\nabla(\tilde{Y}) - \tilde{Y}\cdot\omega^\nabla(\tilde{X}) - \omega^\nabla([\tilde{X},\tilde{Y}]), \\ &= -\omega^\nabla([\tilde{X},\tilde{Y}]) \end{aligned}$$

car $\omega^\nabla(\tilde{X}) = \omega^\nabla(\tilde{Y}) = 0$, et $d(\omega^\nabla)(\tilde{X},\tilde{Y}) = -\omega(R^\nabla_{X,Y}u)$ grâce à 1.4.3, et est égal à $-$ Ré$((\gamma_1^\nabla)_{X,Y})$ par 2.2.1. La relation (2.2.3) s'en déduit immédiatement. □

Remarque. — La relation (2.2.3) s'écrit simplement

$$d(\omega^\nabla) = -\pi^\star(\gamma_1^\nabla)$$

lorsque la forme de Chern γ_1^∇ est réelle, en particulier lorsque la connexion ∇ préserve une structure hermitienne h sur L.

Il résulte de (2.2.3) que la 2-forme $d(\omega^\nabla)$ s'étend différentiablement à L tout entier, et que sa restriction à la section nulle $0(M) = M$ coïncide avec $-Re(\gamma_1^\nabla)$. Toutefois, $d(\omega^\nabla)$ n'est pas à support compact dans les fibres de L. Suivant [Bo-Tu], nous construisons un représentant de la classe de Thom (au sens de de Rham) de L en fixant une structure hermitienne h sur L, en considérant une fonction de troncature $\tilde{\rho}$ sur L définie par

$$\tilde{\rho}(u) = \rho(|u|), \qquad \forall u \in L,$$

où ρ est une fonction $\mathcal{C}^\infty$ définie sur la demi-droite $[0,\infty[$ à valeurs dans le segment $[0,1]$, égale à 1 sur un voisinage de 0 et à 0 sur la demi-droite $[1,\infty]$, et en posant

$$\Theta = -d(\tilde{\rho}\omega^\nabla).$$

La 2-forme Θ est définie sur L tout entier, fermée (mais non exacte, car $\tilde{\rho}\omega^\nabla$ ne s'étend pas à L), à support compact dans chaque fibre, et on a, pour chaque fibre L_x, l'égalité

$$\int_{L_x} \Theta = 1,$$

qui résulte d'un calcul facile sur le 2-plan vectoriel L_x. Ainsi, la 2-forme Θ est un représentant de la classe de Thom dans $H^2_{dR}(M,\mathbf{R})$.

Comme $d\tilde{\rho}$ est nulle et $\tilde{\rho}$ est égal à 1 au voisinage de la section nulle $0(M)$, la restriction de Θ à $0(M)$ coïncide avec celle de $-d(\omega^\nabla)$, i.e. avec Ré(γ_1^∇). Comme Ré(γ_1^∇) est un représentant de $c_1^R(L)$, nous avons ainsi montré la proposition 2.2.2, donc aussi la proposition 2.2.1. □

Remarque. — Lorsque M est compacte et orientée, la classe de Thom de L vit dans $H_c^2(L, \mathbf{Z})$ (cohomologie à support compact) et est égale au dual de Poincaré de la classe fondamentale de la section nulle $0(M)$, via l'isomorphisme de Poincaré, cf. [Hir] th. 4.3.2.

$$H_n(L, \mathbf{Z}) \ \overset{\sim}{\to}\ H_c^2(L, \mathbf{Z}).$$

La classe d'Euler $e(L)$ — donc aussi la classe de Chern $c(L)$ — de L est donc, en tant qu'élément de $H^2(M, \mathbf{Z})$, le dual de Poincaré du cycle déterminé par les zéros d'une section générique de L.

En particulier, si M est une *surface* compacte, $e(L) = c(L)$ est un entier, via l'isomorphisme naturel $H^2(M, \mathbf{Z}) = \mathbf{Z}$, égal au nombre des zéros de toute section ξ de L transverse à la section nulle, chaque zéro étant affecté du signe $+1$ ou -1 suivant l'orientation relative des images $\xi(M)$ et $0(M)$ au point d'intersection considéré. Avec cette convention, $\int_M c_1(L)$ est le nombre algébrique de zéros de toute section ξ de L transverse à la section nulle (formule est à rapprocher de la formule (1.5.8), obtenue dans le cas holomorphe).

2.3. Classes de Chern d'un fibré vectoriel complexe

Ce paragraphe suit de très près [Bot] §3. Une fois définie la classe de Chern $c_1(L)$ d'un fibré en droites L au-dessus d'une variété (réelle) M, nous définissons les classes de Chern $c_j(E)$, $1 \leq j \leq r$, d'un fibré vectoriel (complexe) de rang r de la façon suivante.

Considérons l'espace fibré $\mathbf{P}(E)$ au-dessus de M dont la fibre en x est l'espace projectif complexe $\mathbf{P}(E_x)$. Soit H le fibré en droites sur $\mathbf{P}(E)$ dont la restriction à chaque fibre $\mathbf{P}(E_x)$ est le fibré de Hopf (le fibré tautologique de $\mathbf{P}(E_x)$), cf. §1.5.9. Soit $H^\star$ le dual de H. Soit ζ la classe de Chern $c_1(H^\star)$ de $H^\star$ dans $H^2(\mathbf{P}(E), Z)$.

Par (2.1.4), la restriction de ζ à chaque fibre $\mathbf{P}(E_x)$ coïncide avec la classe de Chern du fibré de Hopf dual de $\mathbf{P}(E_x)$, i.e. avec le générateur positif ζ_x de $H_x^2(\mathbf{P}(E_x), \mathbf{Z})$. Comme ζ_x engendre $H(\mathbf{P}(E_x), \mathbf{Z})$ pour tout point x de M, il en résulte que chaque fibre de $\mathbf{P}(E)$ est *totalement non-homologue à zéro.*

Comme $H^\star(\mathbf{P}(E_x), \mathbf{Z})$ est un $\mathbf{Z}$-module libre, engendré par $1, \zeta_x, \zeta_x^2, \ldots, \zeta_x^{r-1}$, $(\zeta_x^r = 0)$, il résulte alors du théorème de Leray-Hirsch, cf. par ex. [Hus] Ch. 16, que $H^\star(\mathbf{P}(E), Z)$ est un $H^\star(M, \mathbf{Z})$-module libre, engendré par $1, \zeta, \zeta^2, \ldots, \zeta^{r-1}$. En particulier, la puissance r-ième ζ^r du générateur ζ satisfait la relation de dépendance

$$\zeta^r + c_1(E)\zeta^{r-1} + \cdots + c_{r-1}(E)\zeta + c_r(E) = 0, \tag{2.3.1}$$

où chaque $c_j(E)$ appartient à $H^{2j}(M, \mathbf{Z})$ et est entièrement déterminé par (2.3.1). Par définition, $c_j(E)$ est la j-ième *classe de Chern* (entière) de E.

Si $E = L$ est un fibré en droites, $\mathbf{P}(E)$ est réduit à M et H coïncide avec L, dont la classe de Chern est égale à $-c_1(L)$. Il en résulte que la nouvelle définition de $c_1(L)$, donnée par (2.3.1), coïncide avec l'ancienne.

La *classe de Chern totale* $c(E)$ de E est définie, dans l'anneau $H^\star(M, \mathbf{Z})$, par

$$c(E) = 1 + c_1(E) + \cdots + c_r(E).$$

En particulier, la classe de Chern totale d'un fibré en droites L est égale à

$$c(L) = 1 + c_1(L).$$

La fonctorialité (2.1.4) de la classe de Chern c_1 d'un fibré en droites et celle de la construction de $\mathbf{P}(E)$ à partir de E implique la fonctorialité des classes de Chern c_j, i.e. celle de la classe de Chern totale :

(2.3.2) $$c(\psi^\star E) = \psi^\star(c(E)), \quad \forall [E] \in \ \mathrm{Vect}_r M, \quad \forall r \in \mathbf{N},$$

pour toute application (différentiable) ψ de N dans M, cf. § 1.2.1, où $\mathrm{Vect}_r M$ note l'ensemble des classes d'isomorphismes de fibrés vectoriels complexes de rang r sur M et $[E]$ la classe d'isomorphisme de E.

Outre (2.3.2), la propriété fondamentale de la classe de Chern totale c est d'être *multiplicative* en ce sens que

(2.3.3) $$c(E^1 \oplus E^2) = c(E^1)c(E^2), \quad \forall [E^1] \in \ \mathrm{Vect}_{r_1} M, \quad \forall [E^2] \in \ \mathrm{Vect}_{r_2} M,$$

où $E^1 \oplus E^2$ est la *somme* (de Whitney) des fibrés vectoriels E^1 et E^2, i.e. leur produit fibré au-dessus de M, et $c(E^1)c(E^2)$ le produit dans l'anneau $H^\star(M, \mathbf{Z})$. En particulier, si E est totalement décomposable en somme (de Whitney) de fibrés en droites, i.e. s'écrit

(2.3.4) $$E = L^1 \oplus \cdots \oplus L^r,$$

où chaque L^j est un fibré en droites sur M, (2.3.3) implique

(2.3.5) $$c(E) = (1 + c_1(L^1))(1 + c_1(L^2)) \cdots (1 + c_1(L^r)).$$

Inversement, (2.3.3) se déduit de (2.3.5) et du *lemme de décomposition (splitting principle)* que nous verrons un peu plus loin dans ce paragraphe (proposition 2.3.8).

Démonstration de (2.3.5), *cf.* [Bot]. — Observons tout d'abord que (2.3.5) équivaut au fait que le générateur ζ de $H^\star(\mathbf{P}(E), Z)$ satisfait la relation

(2.3.6) $$(\zeta + c_1(L^1)) \cdots (\zeta + c_1(L^r)) = 0.$$

Si π note la projection de $\mathbf{P}(E)$ sur M, le fibré induit $\pi^\star E$ contient (tautologiquement) le fibré H et on a la suite de fibrés vectoriels complexes

(2.3.7) $$0 \to H \to \pi^\star E \to Q \to 0,$$

où Q note le fibré quotient $\pi^\star E/H$. Par produit tensoriel avec $H^\star$, on déduit de (2.3.7) la suite exacte

$$0 \to \mathbf{1} \to \pi^\star E \otimes H^\star \to Q \otimes H^\star \to 0,$$

où $\mathbf{1}$ note le fibré produit $M \times \mathbf{C}$. En particulier, le fibré $\pi^\star E \otimes H^\star$ possède une section ξ sans zéro, dont la projection sur chaque sommant $\pi^\star L^j \otimes H^\star$ est notée ξ_j. Soit U_j l'ouvert de $\mathbf{P}(E)$ — éventuellement vide — sur lequel ξ_j ne s'annule pas.

Il résulte de la définition donnée en §2.1 de la classe de Chern d'un fibré en droites que la restriction de $c_1(\pi^\star L^j \otimes H^\star)$ à l'ouvert U_j est nulle, i.e. que $c_1(\pi^\star L^j \otimes H^\star) = \zeta + c_1(L^j)$ (en identifiant $c_1(L^j)$ à son image $\pi^\star(c_1(L^j))$) appartient à l'image dans $H^2(\mathbf{P}(E), \mathbf{Z})$ de la cohomologie relative $H^2(\mathbf{P}(E), U_j; \mathbf{Z})$.

Comme ξ ne s'annule pas, la réunion des ouverts U_j est la variété $\mathbf{P}(E)$ entière. Il en résulte que le produit $(\zeta + c_1(L^1)) \cdots (\zeta + c_1(L^r))$, qui appartient à l'image de $H^\star(\mathbf{P}(E), \cup U_j\ ;\ Z)$ dans $H^\star(\mathbf{P}(E), Z)$, est nul. Nous obtenons ainsi la relation (2.3.6), donc (2.3.5). □

Sauf si la base M est de dimension 2 (ou 1), il n'est pas vrai que tout fibré vectoriel complexe E puisse se décomposer suivant (2.3.4) mais nous avons la situation suivante, connue sous le nom de *lemme de décomposition*, ou *splitting principle*, qui s'énonce ainsi

PROPOSITION 2.3.8. — *A tout fibré vectoriel complexe E est associée une variété de décomposition M^E et une application q de M^E dans M vérifiant les deux propriétés :*

1. *Le fibré induit $q^\star E$ sur M^E se décompose totalement en somme de Whitney de fibrés en droites,*
2. *La cohomologie entière $H^\star(M, \mathbf{Z})$ se plonge, via $q^\star$, dans la cohomologie entière $H^\star(M^E, \mathbf{Z})$ de M^E.*

Un candidat pour M^E, vérifiant tautologiquement la première condition, est le *fibré en drapeaux* $\mathbf{F}(E)$ associé à E, dont la fibre en x est la variété des drapeaux $\mathbf{F}(E_x)$ de E_x dont les éléments sont les *drapeaux*, i.e. les r-uples de droites vectorielles $\{\ell_j\}$ de $E_x, 1 \leq j \leq r$, telles que les ℓ_j engendrent E_x. On montre que $\mathbf{F}(E)$ vérifie également la deuxième propriété, où q est la projection naturelle de $\mathbf{F}(E)$ sur M, cf. [Bo-Hi], [Hir].

Il résulte de l'existence d'un espace de décomposition pour E et de l'expression (2.3.5) de la classe de Chern totale d'un fibré décomposé que l'anneau de cohomologie entière $H^\star(M, \mathbf{Z})$ peut-être plongé dans le sur-anneau $A = H^\star(M^E, \mathbf{Z})$ de telle sorte que, dans A, la classe de Chern totale $c(E)$ s'écrive sous la forme

$$c(E) = (1 + \beta_1) \cdots (1 + \beta_r), \tag{2.3.9}$$

où les β_j sont des éléments de A, i.e. la classe de Chern $c_j(E)$ est égale à

$$c_j(E) = P_j(\beta_1, \cdots, \beta_r),$$

où P_j est le polynôme symétrique élémentaire d'ordre j. Les β_j sont les *classes de Chern virtuelles* de E. On a, en particulier,

$$c_1(E) = \beta_1 + \cdots + \beta_r, \tag{2.3.10}$$

et $c_r(E) = \beta_1 \cdots \beta_r$.

La *classe d'Euler* $e(E)$ d'un fibré vectoriel réel orienté est elle-même multiplicative, i.e. satisfait, cf. par ex. [Mi-St] :

$$e(E^1 \oplus E^2) = e(E^1)e(E^2),$$

où E^1 et E^2 notent deux fibrés vectoriels orientés sur M et $E^1 \oplus E^2$ leur somme de Whitney muni de l'orientation induite. Il résulte alors de 2.3.8 et (2.3.10) que l'on a l'identité

$$e(E) = c_r(E), \tag{2.3.11}$$

pour tout fibré vectoriel complexe E de rang r, vu, à gauche, comme un fibré réel de rang $2r$, orienté par la structure complexe des fibres.

L'écriture de la classe de Chern totale $c(E)$ en fonction des classes virtuelles β_j donnée par (2.3.9) fournit un procédé automatique pour calculer, à partir des classes de Chern de E, les classes de Chern de tout fibré vectoriel (complexe) $\widetilde{E}$, tel que $E^\star$, End E, $\Lambda^p(E)$, construit de façon fonctorielle à partir de E, cf. § 1.2.4. En effet, toute décomposition (2.3.4) de E induit une semblable décomposition de $\widetilde{E}$ en somme directe de fibrés en droites qui se déduisent fonctoriellement des sommants L^j de E. Nous en déduisons immédiatement, via (2.1.1), l'expression des classes de Chern virtuelles, donc aussi des classes de Chern, de $\widetilde{E}$.

2.3.12. Exemple. — $\widetilde{E} = E^\star$. Les composants de $E^\star$ sont les fibrés duaux $(L^j)^\star$. Les classes de Chern virtuelles de $E^\star$ sont donc

$$\beta_j(E^\star) = -\beta_j(E), \tag{2.3.13}$$

et donc

$$c_j(E^\star) = (-1)^j c_j(E). \tag{2.3.14}$$

2.3.15. Exemple. — $\widetilde{E} = E^1 \otimes E^2$. Si $L^{1,j}$ et $L^{2,k}$ sont les sommants de E^1 et E^2 respectivement, les sommants de $\widetilde{E}$ sont les fibrés en droites $L^{1,j} \otimes L^{2,k}$, et les classes de Chern virtuelles de $\widetilde{E} = E^1 \otimes E^2$ sont donc :

$$\beta_{j,k} = \beta_j(E^1) + \beta_k(E^2), \quad 1 \leq j \leq r_1, \quad 1 \leq k \leq r_2. \tag{2.3.16}$$

2.3.17. Exemple. — $\tilde{E} = \Lambda^p E$. Les sommants de $\Lambda^p E$ sont les fibrés en droites $L^{j_1} \oplus \cdots \oplus L^{j_p}, 1 \leq j_1 < \cdots < j_p < r$. Les classes de Chern virtuelles de $\tilde{E} = \Lambda^p E$ sont donc

$$\beta_{j_1,\ldots,j_p}(\Lambda^p E) = \beta_{j_1} + \cdots + \beta_{j_p}, \quad 1 \leq j_1 < \cdots < j_p < r.$$

En particulier, on a l'identité, pour tout fibré vectoriel (complexe) de rang r,

$$c_1(E) = c_1(\Lambda^r E). \tag{2.3.18}$$

A cause de (2.3.2), la classe de Chern totale $c = 1 + c_1 + \cdots + c_r$ peut être vue comme un *morphisme de cofoncteurs* de $\mathrm{Vect}_\star$ dans $H^\star(., \mathbf{Z})$, l'un et l'autre définis sur la catégorie des variétés différentiables (en fait, sur une catégorie beaucoup plus large d'espaces topologiques, cf. [Hir]). Ce morphisme est *multiplicatif*, i.e. satisfait (2.3.3). Il est *normalisé* par le fait suivant : si $H^\star$ est le fibré de Hopf dual de $\mathbf{P}^m$, on a

$$c(H^\star) = 1 + \text{ le générateur positif de } H^2(\mathbf{P}^m, \mathbf{Z})\ . \tag{2.3.19}$$

Il résulte des propositions 2.1.5 et 2.3.8 que la classe de Chern totale est *caractérisée* par ces propriétés et peut être *définie* à partir d'elles, considérées comme axiomes. cf. [Hir] Ch. I § 3.

Remarque. — La construction précédente des classes de Chern d'un fibré vectoriel complexe s'applique en particulier à l'espace tangent TM d'une variété *presque complexe* (M, J), de dimension (réelle) $n = 2m$, où le fibré tangent TM est considéré, via la structure presque complexe J, comme un fibré vectoriel complexe de rang m. Les classes de Chern $c_j(TM)$ sont alors, par définition, les *classes de Chern de* (M, J), notées $c_j(M, J)$, ou simplement $c_j(M)$.

De (2.3.11), nous déduisons que *la classe de Chern* $c_m(M)$ est égale à la classe d'Euler $e(TM), TM$ étant muni de l'orientation induite par J, i.e. la classe d'Euler de la variété M munie de l'orientation induite par J. En particulier, $c_m(M)$ ne dépend de J que par l'orientation induite par J sur M.

2.4. Le théorème de Chern-Weil.

La *théorie de Chern-Weil* (relative aux classes de Chern) consiste à représenter la classe de Chern (totale) $c(E)$ de E, ou plutôt l'image $c^R(E) = j(c(E))$ de $c(E)$ dans $H^\star_{dR}(M, \mathbf{R})$, par une expression universelle en la courbure R^∇ d'une connexion $\mathbf{C}$-linéaire ∇ sur E. Rappelons que la courbure R^∇ est une 2-forme sur M à valeurs dans le fibré vectoriel (complexe) End E. Il en va de même de la *courbure normalisée* $\tilde{R}^\nabla$ définie par

$$\tilde{R}^\nabla = \frac{1}{2\pi i} R^\nabla.$$

Considérons la *forme de Chern inhomogène* γ^∇, à valeurs complexes, définie sur M par l'expression suivante, où I note l'identité de E,

$$\gamma^\nabla = \det\,(I + \widetilde{R}^\nabla). \tag{2.4.1}$$

Cette expression s'entend de la façon suivante : en tout point x, $I + \widetilde{R}^\nabla$ peut être représentée par une matrice d'ordre r, à coefficients dans l'anneau (commutatif) $\Lambda_x^{\mathrm{pair}} M$, et $\text{dét}(I + \widetilde{R}^\nabla)$ est égal, en x, au déterminant de cette matrice, qui est un élément de $\Lambda_x^{\mathrm{pair}} M$ indépendant de la représentation matricielle choisie de $I + \widetilde{R}^\nabla$.

On observe que, si la connexion ∇ préserve une structure hermitienne h sur E, la courbure normalisée $\widetilde{R}^\nabla$ prend ses valeurs dans le sous-fibré (réel) des éléments h-hermitiens de $\operatorname{End} E$ et la *forme de Chern* correspondante γ^∇ est alors une forme (inhomogène) à valeurs réelles. Le théorème de Chern-Weil (relatif aux classes de Chern) peut alors s'énoncer ainsi :

THÉORÈME 2.4.2. — *Pour toute connexion linéaire ∇ sur E, la forme de Chern inhomogène γ^∇ est fermée. La classe de de Rham $[\gamma^\nabla]$ est réelle et est égale, dans $H^\star_{dR}(M,\mathbf{R})$, à l'image $j(c(E))$ de la classe de Chern totale de E dans $H^\star_{dR}(M,\mathbf{R})$.*

Observons tout d'abord que 2.4.2 a été démontré au § 2.1 dans le cas où $E = L$ est un fibré en droites. L'argument utilisé au § 2.1 pour montrer que γ_1^∇ est fermée, de classe $[\gamma_1^\nabla]$ réelle et indépendante de ∇, s'étend aisément au cas général où E est de rang quelconque r, à l'aide du formalisme développé au § 1.2.8. Pour cela, il est commode de décomposer γ^∇ en ses parties homogènes :

$$\gamma^\nabla = 1 + \gamma_1^\nabla + \cdots + \gamma_r^\nabla,$$

où γ_j^∇ est une forme de degré $2j$, et d'écrire chaque γ_j^∇ sous la forme

$$\gamma_j^\nabla = Q_j^E(\widetilde{R}^\nabla \wedge \cdots \wedge \widetilde{R}^\nabla), \tag{2.4.3}$$

avec les conventions suivantes : $\widetilde{R}^\nabla \wedge \cdots \wedge \widetilde{R}^\nabla$ est le produit extérieur formel (j fois) de $\widetilde{R}^\nabla$ par lui-même, vu comme une $2j$-forme sur M à valeurs dans le produit tensoriel $\operatorname{End}(E) \otimes \cdots \otimes \operatorname{End}(E)$ (j fois) ; Q_j est la composante homogène d'ordre j de la fonction $a \mapsto \text{dét}(I+a)$, vue comme une fonction polynomiale, Ad_G-invariante, définie sur l'algèbre de Lie $g = g\ell(r,\mathbf{C})$ du groupe linéaire $G = GL(r,\mathbf{C})$, où Ad_G note l'action adjointe de G sur g ; de façon précise, Q_j est vue comme une forme j-multilinéaire symétrique sur g, soit encore une forme linéaire définie sur le produit tensoriel $g \otimes \cdots \otimes g$ (j fois) ; Q_j^E est l'élément du dual $(\operatorname{End}(E) \otimes \cdots \otimes \operatorname{End}(E))^\star$ (j fois) déterminé par Q_j ; $Q_j^E(\widetilde{R}^\nabla \wedge \cdots \wedge \widetilde{R}^\nabla)$ est le composé de $\widetilde{R}^\nabla \wedge \cdots \wedge \widetilde{R}^\nabla$ et de Q_j^E.

Nous sommes alors en situation d'appliquer le lemme général suivant. Soit P une fonction polynomiale homogène de degré p, Ad_G-invariante, définie sur g, P^E l'élément induit de $(\mathrm{End}(E) \otimes \cdots \otimes \mathrm{End}(E))^*$ (p fois), $\psi_1, \ldots, \psi_p$, p formes sur M à valeurs dans $\mathrm{End}\, E$ de degrés respectifs $d_1, \ldots, d_p$, d^∇ la différentielle extérieure induite par ∇ sur les $\mathrm{End}\, E$-formes sur M, cf. § 1.2.8.

LEMME 2.4.4. — *La relation*

$$d(P^E(\psi_1 \wedge \cdots \wedge \psi_p)) = P^E(d^\nabla \psi_1 \wedge \cdots \wedge \psi_p) \pm \cdots \pm P^E(\psi_1 \wedge \cdots \wedge d^\nabla \psi_p)$$

est satisfaite.

Dans cet énoncé, les signes, qui dépendent des degrés $d_1, \ldots, d_p$, sont ceux qui figurent dans le cas scalaire (démonstration facile). On observe que le membre de gauche, donc aussi celui de droite, est indépendant de la connexion linéaire ∇. La relation cherchée

$$d(\gamma_j^\nabla) = 0, \qquad 1 \leq j \leq r,$$

est alors, via (2.4.3) et 2.4.4, une conséquence directe de l'identité de Bianchi (1.2.13).

Soient ∇^0 et ∇^1 deux connexions linéaires (distinctes) sur E, et

$$\nabla^t = t\nabla^1 + (1-t)\nabla^0 = \nabla^0 + t\eta, \qquad \forall t \in \mathbf{R},$$

la droite joignant ∇^0 et ∇^1 dans l'espace affine $\mathcal{A}(E)$, cf. (1.4.10). Soient γ^t la forme de Chern, $\widetilde{R}^t$ la courbure normalisée de ∇^t.

De (2.4.3), nous déduisons la suite d'égalités suivante :

$$\begin{aligned} \frac{d}{dt}(\gamma_j^t) &= jQ_j^E\left(\frac{d}{dt}\widetilde{R}^t \wedge \cdots \wedge \widetilde{R}^t\right) && \text{car } Q_j \text{ est symétrique,} \\ &= -jQ_j^E(d^\nabla \eta \wedge \cdots \wedge \widetilde{R}^t) && \text{par (1.4.9),} \\ &= -jd(Q_j^E(\eta \wedge \cdots \wedge \widetilde{R}^t)) && \text{par (1.2.13) et 2.4.4.} \end{aligned}$$

On a donc, pour $1 \leq j \leq r$,

$$\gamma_j^1 - \gamma_j^0 = -jd\left(\int_0^1 Q_j^E(\eta \wedge \widetilde{R}^t \wedge \cdots \wedge \widetilde{R}^t)dt\right),$$

qui implique l'égalité cherchée

$$[\gamma_j^1] = [\gamma_j^0], \qquad 1 \leq j \leq r, \tag{2.4.5}$$

i.e. la classe de de Rham de γ^∇ ne dépend pas de la connexion ($\mathbf{C}$)-linéaire ∇.

Comme $[\gamma^\nabla]$ ne dépend pas de ∇, nous pouvons choisir pour ∇ une connexion h-hermitienne, pour laquelle γ^∇ est réelle. Il en résulte que $[\gamma^\nabla]$ appartient à $H^\star_{dR}(M, \mathbf{R})$. Si E est complètement décomposé, i.e. de la forme (2.3.4), nous pouvons choisir, pour représenter $[\gamma^\nabla]$, une connexion décomposée, pour laquelle la courbure normalisée $\widetilde{R}^\nabla$ est diagonale, i.e. de la forme :

$$\widetilde{R}^\nabla = \begin{pmatrix} \gamma_1^{\nabla^1} & 0 & \cdots & 0 \\ 0 & \gamma_1^{\nabla^2} & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & \gamma_1^{\nabla^r} \end{pmatrix}$$

où $\gamma_1^{\nabla^j}, 1 \leq j \leq r$, est la forme de Chern du fibré en droites L^j relative à une connexion linéaire ∇^j sur L^j. On a donc

$$[\gamma^\nabla] = (1 + c_1^R(L^1)) \cdots (1 + c_1^R(L^r)),$$

qui montre, par (2.3.5), que, lorsque E est totalement décomposé, $[\gamma^\nabla]$ est égale à l'image de $c(E)$ dans $H^\star_{dR}(M, \mathbf{R})$.

Le cas général s'en déduit immédiatement grâce au lemme de décomposition 2.3.8 et à la fonctorialité évidente de la forme de Chern :

$$\gamma^{\psi^\star \nabla} = \psi^\star(\gamma^\nabla),$$

pour toute application ψ de N dans M, cf. § 1.2.1. □

Remarque. — Lorsque M est une variété complexe, de dimension réelle $n = 2m$, et $E = (E, \bar{\partial})$ un fibré vectoriel holomorphe sur M, nous pouvons choisir pour représenter $[\gamma^\nabla]$ dans $H^\star_{dR}(M, \mathbf{R})$ la connexion de Chern ∇ déterminée par la structure holomorphe $\bar{\partial}$ et une structure hermitienne h, cf. § 1.5. La forme de Chern correspondante γ_j^∇ est alors une $2j$-forme (réelle) de type (j, j), i.e. la classe de Chern (réelle) $c_j^R(E) = [\gamma_j^\nabla]$ de E appartient au sous-espace $H^{j,j}_{dR}(M, \mathbf{R})$ de $H^{2j}_{dR}(M, \mathbf{R})$ constitué des classes de de Rham représentables par des $2j$-formes (réelles) de type (j, j), $1 \leq j \leq r$. En général, le sous-espace $H^{j,j}_{dR}(M, \mathbf{R})$ ne coïncide pas avec $H^{2j}_{dR}(M, \mathbf{R})$ et l'observation ci-dessus donne une condition nécessaire pour qu'un fibré vectoriel complexe E au-dessus d'une variété complexe M admette une structure holomorphe.

Ce point peut être précisé dans le cas où $E = L$ est un fibré en droites (complexes) sur une variété complexe M. Dans ce cas, à la suite exacte exponentielle différentiable (2.1.3), s'ajoute la suite exacte exponentielle *holomorphe*

$$0 \longrightarrow M \times \mathbf{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp(2\pi i)} \mathcal{O}^\star \longrightarrow 0, \tag{2.4.6}$$

où $\mathcal{O}$ note le faisceau des fermes de fonctions holomorphes sur M (c'est le *faisceau structural* de la variété complexe M), $\mathcal{O}^\star$ le faisceau (multiplicatif) des germes de fonctions sur M à valeurs dans $\mathbf{C}^\star$, $\exp(2\pi i)$ le morphisme de faisceaux obtenu par restriction à partir de (2.1.3), $M \times \mathbf{Z}$ le faisceau constant relatif au groupe $\mathbf{Z}$.

Comme dans la catégorie différentiable, le groupe $H^1(M, \mathcal{O}^\star)$ s'interprète naturellement comme le groupe des classes d'isomorphisme des fibrés en droites *holomorphes* sur M. La suite exacte "longue" associée à (2.4.6) s'écrit

$$\begin{aligned}(2.4.7)\qquad & 0 \to Z \to \Gamma(\mathcal{O}) \to \Gamma(\mathcal{O}^\star) \to H^1(M, \mathbf{Z}) \\ & \to H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}^\star) \xrightarrow{c_1} H^2(M, \mathbf{Z}) \to H^2(M, \mathcal{O}) \to \cdots\end{aligned}$$

où l'application cobord $H^1(M, \mathcal{O}^\star) \xrightarrow{c_1} H^2(M, \mathbf{Z})$ associe à toute classe d'isomorphisme de fibrés en droites holomorphes sur M la classe de Chern du fibré en droites (complexes) "sous-jacent". En général, cet homomorphisme n'est ni injectif ni surjectif, car le faisceau $\mathcal{O}$, contrairement à C, n'est pas acyclique (il n'existe pas de partitions de l'unité holomorphe !). En termes de *cohomologie de Dolbeault*, $H^q(M, \mathcal{O})$ s'identifie à l'espace de Dolbeault $H^{0,q}_{\mathrm{Dol}}(M, \mathbf{C})$ déterminé par le complexe différentiel de Dolbeault $(\oplus \Lambda^{0,q} M, \bar\partial)$.

L'application $H^2(M, \mathbf{Z}) \to H^2(M, \mathcal{O})$ figurant dans (2.4.7) se factorise, via l'homomorphisme naturel j de $H^2(M, \mathbf{Z})$ dans $H^2_{dR}(M, \mathbf{R})$, par l'homomorphisme de $H^2_{dR}(M, \mathbf{R})$ dans $H^2(M, \mathcal{O}) = H^{0,2}_{\mathrm{Dol}}(M, \mathbf{C})$ défini par $[\psi]_{dR} \to [\psi^{0,2}]_{\mathrm{Dol}}$, où ψ est une 2-forme (réelle) fermée sur M et $\psi^{0,2}$ sa partie de type $(0, 2)$, qui est $\bar\partial$-fermée puisque ψ est fermée. On vérifie aisément que le noyau de cet homomorphisme coïncide avec l'espace $H^{1,1}_{dR}(M, \mathbf{R})$ défini précédemment. Nous obtenons ainsi le résultat suivant, cf. [Che] :

PROPOSITION 2.4.8. — *Un fibré en droites (complexes) L défini sur une variété complexe M admet une structure holomorphe si et seulement si sa classe de Chern réelle $c_1^R(L)$ appartient au sous-espace $H^{1,1}_{dR}(M, \mathbf{R})$ de $H^2_{dR}(M, \mathbf{R})$.*

Lorsque M est une surface de Riemann, la condition ci-dessus est vide car toute 2-forme sur M est alors de type $(1, 1)$. Dans ce cas, *tout* fibré vectoriel complexe E (de rang quelconque) admet une structure holomorphe. De façon précise, une structure hermitienne h (quelconque) étant fixée sur E, l'application $\nabla \to \nabla^{0,1}$ constitue un *isomorphisme d'espaces affines de l'espace des connexions linéaires h-hermitiennes sur l'espace des structures holomorphes sur E*, cf. § 1.5.

Si M est une variété complexe *compacte*, de dimension $n = 2m$, la suite exacte (2.4.7) se simplifie en son début de la façon suivante :

$$\begin{aligned}0 \to H^1(M, \mathbf{Z}) \to H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}^\star) \\ \xrightarrow{c_1} H^2(M, \mathbf{Z}) \to H^2(M, \mathcal{O}) \to \cdots\end{aligned}$$

et le noyau de c_1 dans $H^1(M, \mathcal{O}^\star) = \mathrm{Pic}(M)$, le *groupe de Picard* de la variété complexe M, s'identifie alors avec le quotient $\mathrm{Pic}^0(M) = H^1(M, \mathcal{O})/H^1(M, \mathbf{Z})$, qui est la composante connexe de l'identité de $\mathrm{Pic}(M)$, identifiée à l'ensemble des classes d'isomorphismes de structures holomorphes sur le fibré produit $M \times \mathbf{C}$.

Lorsque, en outre, M est *kählérienne* — mais non en général — le groupe $\mathrm{Pic}^0(M)$ est compact, isomorphes à un tore complexe de dimension (complexe) égale à la moitié du premier nombre de Betti $b_1(M)$ de M.

Soit $\widetilde{E}$ un fibré vectoriel *réel* de rang (réel) r au-dessus de M. Considérons le fibré vectoriel complexe $E = \widetilde{E} \otimes \mathbf{C}$, de rang (complexe) r, obtenu en *complexifiant* $\widetilde{E}$.

Les fibrés vectoriels complexes obtenu de cette manière, i.e. qui admettent une *structure réelle* (un opérateur de *conjugaison* $\mathbf{C}$-antilinéaire de carré 1, dont le fibré des éléments réels coïncide avec $\widetilde{E}$), sont de nature particulière. En particulier, E *est* (non canoniquement) *isomorphe au dual* $E^\star$, via n'importe quelle structure euclidienne (fibrée) sur $\widetilde{E}$, étendue $\mathbf{C}$-linéairement à E. De l'exemple 2.3.12, nous déduisons alors que $2c_{2j+1}(E) = 0$, i.e. que les classes de Chern (entières) d'ordre impair de E sont les éléments de torsion d'ordre 2 de $H^\star(M, \mathbf{Z})$. En particulier, les classes de Chern réelles d'ordre impair de E sont nulles.

La *classe de Pontryagin* $p_j(\widetilde{E})$, d'ordre j, du fibré vectoriel réel $\widetilde{E}$ est l'élément de $H^{4j}(M, \mathbf{Z})$ défini par

$$p_j(\widetilde{E}) = (-1)^j c_{2j}(E), \quad 1 \le j \le [\frac{r}{2}]. \tag{2.4.9}$$

La *classe de Pontryagin totale* $p(\widetilde{E})$ de $\widetilde{E}$ est l'élément de $H^\star(M, \mathbf{Z})$ défini par

$$p(\widetilde{E}) = 1 + p_1(\widetilde{E}) + \cdots + p_{[\frac{r}{2}]}(\widetilde{E}).$$

Malgré le signe $(-1)^j$ figurant en (2.4.9), que nous justifions plus loin, la classe de Pontryagin totale p est multiplicative en ce sens que :

$$p(\widetilde{E}^1 \oplus \widetilde{E}^2) = p(\widetilde{E}^1) \cdot p(\widetilde{E}^2),$$

modulo les éléments de 2-torsion, pour tous fibrés vectoriels réels $\widetilde{E}^1$ et $\widetilde{E}^2$ sur M.

Il existe une théorie de Chern-Weil pour la classe de Pontryagin totale p, formellement similaire à celle que nous avons développée pour la classe de Chern c, en substituant au groupe linéaire complexe $GL(r, \mathbf{C})$ le groupe linéaire réel $GL(r, \mathbf{R})$ et à l'algèbre de Lie $g\ell(r, \mathbf{C})$ l'algèbre de Lie $g\ell(r, \mathbf{R})$. Nous montrons ainsi, de façon formellement identique au cas précédent, que la forme inhomogène φ^∇ définie par

$$\varphi^\nabla = \mathrm{dét}\left(I + \frac{R^\nabla}{2\pi}\right), \tag{2.4.10}$$

où R^∇ est la courbure d'une connexion $\mathbf{R}$-linéaire sur $\widetilde{E}$ est fermée et que sa classe $[\varphi^\nabla]$ dans $H^\star(M, \mathbf{R})$ est indépendante de ∇. En particulier, nous pouvons choisir pour ∇ une connexion préservant une structure euclidienne sur $\widetilde{E}$, de telle sorte que R^∇ prenne ses valeurs dans le fibré des endomorphismes antisymétriques de $\widetilde{E}$. Pour une telle connexion, les éléments homogènes d'ordre impair de φ^∇ sont nuls. On a donc

$$[\varphi^\nabla_{2j+1}] = 0.$$

Après complexification de $\tilde{E}$ en E, la connexion $\mathbf{R}$-linéaire ∇ induit une connexion ($\mathbf{C}$-)linéaire sur E dont la courbure, encore notée R^∇, est l'extension $\mathbf{C}$-linéaire de R^∇. Comme le déterminant de l'extension $\mathbf{C}$-linéaire d'un endomorphisme $\mathbf{R}$-linéaire est égal au déterminant de l'endomorphisme initial, nous constatons que les membres de droite de (2.4.1) et (2.4.10) seraient identiques n'était l'absence du facteur i dans (2.4.10). Il résulte alors de 2.4.2 que

Proposition 2.4.11. — *L'image dans $H^*_{dR}(M,\mathbf{R})$ de la classe de Pontryagin totale $p(\tilde{E})$ est égale à la classe de de Rham $[\varphi^\nabla]$, pour toute connexion $\mathbf{R}$-linéaire ∇ sur $\tilde{E}$.* □

Cet énoncé justifie l'introduction du facteur $(-1)^j$ dans (2.4.9).

Si le fibré vectoriel réel $\tilde{E}$ est lui-même muni d'une structure de fibré vectoriel complexe, de rang $\frac{r}{2}$ on a la décomposition suivante en somme de Whitney de fibrés vectoriels complexes :

$$E = \tilde{E} \oplus \overline{\tilde{E}},$$

où $\overline{\tilde{E}}$ est le *conjugué* de $\tilde{E}$, identique à $\tilde{E}$ en tant que fibré vectoriel réel et muni de l'action conjuguée de $\mathbf{C}$ sur chaque fibre. On a donc la relation

$$\begin{aligned} c(E) &= c(\tilde{E}) \cdot c(\overline{\tilde{E}}) \\ &= (1 + c_1(\tilde{E}) + c_2(\tilde{E}) + \cdots + c_{r/2}(\tilde{E})) \\ &\qquad \cdot (1 - c_1(\tilde{E}) + c_2(\tilde{E}) - \cdots \pm c_{r/2}(\tilde{E})), \end{aligned}$$

à partir de laquelle on détermine aisément les classes de Pontryagin de $\tilde{E}$ à partir des classes de Chern de $\tilde{E}$, vu comme un fibré vectoriel complexe. En particulier, on a

$$p_1(\tilde{E}) = c_1^2(\tilde{E}) - 2c_2(\tilde{E}). \tag{2.4.12}$$

La *signature* $\tau(M)$ d'une variété compacte orientée, de dimension $n = 4k$ multiple de 4, est la signature, au sens des formes quadratiques réelles, de la *forme d'intersection* définie sur l'espace vectoriel $H^{2k}(M, \mathbf{R})$, via le cup-produit

$$H^{2k}(M,\mathbf{R}) \times H^{2k}(M,\mathbf{R}) \to H^{4k}(M,\mathbf{Z})$$

et l'identification $H^{4k}(M,\mathbf{R}) = \mathbf{R}$ déterminée par l'orientation de M. La signature $\tau(M)$ peut s'exprimer au moyen d'une expression universelle en les classes de Pontryagin $p_j(M) = p_j(TM)$ de M, cf. [Hir] Ch. II, [Mi-St] §19. En particulier, si M est une variété orientée de dimension 4, on a

$$\tau(M) = \frac{1}{3}\langle p_1(M), [M]\rangle. \tag{2.4.13}$$

Supposons, en outre, que M soit munie d'une structure presque-complexe J induisant la même orientation. Via l'identification $H^4(M, \mathbf{Z}) = \mathbf{Z}$ déterminée par cette orientation, nous identifions les *classes* $c_1^2(M)$ et $c_2(M)$ avec les *entiers* (*nombres de Chern*) $\langle c_1^2(M), [M]\rangle$ et $\langle c_2(M), [M]\rangle$. Par (2.3.11), on a

$$c_2(M) = \chi(M), \tag{2.4.14}$$

et, par (2.4.12) et (2.4.13),

$$c_1^2(M) = 2\chi(M) + 3\tau(M). \tag{2.4.15}$$

Ainsi, les deux nombres de Chern $c_1^2(M)$ et $c_2(M)$ ne dépendent, via (2.4.14) et (2.4.15), que de la structure topologique et l'orientation induite par J.

2.5. Caractère de Chern et classe de Todd

La classe de Chern totale c a été définie en § 2.3 comme un morphisme de $\mathrm{Vect}_\star$ dans $H^\star(\cdot, \mathbf{Z})$, vus l'un et l'autre comme des cofoncteurs définis sur la catégorie des variétés différentiables (réelles). Plus généralement, une *classe caractéristique* (complexe) F est un morphisme de $\mathrm{Vect}_\star$ dans un cofoncteur quelconque défini sur cette catégorie (ou toute autre catégorie d'espaces topologiques). Nous nous limitons ici au cas où le cofoncteur considéré est $H^\star(\cdot, \mathbf{Z})$ ou $H^\star(\cdot, \mathbf{Q})$, cohomologie singulière (ou de Čech) à coefficients rationnels, et au cas où la catégorie concernée est celle des variétés différentiables réelles (une variété complexe est considérée, comme nous l'avons fait jusqu'à présent implicitement, comme une variété différentiable réelle munie d'une structure supplémentaire).

Une classe caractéristique F est *multiplicative*, resp. *additive*, si elle satisfait la relation

$$F(E^1 \oplus E^2) = F(E^1)F(E^2),$$

resp.

$$F(E^1 \oplus E^2) = F(E^1) + F(E^2),$$

où E^1 et E^2 sont deux éléments quelconques de $\mathrm{Vect}\, M$. Par exemple, la classe de Chern totale c est multiplicative, ainsi que la *dernière classe de Chern* c_r (où r est le rang du fibré vectoriel concerné), mais la *première classe de Chern* c_1 est une classe caractéristique additive.

Parmi les classes caractéristiques (complexes) multiplicatives, la classe de Chern totale c est entièrement déterminée par l'*axiome de normalisation* (2.3.19), et la dernière classe de Chern c_r par l'axiome de normalisation

$$c_r(H^\star) = \text{ le générateur positif de } H^2(\mathbf{P}^m, \mathbf{Z}), \tag{2.5.1}$$

tandis que, parmi les classes caractéristiques (complexes) additives, la première classe de Chern c_1 est déterminée par le même axiome de normalisation

$$c_1(H^\star) = \text{le générateur positif de } H^2(\mathbf{P}^m, \mathbf{Z})$$

(les autres classes de Chern c_j ne sont ni additives, ni multiplicatives).

Grâce aux propositions 2.1.5 et 2.3.8, nous pouvons fabriquer de nombreuses autres classes caractéristiques (complexes), multiplicatives ou additives, normalisées de façon arbitraire sur le fibré de Hopf dual $H^\star$ de $\mathbf{P}^m$. En particulier, soit φ une série entière à coefficients rationnels :

$$\varphi(t) = \varphi_0 + \varphi_1 t + \cdots + \varphi_j t^j + \cdots .$$

A partir de φ, nous fabriquons une classe caractéristique multiplicative $\varphi^\times$ et une classe caractéristique additive φ^+, normalisées l'une et l'autre par

$$\varphi^\times(H^\star) = \varphi^+(H^\star) = \varphi(\zeta),$$

où ζ est le générateur positif de $H^2(\mathbf{P}^m, \mathbf{Z})$, étant entendu que $\zeta^p = 0$ pour $p > m$.

Les classes caractéristiques $\varphi^\times$ et φ^+ sont alors explicitées par

$$\varphi^\times(E) = \varphi(\beta_1) \cdots \varphi(\beta_r), \tag{2.5.2}$$

et pour tout $[E]$ dans $\mathrm{Vect}_r M$,

$$\varphi^+(E) = \varphi(\beta_1) + \cdots + \varphi(\beta_r), \tag{2.5.3}$$

où les β_j sont les classes de Chern virtuelles de E, soit donc encore

$$\varphi^\times(E) = P(c_1(E), \ldots, c_r(E)),$$

et

$$\varphi^+(E) = \widetilde{P}(c_1(E), \ldots, c_r(E)),$$

où P et $\widetilde{P}$ sont des polynômes à coefficients rationnels des classes de Chern $c_j(E)$ de E, déterminés par les expressions symétriques (2.5.2) et (2.5.3) (étant entendu que les formes figurant dans les membres de droites de (2.5.2) et (2.5.3) sont nulles dès que leur degré dépasse la dimension de la variété M considérée).

La série entière φ est appelée la *série génératrice* de la classe caractéristique multiplicative $\varphi^\times$ et de la classe caractéristique additive φ^+. Ainsi, la série génératrice de c est $1 + t$, celle de c_1 et de c_r est t.

Le *caractère de Chern ch* est la classe caractéristique (complexe) additive, à valeurs dans $H^\star(\cdot, \mathbf{Q})$, dont la série génératrice est

$$e^t = 1 + t + \frac{t^2}{2} + \cdots + \frac{t^p}{p!} + \cdots$$

On a donc

$$ch(E) = e^{\beta_1} + \cdots + e^{\beta_r}, \tag{2.5.4}$$

soit donc

$$ch_0(E) = r, \quad ch_1(E) = c_1(E), \quad ch_2(E) = \frac{1}{2}c_1^2(E) - c_2(E), \text{ etc...} \tag{2.5.5}$$

Le caractère de Chern ch est additif, par définition, relativement à la somme de Whitney, et multiplicatif relativement au produit tensoriel :

$$ch(E^1 \otimes E^2) = ch(E^1) \cdot ch(E^2), \qquad \forall [E^1], [E^2] \in \text{ Vect } M,$$

comme on le vérifie aisément à partir de (2.5.4) et de l'exemple 2.3.15.

Une autre classe caractéristique (complexe) fondamentale est la *classe de Todd* qui apparaît dans la formulation générale du théorème de Riemann-Roch-Hirzebruch 3.1.2. La classe de Todd, notée *td*, est la classe caractéristique (complexe) multiplicative, à valeurs dans $H^\star(\cdot, \mathbf{Q})$, dont la série génératrice est

$$td(t) = \frac{t}{1 - e^{-t}}.$$

On peut en déduire l'expression de td en fonction des classes de Chern dont les premiers éléments sont donnés, cf. [Hir] p. 14, par

$$\begin{aligned} td_1(E) &= \tfrac{1}{2}c_1(E) \\ td_2(E) &= \tfrac{1}{12}(c_1^2(E) + c_2(E)) \\ td_3(E) &= \tfrac{1}{24}c_1(E)c_2(E)\ldots \end{aligned} \tag{2.5.6}$$

A toute classe caractéristique (complexe) F, à valeurs dans $H(\cdot, \mathbf{Z})$, est associé le *genre* $\underline{F}$, défini, pour toute variété presque-complexe M, par

$$\underline{F}(M) = \langle F(TM), [M] \rangle$$

où $[M]$ note la classe fondamentale, dans $H_n(M, Z)$ de M muni de l'orientation induite par la structure presque-complexe J.

Si $M = M^1 \times M^2$ est un produit de deux variétés presque-complexes, on a

$$TM = p_1^\star TM^1 \oplus p_2^\star TM^2,$$

où p_1, p_2 notent les projections de M sur M^1, M^2 respectivement. On a donc

$$\underline{F}(M) = \underline{F}(M^1) \cdot \underline{F}(M^2), \tag{2.5.7}$$

ou

$$\underline{F}(M) = \underline{F}(M^1) + \underline{F}(M^2),$$

suivant que F est multiplicative ou additive.

Comme, pour $m = n/2$, $c_m(TM)$ est égale à la classe d'Euler de TM, cf. (2.3.11), on a $\underline{c}(M) = \chi(M)$, où $\chi(M)$ est la caractéristique d'Euler-Poincaré de M (dont il est bien connu qu'elle satisfait (2.5.7) pour toute variété compacte, presque-complexe ou non).

Le genre associé à la classe de Todd td est appelé le *genre de Todd*. Pour toute variété presque-complexe M, le genre de Todd de M sera noté $Td(M)$. Puisque td est une classe multiplicative, on a

$$Td(M_1 \times M_2) = Td(M_1)Td(M_2).$$

2.6. Le genre de Todd des espaces projectifs complexes

Pour calculer les classes de Chern de l'espace projectif complexe $\mathbf{P}^m$, nous considérons la suite exacte de fibrés vectoriels complexes au-dessus de $\mathbf{P}^m$:

$$0 \to H \to \mathbf{1}^{m+1} \to Q \to 0,$$

où H est le fibré en droites *tautologique* (le fibré de Hopf), où $\mathbf{1}^{m+1}$ note le fibré produit $\mathbf{P}^m \times \mathbf{C}^{m+1}$ et Q le fibré quotient $(\mathbf{P}^m \times \mathbf{C}^{m+1})/H$. Par tensorisation avec le fibré de Hopf dual $H^\star$, nous obtenons la suite exacte :

$$0 \to \mathbf{1} \to H^\star \otimes \mathbf{1}^{m+1} \to H^\star \otimes Q \to 0. \tag{2.6.1}$$

Pour chaque x dans $\mathbf{P}^m$ l'espace tangent $T_x\mathbf{P}^m$ s'identifie naturellement à l'espace $\mathrm{Hom}(x, \mathbf{C}^{m+1}/x)$ des homomorphismes de la droite vectorielle (complexe) x dans le quotient $\mathbf{C}^{m+1}/x$. Nous avons donc l'identification naturelle de fibrés vectoriels (complexes)

$$T\mathbf{P}^m = \mathrm{Hom}(H, Q) = H^\star \otimes Q,$$

et la suite exacte (2.6.1) s'écrit encore

$$0 \to \mathbf{1} \to H^\star \otimes \mathbf{1}^{m+1} \to T\mathbf{P}^m \to 0. \tag{2.6.2}$$

Dans la catégorie des fibrés vectoriels complexes (différentiables) toute suite exacte se scinde. Il résulte donc de la suite exacte (2.6.2) qu'on a un isomorphisme (non-canonique)

$$\mathbf{1} \oplus T\mathbf{P}^m = H^\star \oplus \cdots \oplus H^\star \quad (m+1) \text{ fois.} \tag{2.6.3}$$

De la multiplicativité (2.3.3) de la classe de Chern totale c nous déduisons alors que $c(M) = c(T\mathbf{P}^m)$ vaut

$$(1+\zeta)^{m+1} = 1 + (m+1)\zeta + \cdots + \binom{m+1}{j}\zeta^j + \cdots + (m+1)\zeta^m, \tag{2.6.4}$$

où $\zeta = c_1(H^\star)$ est le générateur positif de $H^2(\mathbf{P}^m, \mathbf{Z})$, étant entendu que dans le membre de droite, le terme ζ^{m+1} est nul. En particulier, nous retrouvons le fait que $\chi(\mathbf{P}^m)$ est égale à $(m+1)$. Il résulte de (2.6.3) que, pour toute classe caractéristique multiplicative Φ déterminée par une série entière φ dont le terme constant φ_0 est égal à 1, le genre (multiplicatif) associé, noté $M \mapsto \Phi(M)$, satisfait la propriété :

(2.6.5) $\quad \Phi(\mathbf{P}^m) =$ le coefficient de t^m dans la série entière$(\varphi(t))^{m+1}$.

Comme la série φ est clairement déterminée par les nombres définis par le second membre de (2.6.5), lorsque m décrit l'ensemble des entiers positifs, il résulte de (2.6.5) que

PROPOSITION 2.6.6. — *Le genre (multiplicatif) associé à φ est entièrement déterminé par les valeurs $\Phi(\mathbf{P}^m), m \geq 0$.*

Si la série φ est telle que $t \to \varphi(t)/t$ est une fonction méromorphe sur un voisinage de 0 dans $\mathbf{C}$ nous déduisons immédiatement de (2.6.5) l'expression suivante de $\Phi(\mathbf{P}^m)$:

(2.6.7)
$$\Phi(\mathbf{P}^m) = \frac{1}{2\pi i}\int_\Gamma (\varphi(t)/t)^{m+1} dt,$$

où Γ est toute courbe de degré 1 dans $\mathbf{C} - \{0\}$, contenue dans le voisinage où $\varphi(t)/t$ est définie.

Dans le cas où $\varphi = td$ est la série de Todd, $td(t)/t$ est une fonction méromorphe définie sur $\mathbf{C}$ et on déduit de (2.6.7)

$$Td(\mathbf{P}^m) = \frac{1}{2\pi i}\int_\Gamma \frac{dt}{(1-e^{-t})^{m+1}}.$$

Par changement de variable $t \mapsto u = 1 - e^{-t}$, qui est biholomorphe au voisinage de zéro, on a

$$\begin{aligned} Td(\mathbf{P}^m) &= \frac{1}{2\pi i}\int_\Gamma \frac{du}{(1-u)u^{m+1}} \\ &= \frac{1}{2\pi i}\int_\Gamma (\frac{1}{1-u} + \frac{1}{u} + \cdots + \frac{1}{u^{m+1}}) du \\ &= 1. \end{aligned}$$

Ainsi, le genre de Todd est *le* genre multiplicatif (complexe) tel que

$$Td(\mathbf{P}^m) = 1 \quad \forall m \geq 0.$$

2.7. La formule d'adjonction

Nous considérons la situation de deux variétés presque-complexes compactes (connexes) M et M', de dimensions respectives $n = 2m$ et $n' = n+2$, et d'un plongement (pseudo-)holomorphe i de M dans M', de sorte que M sera considérée comme une sous-variété presque-complexe, de codimension réelle 2, de M'. Nous notons N, pour $N^{M'}(M)$, le *fibré normal* de M dans M', i.e. le fibré quotient de $TM'_{|M} = i^\star TM'$ par le sous-fibré TM. Ainsi défini, le fibré normal N possède une structure naturelle de fibré en droites (complexes) sur M. La variété M est naturellement plongée dans N, via la section nulle 0, et la paire $(0(M), N)$ est difféomorphe à la paire $(M, \mathcal{V}(M))$, où $\mathcal{V}(M)$ est un voisinage tubulaire ouvert de M dans M'. Nous notons $[M]$ la classe fondamentale de M dans $H_n(M, \mathbf{Z})$, déterminée par l'orientation induite par la structure presque-holomorphe de $M, i_\star[M]$ son image dans $H_n(M', \mathbf{Z}), \alpha^M$ le dual de Poincaré de $i_\star[M]$ dans $H^2(M', \mathbf{Z})$.

Nous déduisons de la remarque de la fin du § 2.2 que la classe de Chern $c_1(N)$ de N, qui est aussi la classe d'Euler $e(N)$, est égale à la restriction de α^M à M :

$$c_1(N) = \alpha^M_{|M} = i^\star \alpha^M. \tag{2.7.1}$$

En termes géométriques, α^M est la classe de Chern (entière) d'un fibré en droites L^M sur M' qui est ainsi déterminé, à isomorphisme près, par le plongement i de M dans M', et on a l'isomorphisme (non-canonique)

$$N = L^M_{|M} = i^\star L^M.$$

Par définition de N, on a l'isomorphisme (non-canonique) de fibrés vectoriels complexes sur M :

$$TM'_{|M} = TM \oplus N,$$

d'où nous déduisons, par (2.3.3), la relation suivante

$$c(M')_{|M} = c(M)c(N),$$

soit donc, par (2.7.1), la *formule d'adjonction*

$$c(M')_{|M} = c(M)(1 + \alpha^M_{|M}), \tag{2.7.2}$$

où, rappelons-le, $c(M') = c(TM')$ et $c(M) = c(TM)$ notent les classes de Chern totales de M et M' respectivement, et α^M est le dual de Poincaré de M dans M'.

La relation (2.7.2) détermine entièrement les classes de Chern de M en fonction de celles de M' et de α^M, restreintes à M. En particulier, la première classe de Chern (entière) de M est donnée par

$$c_1(M) = (c_1(M') - \alpha^M)_{|M}. \tag{2.7.3}$$

2.7.4. Exemple : hypersurfaces complexes dans $\mathbf{P}^{m+1}$. — Dans cet exemple, $M' = \mathbf{P}^{m+1}$ est le projectif complexe de dimension (complexe) $m+1$, et M est une hypersurface complexe (lisse) de degré d, i.e. le lieu des zéros d'une section holomorphe générique du fibré en droites $(H^\star)^d$, puissance tensorielle d-ième du fibré de Hopf dual $H^\star$. Dans ce cas, α^M est égale à $d\cdot\zeta$, où est le générateur positif de $H^2(\mathbf{P}^{m+1}, Z)$, et, compte-tenu de (2.6.4), les relations (2.7.2) et (2.7.3) s'écrivent respectivement

$$(1+\zeta_{|M})^{m+2} = c(M)(1+\alpha^M_{|M})$$

et

$$c_1(M) = (m+2-d)\cdot\zeta_{|M}. \tag{2.7.5}$$

Rappelons que, pour toute variété presque-complexe M de dimension réelle $2m$, le *fibré canonique* K^M de M est égal, par définition, à $\Lambda^m(T^\star M)$, où le fibré cotangent est considéré comme un fibré vectoriel complexe de rang m (et la puissance extérieure est relative à $\mathbf{C}$). Il résulte alors de (2.7.5) et (2.3.18) que le fibré canonique K^M de toute hypersurface complexe lisse de degré $(m+2)$ de $\mathbf{P}^{m+1}$ est (topologiquement) trivial. C'est le cas en particulier, des cubiques (lisses) de $\mathbf{P}^2$, qui sont des tores complexes, et des quartiques lisses de $\mathbf{P}^3$, qui appartiennent à la famille des surfaces complexes dites *surfaces* $K3$.

2.7.6. Exemple : la formule du genre. — Dans cet exemple, $M = \Sigma$ est une surface de Riemann (compacte), plongée (pseudo-) holomorphiquement dans une variété presque-complexe M' de dimension 4. Comme $c_1(\Sigma) = c_1(T\Sigma)$ est égale à la classe d'Euler de Σ on a

$$\langle c_1(\Sigma), [\Sigma]\rangle = \chi(\Sigma) = 2-2g, \tag{2.7.7}$$

où $\langle\cdot,\cdot\rangle$ note la dualité naturelle entre $H^2(\Sigma;\mathbf{Z})$ et $H_2(\Sigma;\mathbf{Z})$, et où g est le genre de la surface de Riemann, cf. § 3.2.

Comme α^M est le dual de Poincaré de Σ dans M' on a

$$\langle \alpha^\Sigma_{|\Sigma}, [\Sigma]\rangle = \langle \alpha^\Sigma\cdot\alpha^\Sigma, [M']\rangle, \tag{2.7.8}$$

où $\cdot$ note le cup-produit, i.e. le produit dans l'anneau $H^\star(M';\mathbf{Z})$, et

$$\begin{aligned} \langle c_1(M')_{|\Sigma}, [\Sigma]\rangle &= \langle \alpha^\Sigma\cdot c_1(M'), [M']\rangle \\ &= -\langle \alpha^\Sigma\cdot c_1(K^{M'}), [M']\rangle. \end{aligned} \tag{2.7.9}$$

Nous obtenons ainsi, à partir de (2.7.3), la formule suivante, dite *formule du genre*

$$2g-2 = \langle \alpha^\Sigma\cdot c_1(K^{M'}) + \alpha^\Sigma\cdot\alpha^\Sigma, [M']\rangle, \tag{2.7.10}$$

souvent écrite sous la forme suivante, en posant $K' = K^{M'}$,

$$2g-2 = K'\cdot\Sigma + \Sigma\cdot\Sigma, \tag{2.7.11}$$

où $K'\cdot\Sigma$ est l'*intersection* de K' et Σ, et $\Sigma\cdot\Sigma$ l'*auto-intersection* de Σ, *définis* respectivement par (2.7.9) et (2.7.8).

Remarque. — Lorsque Σ est une *courbe* (au sens complexe) lisse, de degré d, dans le plan projectif $\mathbf{P}^2$, la formule du genre (2.7.10)-(2.7.11) équivaut à la formule suivante donnant le genre de Σ en fonction de degré (dans le cas lisse) :

$$g = \frac{(d-1)(d-2)}{2}.$$

On a, en effet, dans ce cas, $K' \cdot \Sigma = \langle -3\zeta \cdot d\zeta, [\mathbf{P}^2] \rangle = -3d$ et $\Sigma \cdot \Sigma = \langle d\zeta \cdot d\zeta, [\mathbf{P}^2] \rangle = d^2$.

3. Le théorème de Riemann-Roch

3.1. La formule de Riemann-Roch-Hirzebruch

Nous considérons la situation du début de ce chapitre dans le cas particulier où M est une variété complexe de dimension réelle $n = 2m$ et où E est un fibré vectoriel holomorphe de rang r. Nous notons $\mathcal{E}$ le *faisceau* des germes de sections holomorphes de E, $H^q(M, \mathcal{E})$ les groupes de cohomologie (de Čech) de E, cf. § 2.1. Comme M est compacte, les groupes $H^q(M, \mathcal{E})$ sont des espaces vectoriels complexes de dimension (complexe) finie, notées $h^q(M, \mathcal{E})$. La *caractéristique d'Euler-Poincaré* $\chi(M, \mathcal{E})$, relative au faisceau $\mathcal{E}$, est la somme alternée des $h^q(M, \mathcal{E})$:

$$\chi(M, \mathcal{E}) = \sum_{q=0}^{m} h^q(M, \mathcal{E}) \tag{3.1.1}$$

La *formule de Riemann-Roch-Hirzebruch*, établie sous la forme générale suivante par F.Hirzebruch, cf. [Hir] Ch.IV, s'écrit :

PROPOSITION 3.1.2.

$$\chi(M, \mathcal{E}) = \langle ch(E) \cdot td(M), [M] \rangle.$$

Dans cet énoncé, $ch(E)$ est le caractère de Chern de E et $td(M)$ la classe de Todd de M, cf. § 2.5.

Il résulte, en particulier, de 3.1.2 que $\chi(M, \mathcal{E})$ ne dépend pas de la structure holomorphe de E et ne dépend de la structure complexe de M que via les classes de Chern $c_j(M)$ qui sont invariantes par déformation, et l'orientation induite. La formule 3.1.2 a été démontrée par F. Hirzebruch dans le cas où M est une variété projective, i.e. plongeable holomorphiquement dans un espace projectif complexe $\mathbf{P}^N$, de dimension quelconque. Dans le cas général, la formule 3.1.2 se déduit du *théorème de l'indice* d'Atiyah-Singer, en interprétant $\chi(M, \mathcal{E})$ comme l'indice d'un opérateur différentiel elliptique d'ordre 1. Cette interprétation repose sur l'identification déjà considérée en § 2.4 :

$$H^q(M, \mathcal{E}) = H^{0,q}_{\mathrm{Dol}}(M, E) \quad 0 \le q \le m, \tag{3.1.3}$$

où le *groupe de Dolbeault* $H^{0,q}(M,E)$ est le q-ième groupe de cohomologie du complexe différentiel de Dolbeault $(\Omega^{0,\star}(M,E),\bar\partial)$, où $\Omega^{0,q}(M,E)=\Gamma(\Lambda^{0,q}M\otimes E)$ est l'espace des formes différentielles de type $(0,q)$ et $\bar\partial$ l'extension à $\Omega^{0,\star}(M,E)$ de l'opérateur de Cauchy-Riemann déterminant la structure holomorphe de E.

Remarque. — De façon générale, pour tous $0\le p,q\le m$, on a un isomorphisme canonique

$$H^q(M,\Lambda^pT^\star\otimes\mathcal{E})=H^{p,q}_{\mathrm{Dol}}(M,E) \tag{3.1.4}$$

où $\Lambda^pT^\star$ note le faisceau des germes de sections holomorphes du fibré vectoriel (holomorphe) $\Lambda^{p,0}M$ et où le groupe de Dolbeault $H^{p,q}_{\mathrm{Dol}}(M,E)$ est le q-ième espace de cohomologie du complexe différentiel $(\Omega^{p,\star}(M,E),\bar\partial)$, où $\Omega^{p,q}(M,E)=\Gamma(\Lambda^{p,q}M\otimes E)$ est l'espace des formes de type (p,q) sur M et $\bar\partial=\bar\partial^E$ est l'extension naturelle de l'opérateur de Cauchy-Riemann déterminant la structure holomorphe de E à $\Omega^{p,q}(M,E)$. L'isomorphisme (3.1.4) se déduit aisément de l'*exactitude du complexe de Dolbeault*

$$0\to\Lambda^pT^\star\otimes\mathcal{E}\xrightarrow{i}C^{p,0}(E)\xrightarrow{\bar\partial}\cdots\xrightarrow{\bar\partial}C^{p,q}(E)\xrightarrow{\bar\partial}\cdots\xrightarrow{\bar\partial}C^{p,m}(E)\to 0,$$

où $C^{p,q}(E)$ est le faisceau des germes de formes de type (p,q) à valeurs dans E, et de l'acyclicité de ces faisceaux, cf. [Hir] Th. 15.4.1.

Fixons une structure hermitienne g sur M et une structure hermitienne h sur E. Nous pouvons alors *compléter* l'opérateur $\bar\partial$ en un opérateur différentiel *elliptique* D d'ordre 1, en posant $D=\bar\partial+\bar\partial^\star$, où $\bar\partial^\star$ est *l'adjoint* de $\bar\partial$, relatif à g et h, défini par la relation d'adjonction

$$\int_M(\bar\partial^\star\psi_1,\psi_2)\cdot v_g=\int_M(\psi_1,\bar\partial\psi_2)\cdot v_g, \tag{3.1.5}$$

pour toutes les formes $\psi_1\in\Omega^{0,q}(M,E)$, $\psi_2\in\Omega^{0,q+1}(M,E)$, et où $(\cdot,\cdot)$ note le produit scalaire induit par g et h, et v_g la forme-volume induite par g.

Soit $\star$ l'*opérateur de Hodge hermitien*, défini sur $\Lambda^\star M\otimes E$, obtenu en composant l'opérateur de Hodge riemannien induit par g sur $\Lambda^\star M$ et le $\mathbf{C}$-anti-isomorphisme σ induit par h de E sur le dual $E^\star$ de E, de sorte que $\star$ est un $\mathbf{C}$-anti-isomorphisme de $\Lambda^\star M\otimes E$ sur $\Lambda^\star M\otimes E^\star$, et, pour tous p,q, un $\mathbf{C}$-anti-isomorphisme de $\Lambda^{p,q}M\otimes E$ sur $\Lambda^{m-p,m-q}M\otimes E^\star$. On note de même $\star$ l'opérateur de Hodge hermitien défini $\Lambda^\star M\otimes E^\star$ sur $\Lambda^\star M\otimes E$. On a alors $\bar\partial^\star=-\star\circ\bar\partial\circ\star$ et donc $D=\bar\partial-\star\circ\bar\partial\circ\star$. L'opérateur D est auto-adjoint par construction. En particulier, son indice est nul. Nous lui substituons l'opérateur $\bar D$ défini par

$$\bar D=D_{|\Omega^{0,\mathrm{pair}}(M,E)}, \tag{3.1.6}$$

qui est encore un opérateur différentiel elliptique d'ordre 1, dont l'adjoint $\bar{D}^\star$ est égal à

$$\bar{D}^\star = D_{|\Omega^{0,\text{impair}}(M,E)}.$$

L'indice ind $(\bar{D})$ de $\bar{D}$ est donc égal à

$$\text{ind }(\bar{D}) := \text{ dim Ker }(\bar{D}) - \text{ dim Ker }(\bar{D}^\star) = \sum_{q=0}^{m}(-1)^q \text{ dim } B^{0,q}(E),$$

où $B^{0,q}(E)$ note le noyau de D restreint à $\Omega^{0,q}(M,E)$.

De façon générale, l'opérateur de Dolbeault $D = \bar{\partial} + \bar{\partial}^\star$ est défini sur chaque espace $\Omega^{p,q}(M,E)$ et les éléments du noyau correspondant $B^{p,q}(E)$ sont les formes de $\Omega^{p,q}(E)$ qui sont $\bar{\partial}$-fermées ainsi que leurs images par $\star$.

En particulier, l'action de $\star$ induit un **C**-anti-isomorphisme

$$\text{(3.1.7)} \qquad B^{p,q}(E) \xrightarrow{\star} B^{m-p,m-q}(E^\star), \quad 0 \le p,q \le m.$$

Par le théorème de Hodge-Kodaira, cf. par ex. [Ko-Mo] Ch. 3, chaque classe de Dolbeault dans $H^{p,q}_{\text{Dol}}(M,E)$ est représentée par un unique élément de $B^{p,q}(E)$, i.e. on a un isomorphisme

$$\text{(3.1.8)} \qquad H^{p,q}_{\text{Dol}}(M,E) = B^{p,q}(E), \qquad 0 \le p,q \le m.$$

De (3.1.4) et (3.1.7), appliqués au cas $p = 0$, nous déduisons :

PROPOSITION 3.1.9.

$$\chi(M,\mathcal{E}) = \text{ ind } (\bar{D}).$$

Nous avons ainsi identifié, à l'aide de la théorie de Hodge élémentaire, la caractéristique d'Euler-Poincaré $\chi(M,E)$ à l'indice de l'opérateur elliptique $\bar{D}$. La formule 3.1.2 résulte alors directement de l'égalité

$$\text{ind } (\bar{D}) = \langle \text{ch } (E) \cdot td(M), [M] \rangle,$$

déduite de la formule générale donnée par le théorème de l'indice d'Atiyah-Singer, cf. par exemple [Sch] § 25.4.

3.1.10. Remarque. — L'opérateur de Hodge hermitien $\star$ est *défini* par l'identité

$$\psi_1 \wedge \star\psi_2 = (\psi_1, \psi_2) \cdot v_g, \qquad \forall \psi_1, \psi_2 \in \Omega^{p,q}(M,E),$$

où $\psi_1 \wedge \star\psi_2$ est le produit extérieur *contracté*, i.e. le produit extérieur formel, à valeurs dans le produit tensoriel $E \otimes E^\star$, suivi de la contraction naturelle de $E \otimes E^\star$

dans **C**. Il résulte alors de (3.1.7) et (3.1.8) que l'homomorphisme de $H^{p,q}(M,E) \otimes H^{m-p,m-q}(M,E^\star)$ dans **C** défini par

$$[\psi_1] \otimes [\psi_2] \mapsto \int_M \psi_1 \wedge \psi_2, \tag{3.1.11}$$

constitue une *dualité* entre $H^{p,q}(M,E)$ et $H^{m-p,m-q}(M,E^\star)$, i.e. induit un *isomorphisme* d'espaces vectoriels complexes de $H^{p,q}(M,E)$ sur le dual de $H^{m-p,m-q}(M,E)$, donc aussi, via (3.1.4), un isomorphisme

$$H^q(M, \Lambda^p \mathcal{T}^\star \otimes \mathcal{E}) \tilde{\to} (H^{m-q}(M, \Lambda^{m-p}\mathcal{T}^\star \otimes \mathcal{E}^\star))^\star, \quad 0 \leq p, q \leq m, \tag{3.1.12}$$

pour tout fibré vectoriel holomorphe E sur M.

Cet isomorphisme, que nous avons établi à partir du théorème de Hodge-Kodaira, mais qui est défini, via (3.1.11), indépendamment de toute métrique sur M ou E, est connu sous le nom de *dualité de Serre.*

Pour $p = 0$, l'isomorphisme (3.1.12) s'écrit

$$H^q(M, \mathcal{E}) \tilde{\to} (H^{m-q}(M, \mathcal{K} \otimes \mathcal{E}^\star))^\star, \quad 0 \leq q \leq m, \tag{3.1.13}$$

où $\mathcal{K} = \Lambda^m \mathcal{T}^\star$ est le faisceau des germes de sections holomorphes du fibré canonique $K = K^M$ de la variété complexe M.

3.1.14. Remarque. — Lorsque $E = \mathbf{1}$ est le fibré produit, muni de sa structure holomorphe naturelle, le faisceau $\mathcal{E}$ est le faisceau structural $\mathcal{O}$ de la variété complexe M et la formule 3.1.2 s'écrit

$$\chi(M, \mathcal{O}) = Td(M), \tag{3.1.15}$$

où $Td(M)$ est le genre de Todd de la variété complexe. En particulier, le genre de Todd d'une variété complexe (compacte) est un entier. Ce fait reste vrai lorsque $M = (M, J)$ est une variété presque-complexe. En effet, bien que le fibré tangent TM ne puisse être considéré comme un fibré "holomorphe", l'opérateur $\bar\partial$ est encore défini, par

$$\bar\partial\psi = \text{ composante de type } (0, q+1) \text{ de } d\psi, \quad \forall \psi \in \Omega^{0,q}M,$$

ainsi que son adjoint $\bar\partial^\star$, relatif à une métrique hermitienne quelconque g sur M, et l'opérateur $D = \bar\partial + \bar\partial^\star$, dont le symbole principal est le même que dans le cas intégrable. Si $\bar D$ est l'opérateur défini à partir de D par (3.1.6), on a donc, comme dans le cas intégrable, ind $(\bar D) = Td(M)$. En particulier

PROPOSITION 3.1.16. — *Le genre de Todd $Td(M)$ de toute variété presque-complexe compacte M est un entier.* □

Lorsque M est de dimension (réelle) 4, le genre de Todd $Td(M)$ est égal, par (2.5.6), (2.4.14) et (2.4.15), à

$$Td(M) = \frac{\chi(M) + \tau(M)}{4}. \tag{3.1.17}$$

On en déduit le fait suivant :

COROLLAIRE 3.1.18. — *Une variété compacte, orientée, de dimension* 4 *n'admet aucune structure presque-complexe si la somme* $\chi(M) + \tau(M)$ *de sa caractéristique d'Euler-Poincaré et de sa signature n'est pas divisible par* 4.

Une telle situation est illustrée par la sphère S^4, pour laquelle on a $\chi(S^4) = 2$ et $\tau(S^4) = 0$, et pour le plan projectif complexe, muni de l'orientation inverse, $\underline{\mathbf{P}}^2$, pour lequel on a $\chi(\underline{\mathbf{P}}^2) = 3$ et $\tau(\underline{\mathbf{P}}^2) = -1$.

On observe que la somme $\chi(M) + \tau(M)$ est toujours un nombre pair. De façon précise, on a

$$\chi(M) + \tau(M) = 2(1 - b_1(M) + b_+(M)), \tag{3.1.19}$$

où $b_1(M) = \dim H^1(M; \mathbf{R})$ est le premier *nombre de Betti* de M et $b_+(M)$ la dimension d'un sous-espace maximal de $H^2(M; \mathbf{R})$ sur lequel la forme d'intersection est définie positive. On a donc l'équivalence

$$\chi(M) + \tau(M) \text{ est divisible par } 4 \Leftrightarrow b_+(M) - b_1(M) \text{ est impair.}$$

Lorsque M est une *surface complexe*, i.e. une variété complexe de dimension complexe 2, compacte, la formule (3.1.15) est la *formule de Noether*. Compte-tenu de (3.1.17) et (3.1.19), elle s'écrit encore, en posant $h^q(M, 0) = h^{0,q}(M) = h^{0,q}$ (et en omettant la référence à M) :

$$1 - h^{0,1} + h^{0,2} = \frac{1}{2}(1 - b_1 + b_+). \tag{3.1.20}$$

On déduit aisément de (3.1.20) l'alternative suivante, cf. [Kod] :

$$b_1 \text{ impair } : \ b_1 = 2h^{0,1} - 1, \quad b_+ = 2h^{0,2}, \tag{3.1.21}$$

ou bien

$$b_1 \text{ pair } : \ b_1 = 2h^{0,1} - 1, \quad b_+ = 2h^{0,2} + 1. \tag{3.1.22}$$

Une surface complexe compacte M admet une structure kählérienne si et seulement si elle est du type (3.1.22), cf. [B-P-V] et références incluses.

3.2. La formule de Riemann-Roch sur une surface de Riemann

Dans ce paragraphe, nous supposons que $M = \Sigma$ est une *surface de Riemann* compacte, i.e. une surface réelle compacte munie d'une structure presque complexe J (qui est toujours intégrable). Comme Σ est de dimension (réelle) 2, la donnée de J équivaut à la donnée conjointe d'une orientation et d'une structure conforme (définie-positive) c. Le fibré canonique K de Σ est considéré indifféremment comme le fibré cotangent (réel) $(T^\star\Sigma, J)$ ou comme le fibré $\Lambda^{1,0}\Sigma$ des formes de type $(1,0)$, en identifiant un covecteur (réel) α à sa partie de type $(1,0)$, $\alpha^{1,0}$. Nous avons le fait suivant.

PROPOSITION 3.2.1. — *Une 1-forme α est harmonique, relativement à toute métrique de la classe conforme c, si et seulement si elle est holomorphe en tant que section du fibré canonique K.*

L'argument est le suivant. Pour toute métrique riemannienne de c, l'opérateur de Hodge riemannien $\star$ coïncide avec l'opérateur J :

$$\star\alpha = J\alpha,\ \forall\alpha \in T_x, \quad \forall x \in \Sigma,$$

en particulier, est indépendante de la métrique choisie (dans c). Pour toute métrique dans c, on a donc les équivalences

$$\alpha \text{ est harmonique } \Leftrightarrow d\alpha = d(J\alpha) = 0 \Leftrightarrow \bar{\partial}\alpha^{1,0} = 0.$$

De 3.2.1 nous déduisons l'identification (d'espaces $\mathbf{R}$-vectoriels)

$$H^1_{dR}(\Sigma, \mathbf{R}) = H^0(\Sigma, \mathcal{K}),$$

et donc l'égalité

$$b_1 = 2g, \tag{3.2.2}$$

où b_1 est le premier nombre de Betti de Σ et où le *genre* g de Σ est *défini* par

$$g = \dim_{\mathbf{C}} H^0(\Sigma, \mathcal{K}),$$

i.e. le genre g est, par définition, la dimension (complexe) de l'espace $H^0(\Sigma, \mathcal{K})$ des 1-formes holomorphes sur Σ. De (3.2.2), nous déduisons directement l'égalité (2.7.7). Pour tout fibré vectoriel holomorphe E sur (de rang r), la dualité de Serre (3.1.13) implique l'égalité

$$h^1(\Sigma, \mathcal{E}) = h^0(\Sigma, \mathcal{K} \otimes \mathcal{E}^\star),$$

qui se réduit à

$$h^1(\Sigma, \mathcal{O}) = g, \tag{3.2.3}$$

lorsque $\mathcal{E}$ est le faisceau structural $\mathcal{O}$.

Par ailleurs, les groupes de cohomologie $h^q(\Sigma, \mathcal{E})$ sont nuls dès que q est supérieur à 1, comme il résulte clairement de l'isomorphisme de Dolbeault (3.1.3). On a donc

$$\chi(\Sigma, \mathcal{E}) = h^0(\Sigma, \mathcal{E}) - h^1(\Sigma, \mathcal{E}).$$

Comme Σ est de dimension complexe égale à 1, on a par (2.5.5) et (2.5.6) :

$$ch(E) = r + c_1(E), \; td(\Sigma) = 1 + \frac{1}{2}c_1(\Sigma).$$

La formule de Riemann-Roch-Hirzebruch 3.1.2 se réduit donc, lorsque la base est une surface de Riemann de genre g, à la *formule de Riemann-Roch*

$$h^0(\Sigma, \mathcal{E}) - h^1(\Sigma, \mathcal{E}) = r(1 - g) + d(E), \tag{3.2.4}$$

où $d(E)$ est le *degré* de E, défini par

$$d(E) = \langle c_1(E), [\Sigma] \rangle.$$

Lorsque $E = L$ est de rang 1, la formule (3.2.4) s'écrit

$$h^0(\Sigma, \mathcal{L}) - h^1(\Sigma, \mathcal{L}) = 1 - g + d(L), \tag{3.2.5}$$

qui, pour $L = \mathbf{1}$, se réduit à (3.2.3).

Le reste du paragraphe est consacré à une démonstration directe des formules (3.2.4) et (3.2.5), extraite de [Gun1] et [Gun2]. Puisque (3.2.3) a été établi, il suffit, pour démontrer (3.2.5), d'établir le fait suivant. Pour tout fibré vectoriel holomorphe E, définissons $\delta(E)$ par

$$\delta(E) = h^0(\Sigma, \mathcal{E}) - h^1(\Sigma, \mathcal{E}) - d(E).$$

PROPOSITION 3.2.6. — *$\delta(L)$ est indépendant de L.*

Un *diviseur* sur Σ est une somme formelle

$$\nu = \Sigma_{j=1}^{N} p_j \cdot x_j,$$

où les x_j sont des points distincts de Σ, où les *multiplicités* p_j sont des nombres entiers et N un entier positif indéterminé. A un tel diviseur ν est associé un fibré en droites holomorphe $L^{(\nu)}$, défini, à isomorphisme près, de la façon suivante. Choisissons, pour chaque indice j, un ouvert $\mathcal{U}_j$ de Σ contenant le point x_j et une carte locale $z_j \; : \; \mathcal{U}_j \to \mathbf{C}$. Nous supposons que les ouverts $\mathcal{U}_j$ sont disjoints deux à deux et nous complétons le recouvrement ouvert de Σ avec l'ouvert $\mathcal{U}_0 = \Sigma - \cup\{\mathcal{U}_j\}$. Le fibré $L^{(\nu)}$ est alors déterminé (à isomorphisme près) par les fonctions de transitions relatives à ce recouvrement, définies, cf. § 1.1, par

$$\Phi_{j,0} = (z_j)^{p_j}, \text{ sur } \mathcal{U}_j - \{x_j\}.$$

En outre, $\xi^{(\nu)}$ est muni d'une section méromorphe non-triviale ξ, dont l'expression locale, cf. (1.1.3), est

$$\begin{aligned} \xi^{(j)} &= (z_j)^{p_j}, \text{ sur } \mathcal{U}_j \\ \xi^{(0)} &= 1, \text{ sur } \mathcal{U}_0 \end{aligned} \tag{3.2.7}$$

et dont ν est le diviseur (x_j est un *zéro* d'ordre p_j si p_j est positif, un *pôle* d'ordre $-p_j$ si p_j est négatif).

Inversement, si un fibré en droites holomorphes L admet une section méromorphe non-triviale ξ, de diviseur ν, on a clairement l'isomorphisme :

$$L = L^{(\nu)}. \tag{3.2.8}$$

Il résulte de (1.5.8), cf. aussi § 2.7, que l'on a

$$d(L^{(\nu)}) = d(\nu), \tag{3.2.9}$$

où $d(\nu)$ est le *degré* du diviseur ν, défini par

$$d(\nu) = \Sigma_{j=1}^{N} p_j. \tag{3.2.10}$$

Nous nous proposons d'établir les deux propositions suivantes.

PROPOSITION 3.2.11. — *Pour tout fibré vectoriel holomorphe E et tout diviseur ν, on a*

$$\delta(E \otimes L^{(\nu)}) = \delta(E).$$

PROPOSITION 3.2.12. — *Tout fibré vectoriel holomorphe E sur une surface de Riemann compacte Σ admet une section méromorphe non-triviale. En particulier, tout fibré en droites holomorphe sur Σ est de la forme* (3.2.8).

Il est clair que 3.2.11 et 3.2.12 mis ensemble impliquent 3.2.6, donc (3.2.5). Par ailleurs, 3.2.11 implique 3.2.12. En effet, 3.2.11 peut s'écrire sous la forme

$$h^0(\Sigma, \mathcal{E} \otimes \mathcal{L}^{(\nu)}) = h^1(\Sigma, \mathcal{E} \otimes \mathcal{L}^{(\nu)}) + h^0(\Sigma, \mathcal{E}) - h^1(\Sigma, \mathcal{E}) + d(\nu), \tag{3.2.13}$$

puisque $d(E \otimes L^{(\nu)})$ est égal à $d(E) + d(\nu)$ par (3.2.9). Il résulte évidemment de (3.2.13) que $h^0(\Sigma, \mathcal{E} \otimes \mathcal{L}^{(\nu)})$ est positif, i.e. $E \otimes L^{(\nu)}$ admet des sections *holomorphes* non-triviales, dès que $d(\nu)$ est assez grand. Comme E est isomorphe à $(E \otimes L^{(\nu)}) \otimes L^{(-\nu)}$ et comme $L^{(-\nu)}$ possède une section méromorphe non-triviale par construction, E possède lui-même une section méromorphe non-triviale. □

Pour démontrer 3.2.11, nous considérons le fibré $L^{(x)}$, i.e. le fibré en droites holomorphe déterminé par le diviseur réduit au seul point x, affecté de la multiplicité 1, et la section *holomorphe* $\xi^{(x)}$ de $L^{(x)}$ déterminée par (3.2.7), ayant le point x comme unique zéro (simple). La section holomorphe $\xi^{(x)}$ détermine un morphisme de faisceaux de $\mathcal{E} \otimes \mathcal{L}^{(-x)}$ dans le faisceau $\mathcal{E}$, qui, à tout germe de section holomorphe φ de $E \otimes L^{(-x)}$, associe le germe de la section holomorphe $\varphi \otimes \xi^{(x)}$, vue comme un germe de section holomorphe de E via l'isomorphisme canonique de $L^{(-x)} \otimes L^{(x)}$ avec le fibré holomorphe trivial $\mathbf{1}$. Ce morphisme de faisceaux est injectif et est un isomorphisme de faisceaux en dehors du point x. En x, l'image de ce morphisme est constitué des germes de sections holomorphes de E qui s'annulent en x.

On a donc la suite exacte de faisceaux suivante

$$(3.2.14)\qquad 0 \longrightarrow \mathcal{E}\otimes\mathcal{L}^{(-x)} \xrightarrow{\xi^{(x)}} \mathcal{E} \longrightarrow \mathcal{E}_x \longrightarrow 0,$$

où $\mathcal{E}_x$ note le "faisceau gratte-ciel" sur Σ dont la fibre est E_x en x, $\{0\}$ sur $\Sigma - \{x\}$, et où la dernière flèche associe à tout germe de section holomorphe de E sa valeur en x. On vérifie aisément que l'on a :

$$H^0(\Sigma, \mathcal{E}_x) = E_x,\ H^q(\Sigma, \mathcal{E}_x) = \{0\},\ \forall q > 1.$$

La suite exacte longue associée à (3.2.14) s'écrit donc

$$(3.2.15)\qquad \begin{aligned} &0 \to H^0(\Sigma, \mathcal{E}\otimes L^{(-x)}) \to H^0(\Sigma, \mathcal{E}) \to \mathcal{E}_x \to \\ &H^1(\Sigma, \mathcal{E}\otimes\mathcal{L}^{(-x)}) \to H^1(\Sigma, \mathcal{E}) \to 0. \end{aligned}$$

Puisque (3.2.16) est une suite exacte, la somme alternée des dimensions des espaces qui y figurent est nulle. Par ailleurs, pour tout fibré vectoriel (complexe) E de rang r et tout fibré en droites (complexes) L, on a $\Lambda^r(E\otimes L) = \Lambda^r E \otimes L^r$, et donc, par (2.3.18),

$$(3.2.16)\qquad d(E\otimes L) = d(E) + rd(L).$$

Puisque le degré de $L^{(-x)}$ est égal à -1, cf. (3.2.9) et (3.2.10), il résulte de (3.2.16) que $d(E\otimes L^{(-x)})$ est égal à $d(E) - r$. Nous obtenons ainsi l'égalité

$$(3.2.17)\qquad \delta(E\otimes L^{(-x)}) = \delta(E),\ \forall E \in \ \mathrm{Vect}_*\Sigma,\ \forall x\in\Sigma.$$

Il est clair que (3.2.17) implique 3.2.11. □

La démonstration de (3.2.4) se fait par récurrence sur le rang du fibré E, à partir de 3.2.5, qui vient d'être établie, et du fait suivant.

PROPOSITION 3.2.18. — *Tout fibré vectoriel holomorphe E sur Σ admet un sous-fibré holomorphe de rang* 1.

La proposition 3.2.18 est une conséquence facile de 3.2.12. En effet, comme Σ est de dimension complexe 1, toute section méromorphe non-triviale ξ de E détermine une section *holomorphe* $\tilde{\xi}$ de la fibration (holomorphe) $\mathbf{P}(E) \to \Sigma$, obtenue de la façon suivante. Si x n'est ni un zéro, ni un pôle de ξ, $\tilde{\xi}(x)$ est la droite complexe de E_x engendrée par $\xi(x)$; si x est un zéro d'ordre p de ξ, $\tilde{\xi}(x)$ est la droite complexe de E_x engendrée par $z^{-p}\xi(x)$, où z est une carte holomorphe quelconque au voisinage de x ; si x est un pôle d'ordre $p, \tilde{\xi}(x)$ est la droite complexe de E_x engendrée par $z^p\xi(x)$.

Par ailleurs, la donnée d'un sous-fibré holomorphe de rang 1 de E *équivaut* à la donnée d'une section holomorphe de la fibration $\mathbf{P}(E) \to \Sigma$. □

L'argument de récurrence repose sur la suite exacte de faisceaux

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{Q} \to 0,$$

où $\mathcal{Q}$ est le faisceau des germes de sections holomorphes du fibré vectoriel (holomorphe) quotient $Q = E/L$, dont la suite exacte "longue" s'écrit

$$\begin{aligned} &0 \to H^0(\Sigma, \mathcal{L}) \to H^0(\Sigma, \mathcal{E}) \to H^0(\Sigma, \mathcal{Q}) \to H^1(\Sigma, \mathcal{L}) \to \\ &\qquad H^1(\Sigma, \mathcal{E}) \to H^1(\Sigma, \mathcal{Q}) \to 0. \end{aligned} \tag{3.2.19}$$

De (3.2.19), nous déduisons l'égalité :

$$\begin{aligned} h^0(\Sigma, \mathcal{E}) - h^1(\Sigma, \mathcal{E}) &= h^0(\Sigma, \mathcal{L}) - h^1(\Sigma, \mathcal{L}) + h^0(\Sigma, \mathcal{Q}) - h^1(\Sigma, \mathcal{Q}) \\ &= 1 - g + d(L) \quad \text{par (3.2.5)} \\ &\quad +(r-1)(1-g) + d(Q) \text{ par hypothèse de récurrence} \\ &= r(1-g) + d(E) \text{ car } d(E) = d(Q) + d(L) \end{aligned}$$

puisque, topologiquement (mais non holomorphiquement) E est isomorphe à la somme de Whitney $Q \oplus L$. Ceci achève la démonstration de la formule (3.2.4). □

Remarque. — Il résulte du théorème de Koszul-Malgrange, cf. §5, que *toute* connexion **C**-linéaire ∇ opérant sur les sections d'un fibré vectoriel complexe E (de rang quelconque) au-dessus de la surface de Riemann Σ détermine une structure holomorphe sur E, dont l'opérateur de Cauchy-Riemann $\bar\partial^\nabla$ est égal à $\nabla^{0,1}$.

Par ailleurs, l'opérateur de Cauchy-Riemann $\bar\partial^\nabla$ est *lui-même*, dans ce cas, un opérateur différentiel elliptique (d'ordre 1), dont l'indice coïncide avec l'indice de l'opérateur de Dolbeault D, défini au § 3.1 (pour toute métrique hermitienne g sur Σ et toute structure hermitienne h sur E).

Le théorème de Riemann-Roch (3.2.4) peut alors s'énoncer sous la forme suivante.

THÉORÈME 3.2.20. — *Soit E un fibré vectoriel complexe de rang r et de degré $d(E)$ au-dessus d'une surface de Riemann compacte Σ de genre g. Pour toute connexion* **C**-*linéaire ∇ sur E, on a*

$$\text{ind}\,(\bar\partial^\nabla) = r(1-g) + d(E).$$

En particulier, ind $(\bar\partial^\nabla)$ *est indépendant de la connexion* ∇.

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Part 3

Pseudo-holomorphic curves and applications

Chapter V

Some properties of holomorphic curves in almost complex manifolds

Jean-Claude Sikorav

The study of holomorphic curves in almost complex manifolds can be viewed as the confluence of two fields.

Firstly, there is the theory of pseudoanalytic (or generalised analytic) functions from **C** to **C**, initiated by T. Carleman in the thirties and developed in the fifties by L. Bers [2] and I. N. Vekua [11]. This was later extended by many authors to a study of more and more general systems of partial differential equations for maps from $\mathbf{R}^2$ to $\mathbf{R}^n$ which are of order one and of elliptic type [so that n is necessarily even] (see [12]). It is also closely related with the Beltrami equation which occurs in the uniformisation of complex structures in dimension one (see [1]).

Secondly, there is the theory of holomorphic curves in complex manifolds, especially Kähler or algebraic. In this setting one can consider holomorphic curves as analytic subsets of dimension one. The use of these objects to study algebraic manifolds is well known, and the literature is enormous. From our point of view one of the most important properties, in the complex framework, is the compactness theorem of E. Bishop [3] for analytic sets with bounded volume.

In this chapter we shall first (in § 1) quote the basic properties of the classical Cauchy-Riemann operator in one dimension. As is usual in elliptic theory, they involve the Sobolev spaces $L^{k,p}$ and the Hölder spaces $C^{k,\alpha}$. This will be used in § 2 to give a proof of the elliptic regularity properties for holomorphic curves. Since one obtains a priori bounds, the proof shows that all natural topologies on spaces of holomorphic curves are identical. In § 3 we prove the existence of local holomorphic curves and some "analyticity" properties (finiteness of the intersection between two curves and of the critical points of one curve). In § 4 we give the main properties of the area of holomorphic curves, and in § 5 we sketch a proof of Gromov's compactness theorem.

Most of the results of the last three sections are proved with more details by D. McDuff, M.-P. Muller and P. Pansu in chapters VI, VII and VIII, but our approach is different and we hope complementary to theirs. Also, we pay a special attention to curves with boundary for which we try to give the results with more "realistic" hypotheses (i.e. without requiring the analyticity of the totally real submanifold or the integrability of the almost complex structure in the neighbourhood).

1. The equation $\overline{\partial} f = g$ in $\mathbf{C}$

1.1. Notations

For a differentiable map from a domain $U \subset \mathbf{C}$ to $\mathbf{C}$, we define the maps

$$\begin{aligned}\partial f &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)\\ \overline{\partial} f &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).\end{aligned}$$

These definitions obviously extend componentwise for a map from U to $\mathbf{C}^n$.

Let $D \subset \mathbf{C}$ be the closed unit disc. For $k \in \mathbf{N}$ and $1 < p < \infty$, denote by $L^{k,p}(D, \mathbf{C})$ the space of measurable maps from D to $\mathbf{C}$ having k derivatives in L^p as distributions. It can be obtained by completing the space of $\mathcal{C}^\infty$ maps with respect to the norm

$$\|g\|_{L^{k,p}} = \sum_{i=0}^{k} \|D^i g\|_{L^p}.$$

Let $r > 0$ *not an integer.* We write $r = k + \alpha$ with $k \in \mathbf{N}$ and $0 < \alpha < 1$. Denote by $\mathcal{C}^r(D, \mathbf{C}) = \mathcal{C}^{k,\alpha}(D, \mathbf{C})$ the space of k times differentiable maps from D to $\mathbf{C}$ whose k-th derivative is of Hölder class $\mathcal{C}^\alpha$. We define in this manner a Banach space for the norm

$$\|g\|_{\mathcal{C}^r} = \|g\|_{\mathcal{C}^{k,\alpha}} = \sum_{i=0}^{k} \|D^i g\|_{L^\infty} + \sup_{x \neq y} \frac{|D^k g(x) - D^k g(y)|}{|x-y|^\alpha}.$$

We denote by $L_0^{k,p}(D, \mathbf{C})$, resp. $\mathcal{C}_0^r(D, \mathbf{C})$ the closure in $L^{k,p}(D, \mathbf{C})$, resp. in $\mathcal{C}^r(D, \mathbf{C})$, of the space of $\mathcal{C}^\infty$ maps with compact support in $\operatorname{Int} D$. Notice that $L_0^p(D, \mathbf{C}) = L^p(D, \mathbf{C})$. Similarly, we define the spaces $L^{k,p}(D, \mathbf{C}^n)$, $\mathcal{C}^{r+1}(D, \mathbf{C}^n)$...

We define $L^{k,p}(D, \partial D; \mathbf{C}, \mathbf{R})$ and $\mathcal{C}^{r+1}(D, \partial D; \mathbf{C}, \mathbf{R})$ as being the closures in $L^{k,p}(D)$ and $\mathcal{C}^{r+1}(D)$ of the space of all smooth functions $f : D \to \mathbf{C}$ such that $f(\partial D) \subset \mathbf{R}$.

1.2. A priori estimates

We can now summarise the results that we shall need in the study of holomorphic curves.

PROPOSITION 1.2.1. — *There exists continuous linear operators $T : L^2(D) \to L^2(D)$ and $P : L^2(D) \to L^{1,2}(D)$ and positive constants $A_{k,p}$, C_p, $k \in \mathbf{N}$, $1<p<+\infty$ and B_r, $r \in \mathbf{R}^+ \setminus \mathbf{N}$, with the following properties:*

(i) $\overline{\partial} \circ P = \mathrm{Id}$, $\quad \partial \circ P = T$,

(ii) $P \circ \overline{\partial} \mid L_0^{1,2}(D) = \mathrm{Id}$, $\quad T \circ \overline{\partial} \mid L_0^{1,2}(D) = \partial$,

(iii) $\|Tg\|_{L^{k,p}} \leq A_{k,p} \|g\|_{L^{k,p}}$, $\quad \|Pg\|_{L^{k+1,p}} \leq (1 + A_{k,p}) \|g\|_{L^{k,p}}$,

(iv) $\|Tg\|_{\mathcal{C}^r} \leq B_r \|g\|_{\mathcal{C}^r}$, $\quad \|Pg\|_{\mathcal{C}^{r+1}} \leq (1 + B_r) \|g\|_{\mathcal{C}^r}$,

(v) $2<p<+\infty \Rightarrow \|Pg\|_{\mathcal{C}^{1-2/p}} \leq C_p \|g\|_{L^p}$.

For the proof, we refer to the book [11] of N. Vekua (Chapter I, §§6, 8, 9). Let us just indicate the definitions of P for $g \in \mathcal{C}^0(D)$ and of T for $g \in \mathcal{C}_0^1(D)$:

$$Pg(z) = \frac{1}{2\pi i} \iint_D \frac{g(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta} = -\frac{1}{\pi} \iint_D \frac{g(\zeta)}{\zeta - z}\, d\xi \wedge d\eta$$

$$Tg(z) = p.v. \left(\frac{1}{2\pi i} \iint_D \frac{g(\zeta)}{(\zeta - z)^2}\, d\zeta \wedge d\bar{\zeta} \right) = \lim_{\varepsilon \to 0} \iint_{\{\zeta \,|\, |\zeta - z| \geq \varepsilon \, |\zeta| \leq 1\}} \frac{g(\zeta)}{(\zeta - z)^2}\, d\zeta \wedge d\bar{\zeta}.$$

Here *p.v.* means the Cauchy principal value. Properties (i) and (ii) are easy to show for functions of class $\mathcal{C}^0$ and $\mathcal{C}^1$ respectively. The proof that T is continuous on $L^2(D)$ is based on a "symplectic" argument. Assuming that g is of class $\mathcal{C}^1$ on $\mathbf{C}$ with support in D, set $f = Pg$. Setting $f(\infty) = 0$ extends f to a $\mathcal{C}^1$ map from $\mathbf{C} \cup \{\infty\} = \mathbf{CP}^1$ to $\mathbf{C}$. Thus we have

$$\iint_{\mathbf{C}} \left(|\partial f|^2 - |\bar{\partial} f|^2 \right) d\zeta \wedge d\bar{\zeta} = \iint_{\mathbf{CP}^1} f^\star \left(d\zeta \wedge d\bar{\zeta} \right) = 0$$

since $d\zeta \wedge d\bar{\zeta}$ is closed. Thus

$$\iint_D |Tg|^2 = \iint_D |\partial f|^2 \leq \iint_{\mathbf{C}} |\partial f|^2 = \iint_{\mathbf{C}} |\bar{\partial} f|^2 = \iint_D |g|^2.$$

The inequalities (iii) and (iv), which are due respectively to G. Giraud (1934) and to A. Calderon-A. Zygmund (1952), can be found in [11], p.56 and 64ff.

For the inequality (v), see [11] p.38: in fact we have then the very important consequence: $L^{1,p} \subset C^{1-2/p}$. This last estimate will play a crucial role in the proof of the regularity theorem (step (a) in the proof of 2.3.6 (i)).

Remarks.

(i) If r is not an integer or if $p = +\infty$ then the a priori bounds (iii) and (iv) are no longer valid. For instance, if $f(z) = z \log\log |z|^2$ then $\bar{\partial} f \in \mathcal{C}^0$ (as a distribution) but f is not even Lipschitz. Indeed one has

$$\partial f = \log\log |z|^2 + \frac{1}{\log |z|^2}$$

and

$$\bar{\partial} f = \frac{z}{z \log |z|^2}.$$

(ii) If g is real analytic in D, then we can find Pg and Tg (see [11] p.27) using

$$\begin{aligned} P(z^n \bar{z}^m) &= \frac{1}{m+1}(z^n \bar{z}^{m+1} - z^{n-m-1}) \text{ for } n \geq m+1 \\ &= \frac{1}{m+1}(z^n \bar{z}^{m+1}) \text{ for } n < m+1. \end{aligned}$$

(iii) Everything in this section remains obviously valid for maps from $\mathbf{C}$ to $\mathbf{C}^n$.

1.3. Boundary conditions

PROPOSITION 1.3.1. — *There exists unique continuous linear operators T_0 : $L^2(D) \to L^2(D)$ and $P_0 : L^2(D) \to L^{1,2}(D, \partial D; \mathbf{C}, \mathbf{R})$ such that*

(i) $f = P_0(\overline{\partial} f) + a_0(f) \quad \partial f = T_0(\overline{\partial} f)$ *for every* $f \in L^{1,2}(D, \partial D; \mathbf{C}, \mathbf{R})$, *where* $a_0(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})d\theta$ *is the constant Fourier coefficient of* $f|\partial D$. *Moreover we have the following properties:*

(ii) $\partial \circ P_0 = T_0$ *and* $\overline{\partial} \circ P_0 = \mathrm{Id}$, *and* T_0, P_0 *satisfy inequalities analogous to those of proposition* 1.2.1.

The proof of these results can be found in [11], Chapter IV, § 7. The operators P_0 and T_0 are defined by

$$P_0 = P - zP^\star, \quad T_0 = T - zT^\star,$$

where

$$P^\star g(z) = \frac{1}{2\pi i}\iint_D \frac{\overline{g}(\zeta)}{1 - \overline{\zeta} z}\, d\zeta \wedge d\overline{\zeta},$$

$$T^\star g(z) = \frac{1}{2\pi i}\iint_D \frac{\overline{g}(\zeta)}{(1 - \overline{\zeta} z)^2}\, d\zeta \wedge d\overline{\zeta}.$$

In the definition of P_0 and T_0, we have a minus sign instead of the plus sign of Vekua because our boundary condition is $f(\partial D) \subset \mathbf{R}$ instead of $\mathrm{Re}\, f_{|\partial D} = 0$.

Remark. — Everything in this section remains valid if we replace $(\mathbf{C}, \mathbf{R})$ by $(\mathbf{C}^n, \mathbf{R}^n)$.

2. Regularity of holomorphic curves

2.1. Definitions

Let V be a differentiable manifold of class C^1. An *almost complex* structure on V is a complex structure on the tangent bundle, i.e. a continuous section J of $\mathrm{End}\, TV$ such that $J^2 = -\mathrm{Id}$ (see chapter II).

A *totally real* submanifold of (V, J) is a submanifold W of class C^1 and of half dimension such that $T_w W$ never contains a J-complex line. Equivalently: $T_W V = TW \oplus JTW$.

Let S be a Riemann surface. A map $f : S \to (V, J)$ is *J-holomorphic* if it is differentiable and its differential $df : TS \to TV$ is a complex homomorphism at each point.

Remark on the differentiability assumption. — One usually finds the requirement that f should be of class C^1. We will see that this is automatic if J is of class C^r for some $r > 0$.

2.2. Statement of the results

The linearisation of the equation for holomorphic maps is an elliptic operator of order 1. This implies the regularity of solutions for Hölder and Sobolev norms. The proof gives a priori bounds which imply that all natural topologies on spaces on J-holomorphic maps coincide.

THEOREM 2.2.1. — *Let (V, J) be an almost complex manifold without boundary of class C^r with $r \in \mathbf{R}^+ \setminus \mathbf{N}$ and W be a totally real submanifold of class C^{r+1} (maybe empty). Let S be a Riemann surface and $f : (S, \partial S) \to (V, W)$ be a J-holomorphic map. Then*

(i) *f is of class C^{r+1},*

(ii) *the topologies C^0 and C^{r+1} of uniform convergence over compact subsets coincide on the space $\mathrm{Hol}_J(S, \partial S; V, W)$ of J-holomorphic maps $(S, \partial S) \to (V, W)$. In particular, this space is closed for the C^0 topology.*

The proof will be given in the next paragraphs.

Remark. — The proof will show that if f is continuous everywhere, differentiable and J-holomorphic except on a discrete subset, then it is differentiable everywhere (and thus J-holomorphic) if J is of class C^r for some $r > 0$. This will be useful for the theorem on removable singularity (see 4.5).

2.3. Local version of the equation of J-holomorphic maps

Let $f : S \to (V, J)$ be a map such that $f(z_0) = v_0$.

Interior points. — Choose coordinate charts $\varphi : (D, 0) \to (S, z_0)$ and $\Phi : (B, 0) \to (V, v_0)$ where B is the closed unit ball in $\mathbf{R}^{2n} = \mathbf{C}^n$, such that $f(\varphi(D)) \subset \Phi(B)$. From now on we identify $\varphi(D) = D$ and $\Phi(B) = B$ so that we get a map $f : (D, 0) \to (B, 0)$.

The almost complex structure on B is given by a map $J : B \to \mathrm{End}_{\mathbf{R}}(\mathbf{C}^n)$ such that $J^2 = -\mathrm{Id}$. We can assume that $J(0) = i\,\mathrm{Id}$, the standard complex structure on $\mathbf{C}^n$. We shall write also $i\,\mathrm{Id} = i$ if there is no danger of confusion. We shall also assume that B was chosen small enough so that $J(v) + i$ is always invertible.

The equation expressing in these charts that f is J-holomorphic is

$$\frac{\partial f}{\partial y} = J(f)\frac{\partial f}{\partial x}. \tag{2.3.1}$$

Using the identities

$$\frac{\partial f}{\partial y} = \frac{i}{2}\left(\partial f - \bar{\partial} f\right)$$

and

$$\frac{\partial f}{\partial x} = \frac{1}{2}\left(\partial f + \bar{\partial} f\right),$$

the equation becomes $(i+J(f))\,\overline{\partial}f = (i-J(f))\,\partial f$. Since $i+J(v)$ is always invertible, this can be written

$$\overline{\partial}f + q(f)\partial f = 0. \tag{2.3.2}$$

Here $q : B \to \mathrm{End}_{\mathbf{R}}(\mathbf{C}^n)$ is defined by

$$q(v) = (i + J(v))^{-1}\,(i - J(v))\,. \tag{2.3.3}$$

It has the same regularity as J and satisfies $q(0) = 0$.

Comment. — Equation (2.3.2) is a quasilinear partial differential equation of order 1 (or rather a system in the usual terminology). The associated linear equation $\overline{\partial}h + q_1(z)\partial h = 0$, where $q_1 = q \circ f$ and f is a fixed solution, is elliptic precisely if $q_1(z)$ has no eigenvalues of modulus 1, in particular if $\|q_1\|_{L^\infty} < 1$.

Remark. — It is easy to see that q is anti-complex ($qi + iq = 0$, and similarly $qJ + Jq = 0$), and that the map $J \mapsto q = (i + J)^{-1}\,(i - J)$ gives a chart near i of the manifold $\mathcal{J}_n$ of almost complex structures on $\mathbf{C}^n$.

If J is of class C^r, then so is q. Moreover, we can assume by "renormalisation" that $\|q\|_{C^r}$ is as small as we wish. Indeed, fix $\varepsilon > 0$ and replace f by $f_{\alpha,\beta}(z) = \beta^{-1}f(\alpha z)$ and q by $q_\beta(v) = q(\beta v)$. We choose α, β such that

$$\|q_\beta\|_{C^r} \leq \varepsilon \text{ and } f(D(\alpha)) \subset B(\beta). \tag{2.3.4}$$

Then $f_{\alpha,\beta}$ and q_β satisfy (2.3.2), and it is equivalent to prove the regularity at 0 of f or of $f_{\alpha,\beta}$. Notice also that α and β can be found once we know a modulus of continuity of f at z_0.

Boundary points. — We choose a chart near z_0 of the form $\varphi : (D^+, \partial D^+, 1) \to (S, \partial S, z_0)$ where $D^+ = \{z \in D \mid \mathrm{Re}\, z > 0\}$, and a chart near v_0 of the form $\Phi : (B, B \cap \mathbf{R}^n, 0) \to (V, W, v_0)$, and such that $f(\varphi(D^+)) \subset \Phi(B)$. As W is totally real, we can assume $J(0) = i$.

Here again we may assume by "renormalisation" that $\|q\|_{C^r}$ is as small as we wish. For this, we use instead of the homothety αz for small α the homography $\varphi_\alpha(z) = (z + \alpha)/(1 + \alpha z)$ for $\alpha \in]0, 1[$ tending to 1. Then we replace f and q by $f_{\alpha,\beta} = \beta^{-1} f \circ \varphi_\alpha$ and $q_\beta(v) = q(\beta v)$ where α, β are chosen so that

$$\|q_\beta\|_{C^r} \leq \varepsilon \text{ and } f \circ \varphi_\alpha(D^+) \subset B(\beta). \tag{2.3.5}$$

Again, α and β can be chosen once we know a modulus of continuity of f at z_0. Theorem 2.2.1 will then follow from the

PROPOSITION 2.3.6. — *Fix $r \in \mathbf{R}^+ \setminus \mathbf{N}$, and $0 < \eta < 1$. Then there exists $\varepsilon > 0$ with the following property. Let $q : B \to End_{\mathbf{R}}(\mathbf{C}^n)$ be of class C^r and satisfy $\|q\|_{C^r} \leq \varepsilon$.*

(i) *If $f : D \to B$ is a differentiable map such that $\overline{\partial} f + q(f)\partial f = 0$, then f is of class C^{r+1} on $D(1-\eta)$. Moreover we have the a priori bound*

$$\|f_{|D(1-\eta)}\|_{C^{r+1}} \leq C(r,\eta)\, \|f\|_{L^\infty}.$$

(ii) *If $f : (D^+, \partial D^+) \to (B, B \cap \mathbf{R}^n)$ is a differentiable map such that $\overline{\partial} f + q(f)\partial f = 0$, then f is of class C^{r+1} on $D^+(\eta) = \{z \in D^+ \mid \operatorname{Re} z \geq \eta\}$. Moreover we have the a priori bound*

$$\|f_{|D^+(\eta)}\|_{C^{r+1}} \leq C'(r,\eta)\, \|f\|_{L^\infty}.$$

The proof of the proposition rests upon the properties of the operator $\overline{\partial}$ in $\mathbf{C}$ which we recalled in § 1.

2.4. Proof of 2.3.6 (i)

Set $f_1 = \rho_1 f$ where $\rho_1 : D \to [0,1]$ is C^∞, supported in $\operatorname{Int} D$ and such that $\rho_1 | D(1-\eta/4) = 1$. This is possible with $\|\rho_1\|_{C^r} \leq C(r,\eta)$. Then we obtain

$$\overline{\partial} f_1 + q_1 \partial f_1 = g_1 \tag{2.4.1}$$

where $q_1 = q \circ f$ and $g_1 = -\left(\overline{\partial}\rho_1 + \partial\rho_1\, q_1\right) f$. Observe that $g_1 \in L^\infty(D)$ with a norm bounded by $2C(r,\eta)\|f\|_{L^\infty}$, and also that $\|q_1\|_{L^\infty} \leq \varepsilon$.

We first show that $f_1 \in L^{1,2}$. This will follow from the

LEMMA 2.4.2. — *Let $f : (D, \partial D) \to (\mathbf{C}^n, \mathbf{R}^n)$ be continuous, differentiable except on a finite subset. Assume that there exists $\varepsilon, C > 0$ such that at all points of differentiability we have*

$$|\overline{\partial} f(z)| \leq (1-\varepsilon)|\partial f(z)| + C.$$

Then $f \in L^{1,2}$.

Proof. — We have to show that $\iint_D \left(|\partial f|^2 + |\overline{\partial} f|^2\right) dxdy < +\infty$. The hypothesis implies

$$|\partial f|^2 + |\overline{\partial} f|^2 \leq A(|\partial f|^2 - |\overline{\partial} f|^2) + B \tag{2.4.3}$$

for suitable A, B. The 2-form $\beta = (|\partial f|^2 - |\overline{\partial} f|^2)\, dx \wedge dy$ is equal to $f^\star \omega_0$ where $\omega_0 = d\lambda_0$ is the standard symplectic structure on $\mathbf{C}^n$. We have $\beta = d\alpha$ where $\alpha = f^\star \lambda_0$ is a 1-form vanishing on ∂D. Moreover we have the inequality $d\alpha \geq -B/A > -\infty$. Then Stokes' theorem for forms with differentiable coefficients applies (see the appendix), and gives that $d\alpha$ is integrable with $\iint_D d\alpha = 0$. The inequality (2.4.3) gives then the desired result.

Using (ii) of proposition 1.2.1, this implies that $\partial f_1 = T\left(\overline{\partial} f_1\right)$ and $f_1 = P\left(\overline{\partial} f_1\right)$ so that the equation (2.4.1) can be written

$$(\mathrm{Id} + q_1 T)\left(\overline{\partial} f_1\right) = g_1. \tag{2.4.4}$$

Here $\overline{\partial} f_1$ and g_1 are considered as elements of $L^2(D, \mathbf{C}^n)$ and q_1 as an element of $L^\infty(D, \mathrm{End}_{\mathbf{R}}(\mathbf{C}^n))$. We will show that this equation (or a slight modification) holds in various spaces L^p and $\mathcal{C}^s$ and that the operator $\mathrm{Id} + q_1 T$ is invertible in these spaces, thus improving at each step the regularity of $\overline{\partial} f_1$, $f_1 = P\left(\overline{\partial} f_1\right)$, q_1 and g_1 ("bootstrap").

(a) Let us now prove 2.3.6 (i) when $[r] = 0$, i.e. $0 < r < 1$.

Step 1: $f|D(1-\eta/4) \in \mathcal{C}^{1/3}$. — By (iii) of proposition 1.2.1, T is continuous on $L^3(D)$ with a norm $\leq A_{0,3}$. Thus if $\varepsilon \leq (2A_{0,3})^{-1}$, $\mathrm{Id} + q_1 T$ is invertible on $L^3(D)$ with an inverse ≤ 2 in norm. Equation (2.4.3) implies that $\overline{\partial} f_1 \in L^3$. Since $f_1 = P\left(\overline{\partial} f_1\right) \in \mathcal{C}^{1/3}$, 1.2.1(v) implies that $f_1 \in \mathcal{C}^{1/3}$ and

$$\|f_1\|_{\mathcal{C}^{1/3}} \leq C_3 \|\overline{\partial} f_1\|_{L^3} \leq 2C_3 \|g_1\|_{L^3} \leq C_1(r,\eta)\|f\|_{L^\infty}.$$

Step 2: $f|D(1-\eta/2) \in \mathcal{C}^{1+r/3}$. — Consider $f_2 = \rho_2 f = \rho_2 f_1$ where ρ_2 is supported in $D(1-\eta/4)$ and $\rho_2|D(1-\eta/2) = 1$. Then f_2 satisfies

$$(\mathrm{Id} + q_2 T)(\overline{\partial} f_2) = g_2$$

where $q_2 = q \circ f_1$ and $g_2 = -(\overline{\partial}\rho_2 + \partial\rho_2\, q_2)\, f = -(\overline{\partial}\rho_2 + \partial\rho_2\, q_2)\, f_1$. Step 1 implies that q_2 and g_2 are in $\mathcal{C}^{r/3}$, with norms bounded respectively by by $C_2(r,\eta)\varepsilon$ and $C_3(r,\eta)$. By 1.2.1 (iv), T is continuous on $\mathcal{C}^{r/3}(D)$ and bounded by $B_{r/3}$. Let ε be small enough so that

$$B_{r/3} C_2(r,\eta)\,\varepsilon \leq \frac{1}{2}.$$

Then $\mathrm{Id} + q_2 T$ is invertible on $\mathcal{C}^{r/3}(D)$ with an inverse ≤ 2 in norm. This implies that $\overline{\partial} f_2$ is in $\mathcal{C}^{r/3}(D)$, so 1.2.1 (iv) gives $f_2 \in \mathcal{C}^{1+r/3}(D)$, with

$$\|f_2\|_{\mathcal{C}^{1+r/3}} \leq C_4(r,\eta)\|f\|_{L^\infty}.$$

Step 3: $f|D(1-\eta/2) \in \mathcal{C}^{r+1}$. — Consider $f_3 = \rho_3 f = \rho_3 f_2$ where ρ_3 is supported in $D(1-\eta/2)$ and $\rho_3|D(1-\eta) = 1$. Then f_3 satisfies

$$(d + q_3 T)\left(\overline{\partial} f_3\right) = g_3$$

where $q_3 = q \circ f_2$ and $g_3 = -\left(\overline{\partial}\rho_3 + \partial\rho_3\, q_3\right) f = -\left(\overline{\partial}\rho_3 + \partial\rho_3\, q_3\right) f_2$. Then Step 2 implies that q_3 and g_3 are in $\mathcal{C}^r$ with norms bounded respectively by $C_5(r,\eta)\varepsilon$ and $C_6(r,\eta)$. Thus if

$$B_r C_5(r,\eta)\varepsilon \leq \frac{1}{2},$$

$\mathrm{Id} + q_3 T$ is invertible in $\mathcal{C}^r$ with an inverse ≤ 2, so that $\overline{\partial} f_3 \in \mathcal{C}^r$ and $f_3 \in \mathcal{C}^{r+1}$ with

$$\|f_3\|_{\mathcal{C}^{1+r/3}} \leq C_7(r,\eta)\|f\|_{L^\infty}.$$

This finishes the proof for $[r] = 0$.

(b) Assume by induction that $[r] = k \geq 1$ and that the proposition is true for $[r] = k-1$. This implies that f is already of class $\mathcal{C}^r$ on $D(1-\eta/2)$ and the same is true for $q_2 = q \circ f_1$ (we keep the notations of (a)). Also $\|q_2\|_{\mathcal{C}^r}$ and $\|g_2\|_{\mathcal{C}^r}$ are bounded by $C\varepsilon$. Then we may assume that $\|q_2\|_{\mathcal{C}^r}$ is small enough for $\mathrm{Id} + q_2 T$ to be invertible on $\mathcal{C}^r(D)$: this implies that f_2 is in $\mathcal{C}^{r+1}(D)$ so that f is $\mathcal{C}^{r+1}$ on $D(1-\eta)$.

Remarks.

1. A more geometrical way to bound the higher derivatives is to observe that the differential of f is $\tilde{J}$-holomorphic for some suitable structure $\tilde{J}$ on TV of class $\mathcal{C}^{r-1}$ (see the appendix by Gauduchon in chapter II).

2. If q is only L^∞, Step 1 can be refined to prove that $f \in L^{1,p}$ for all $p < \infty$ and consequently $f \in C^r$ for all $r < 1$.

3. In fact, a solution of (2.3.2) is of class $\mathcal{C}^{r+1}$ on $\mathrm{Int}\, D$ or D^+ respectively as soon as q is of class $\mathcal{C}^r$ and $q(f(z))$ never has eigenvalues of modulus 1, in particular if $\|q\|_{L^\infty} < 1$). The a priori bounds are valid with the constants depending also on $\min_z \mathrm{dist}\,(\mathrm{Spec}\, q(z), S^1)$. It suffices to prove that if $q(0)$ has no eigenvalues of modulus one then f is of class $\mathcal{C}^{r+1}$ near 0 (in case (i)). For this, the crucial remark is that the operator $\mathrm{Id} + q(0)T$ (see §1 and 2.4) is an isomorphism in suitable spaces, and every small enough perturbation will also be an isomorphism. Then the "bootstrap" method can be applied exactly as in § 2.4.

2.5. Proof of 2.3.6 (ii)

Since it is completely analogous with the proof of 2.3.6 (i) we shall merely indicate how to modify it.

Set $f_1 = \rho f$ where ρ has support in $\{\mathrm{Re}\, z \geq \eta/4\}$ and equals 1 in $D^+(\eta/2)$. Then f_1 is a differentiable map $(D, \partial D) \to (\mathbf{C}^n, \mathbf{R}^n)$ which satisfies an equation of the type $\overline{\partial} f_1 + q_1 \partial f_1 = g_1$. Lemma 2.4.2 implies that f_1 is in $L^{1,2}$. Thus 1.2.1 (i) implies that $f_1 = P_0(\overline{\partial} f_1) + a_0(f_1)$ and $\partial f_1 = T_0(\overline{\partial} f_1)$ so that the equation becomes

$$(\mathrm{Id} + q_1 T_0)(\overline{\partial} f_1) = f_1.$$

The proof then goes on exactly as above, with T_0 and P_0 replacing T and P, and using the properties of § 1.3 instead of § 1.2.

2.6. On the equation $\overline{\partial}_J f = g$

If f is a differentiable map $S \to V$, we define

$$\overline{\partial}_J f = \frac{1}{2}(df + J(f)dfi).$$

We consider it as a section over graph (f) of the vector bundle $E = \overline{\mathrm{Hom}_J}(TS, TV)$ over $S \times V$ (the fiber $E_{z,v}$ is the space of anti-complex morphisms from T_zS to T_vV). This gives a meaning to the equation $\overline{\partial}_J f = g$ if g is a global section of E.

An essential observation is that this equation is equivalent to the fact that the map $\mathrm{Id} \times f : S \to S \times V$ is J_g-holomorphic for the structure J_g on $S \times V$ defined by

$$J_g(\xi, X) = (i\xi, JX + g\xi)\,.$$

3. Other local properties

3.1. Existence of local holomorphic maps. Perturbations, normal coordinates

Notations. — We denote by $\mathcal{J}^r(V)$ the set of all almost complex structures on V of class C^r. It is naturally a smooth Fréchet manifold, and a smooth Banach manifold if V is compact. A chart near J_0 is given by the formula $J \mapsto (J_0 + J)^{-1}(J_0 - J)$ which takes values in the space of all C^r sections of $\overline{\mathrm{End}_{J_0}}(TV)$. This last is the bundle over V whose fiber at v is the space of J_0-anticomplex endomorphisms of T_vV.

THEOREM 3.1.1. — *Let (V, J_0) be an almost complex manifold of class C^r for some $r > 1$. Fix a point $v \in V$. Then*

(i) *For every $X \in T_vV$ small enough there exists a J_0-holomorphic map $f : (D, 0) \to (V, v)$ such that $df(0) \cdot 1 = X$.*

(ii) *Let $f_0 : D \to V$ be a J_0-holomorphic map. Then there exists $\alpha > 0$, a neighbourhood $\mathcal{U}$ of J_0 in $\mathcal{J}^r(V)$ and a map of class C^r*

$$F : \mathcal{U} \times D(\alpha) \to V$$

such that $F(J_0, .) = f_0$ and $F(J, .)$ is J-holomorphic.

Proof. — We follow [7] (see also chapter VI) and use the results of § 1.2. The key observation is that the equation $\overline{\partial} f + q(f)\partial f = 0$ is equivalent to the holomorphicity (in the standard sense!) of $h = (\mathrm{Id} + Pq(f)\partial)\, f$. This will imply that locally pseudo-holomorphic and holomorphic maps are in a 1-1 correspondence.

We work in charts D and B as in § 2. Define

$$\mathcal{U} = \{J \in \mathcal{J}^r(B) \mid J(v) - i \text{ is invertible}\}.$$

To each $J \in \mathcal{U}$ is associated $q_J = (i+J)^{-1}(i-J)$. Consider the map

$$\begin{aligned} \Phi : \mathcal{U} \times [0,1] \times \mathcal{C}^{r+1}(D,B) &\longrightarrow \mathcal{C}^{r+1}(D,\mathbf{C}^n) \\ (J,\varepsilon,f) &\longmapsto (\mathrm{Id} + Pq_J(\varepsilon f)\partial)\, f. \end{aligned}$$

Then Φ is of class $\mathcal{C}^r$ and $\Phi(J,0,.)$ is the canonical embedding $\mathcal{C}^{r+1}(D,B) \hookrightarrow \mathcal{C}^{r+1}(D,\mathbf{C}^n)$. Thus the implicit function theorem implies that we can find $\varepsilon > 0$ such that, for $\varepsilon \le \varepsilon_0$, $\Phi(J_0,\varepsilon,.)$ is a $\mathcal{C}^r$-embedding $\mathcal{C}^{r+1}(D,B) \hookrightarrow \mathcal{C}^{r+1}(D,\mathbf{C}^n)$ whose image contains a neighbourhood $\mathcal{V}$ of the origin.

Let $v \in \mathbf{C}^n$ be small enough so that $h_v \in \mathcal{V}$, where h_v is the linear map $h_v(z) = zv$. Define $f_{\varepsilon,v} = (\Phi(J_0,\varepsilon,.))^{-1}(h_v)$. Then $g_{\varepsilon,v} = \varepsilon^{-1} f_{\varepsilon,v}$ is a J_0-holomorphic map. Moreover $f_{0,v} = h_v$ so that $df_{0,v}(0) = v$. Thus for $\varepsilon \le \varepsilon_1$, the map $v \mapsto dg_{\varepsilon,v}(0)$ is a diffeomorphism of $B(\frac{1}{2})$ onto its image, which is a neighbourhood of 0. This clearly implies (i).

To prove (ii), recall the definition $f_{\alpha,\beta}(z) = \beta^{-1} f(\alpha z)$. Choose $\beta = \varepsilon_0$ and α small enough so that $h = \Phi(J_0,\varepsilon_0,.)(f_{\alpha,\varepsilon_0})$ is in $\mathcal{V}$. Then we have $f_{\alpha,\varepsilon_0} = \Phi(J_0,\varepsilon_0,.)^{-1}(h)$ so that $f(z) = \varepsilon_0 \Phi(J_0,\varepsilon_0,.)^{-1}(h)(\alpha^{-1}z)$. It then suffices to define

$$F(J)(z) = \varepsilon_0 \Phi(J,\varepsilon_0,.)^{-1}(h)\,(\alpha^{-1}z). \quad \square$$

Question. — If J is less regular, does there still exist local J-holomorphic maps? For $n = 1$ this is true if J is $\mathcal{C}^r$ with $r > 0$, and even if J is in L^∞ provided we generalise the notion of J-holomorphicity to almost everywhere differentiable maps. This follows from the Ahlfors-Bers study of the Beltrami equation $\overline{\partial} f = \mu(z)\partial f$ [1].

An immediate corollary of the existence of local homolorphic curves depending differentiably on a tangent vector of V is the existence of *normal coordinates.*

COROLLARY 3.1.2. — *Let (V,J) be an almost complex manifold where J is of class $\mathcal{C}^r$ with $r > 1$ not an integer. Then near each point v_0 there exists $\mathcal{C}^{r+1}$ coordinates $(w_1,w') \in \mathbf{C} \times \mathbf{C}^{n-1}$ such that J is of the form*

$$J(w_1,w') = \begin{pmatrix} i & B(w_1,w') \\ 0 & D(w_1,w') \end{pmatrix}.$$

For $n = 2$ one has also the following normal form:

$$J(w_1,w_2) = \begin{pmatrix} A(w_1,w_2) & 0 \\ 0 & D(w_1,w_2) \end{pmatrix}.$$

Remark. — This could be useful in dimension 4 because the local equation of a J-holomorphic map $(f_1, f_2) : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ becomes

$$\frac{\partial f_1}{\partial y} = A(f_1, f_2)\frac{\partial f_1}{\partial x}\ ,\ \frac{\partial f_2}{\partial y} = D(f_1, f_2)\frac{\partial f_2}{\partial x}$$

so that each coordinate function satisfies a Beltrami equation with an L^∞ coefficient (cf. [11] p.211). Notice also that in the normal coordinates, the standard symplectic form $\omega_0 = \frac{i}{2}(dw_1 \wedge d\overline{w}_1 + dw_2 \wedge d\overline{w}_2)$ is *calibrating* for J, i.e. $\omega_0(X, JY)$ is a Riemannian metric. This could make possible the application of ideas of the theory of minimal surfaces.

3.2. Isolation of critical points and of intersections

PROPOSITION 3.2.1. — *Let (V, J) be an almost complex manifold of class C^r with $r > 0$ and let S be a Riemann surface.*

(i) *Let $f, g : S \to V$ be two J-holomorphic maps. Then the points where $f(z) = g(z)$ are isolated.*

(ii) *Assume that J is of of class C^1 and let $f : S \to V$ be J-holomorphic. Then its critical points are isolated.*

Proof. — We follow [4], who generalises to higher dimensions a classical method of Carleman (cf. also [11] p. 211). Obviously, the problem is local, so we can consider maps $(D, 0) \to (B, 0)$ as in § 2.

(i) The maps f and g satisfy

$$\frac{\partial f}{\partial y} = J(f)\frac{\partial f}{\partial x} \text{ and } \frac{\partial g}{\partial y} = J(g)\frac{\partial g}{\partial x}.$$

The difference $h = f - g$ satisfies

$$\frac{\partial h}{\partial y} = J(f)\frac{\partial h}{\partial x} + (J(f) - J(g))\frac{\partial g}{\partial x}.$$

This can be written in the form

$$\frac{\partial h}{\partial y} = J(f)\frac{\partial h}{\partial x} + A(z)h \tag{3.2.2}$$

where A is a continuous map from D to $\mathrm{End}_{\mathbf{R}}(\mathbf{C}^n)$.

Assuming $J - i$ small enough we can find $\Phi : B \to GL_{\mathbf{R}}(\mathbf{C}^n) \subset \mathrm{End}_{\mathbf{R}}(\mathbf{C}^n)$ such that $J(v) = \Phi(v)\, i\, \Phi(v)^{-1}$ and having the same regularity as J. Setting $J_1 = J(f)$, $\Phi_1 = \Phi(f)$ and $h = \Phi_1 h_1$ and using $J_1\Phi_1 = \Phi_1 i$, we have

$$\frac{\partial h}{\partial y} - J_1\frac{\partial h}{\partial x} = \left(\frac{\partial \Phi_1}{\partial y} - J_1\frac{\partial \Phi_1}{\partial x}\right) h_1 + \Phi_1\left(\frac{\partial h_1}{\partial y} - i\frac{\partial h_1}{\partial x}\right)$$

thus the equation becomes

$$\overline{\partial} h_1 + B(z)h_1 = 0$$

where B is a continuous map from D to $\mathrm{End}_{\mathbf{R}}(\mathbf{C}^n)$. We can assume by renormalisation that its norm is as small as we wish. In particular we have the inequality

$$|\overline{\partial} h_1| \leq \varepsilon |h_1| \tag{3.2.3}$$

with ε as small as we wish.

LEMMA 3.2.4. — *For ε small enough there exists a continuous map $\Phi_2 : D \to GL_{\mathbf{R}}(\mathbf{C}^n)$ such that $h_1 = \Phi_2 H$ where H is a standard holomorphic map.*

Proof. — The inequation (3.2.3) implies

$$\partial \overline{h}_1 + C(z)h_1 = 0,$$

where now C is a L^∞ map from D to $\mathrm{End}_{\mathbf{C}}(\mathbf{C}^n) = M_n(\mathbf{C})$. Indeed, one can for instance define $C(z)$ as the linear map such that $C(z)\cdot h_1(z) = -\overline{\partial} h_1(z)$ and $C(z) = 0$ on the orthogonal of $h_1(z)$.

Consider the map $u \mapsto u^{-1}\overline{\partial} u$ from

$$E = \{u \in L^{1,3}(D, \partial D; GL_n \mathbf{C}, GL_n \mathbf{R}) \mid a_0(u) = \mathrm{Id}\}$$

to $L^3(D, M_n(\mathbf{C}))$. It is a $\mathcal{C}^\infty$ map, and proposition 1.3.1 implies that the derivative at Id is an isomorphism. Using the implicit function theorem, we find for ε small enough a map $\Phi_2 \in L^{1,3}(D, GL_n \mathbf{C})$ such that $\overline{\partial}\Phi_2 \, \Phi_2^{-1} = -C$, that is $\overline{\partial}\Phi_2 = -C\Phi_2$. If we write $h_1 = \Phi_2 H$, the equation becomes $\overline{\partial} H = 0$ which proves the lemma. □

Thus the difference $h = f - g$ can be written $h = \Psi H$, where $\Psi = \Phi_1\Phi_2$ is a continuous map from D to $GL_{\mathbf{R}}(\mathbf{C}^n)$ and H is holomorphic. If h is not identically zero, it has an isolated zero at the origin, which finishes the proof of (i). □

(ii) Let $u = \partial f/\partial x$. Then differentiating the equation $\partial f/\partial y = J(f)\partial f/\partial x$ gives

$$\frac{\partial u}{\partial y} = J(f)\frac{\partial u}{\partial x} + \left(\partial J \partial f + \overline{\partial} J \overline{\overline{\partial} f}\right) u,$$

which can be written in the form

$$\frac{\partial u}{\partial y} = J(f)\frac{\partial u}{\partial x} + A(z)u,$$

where A is a continuous map from D to $\mathrm{End}_{\mathbf{R}}(\mathbf{C}^n)$. This is exactly the same type of equation as (3.2.2). So we may apply the same arguments as to h in part (i), and obtain that either u is identically zero, in which case f is locally constant, or u has an isolated zero at the origin: thus f has isolated critical points if it is not locally constant. □

Remark. — If J is $\mathcal{C}^\infty$ one can use Aronszajn's lemma to prove these results (see [5], [6] and chapter VI).

4. Properties of the area of holomorphic curves

To investigate geometric properties of holomorphic curves, Gromov uses a Hermitian metric on (V, J) (if V is non compact, it should have "bounded geometry" properties: bounded curvature, injectivity radius bounded below). For this approach, see chapter VII. Here we shall proceed differently, working on a "reasonable" category of almost complex manifolds which includes all the closed manifolds. Our approach has the advantage that it requires less regularity on J.

4.1. Tame almost complex manifolds

DEFINITION 4.1.1. — *Let (V, J, μ) be an almost complex manifold with a Riemannian metric. It is tame if μ is complete and there exists positive constants r_0, C_1, C_2 with the following properties:*

(T1) *For all $x \in V$, the map $\exp_x : B(0, r_0) \to B(x, r_0)$ is a diffeomorphism.*

(T2) *Every loop γ in V contained in a ball $B = B(x, r)$ with $r \leq r_0$ bounds a disc in B of area less than C_1 length$(\gamma)^2$.*

(T3) *On every ball $B = B(x, r_0)$, there exists a symplectic form ω_x such that $\|\omega_x\| \leq 1$ and $|X|^2 \leq C_2 \omega_x(X, JX)$ (taming property).*

Comments.

1. Properties (T1) and (T2) obviously hold if V is closed and the metric is of class $\mathcal{C}^2$. In fact, it suffices that μ be complete, that its sectional curvature be bounded above by some (positive) constant K and the injectivity radius be bounded below by some positive ρ_0. One can then take $C_1 = 1/\pi$ and $\rho_0 = \min\left(r_0, \pi/\sqrt{K}\right)$. Property (T3) holds if in addition J is uniformly continuous with respect to μ. Notice also that (T2) implies that any closed curve (not necessarily connected) bounds a surface (union of discs) with the same property.

2. Another way to define tameness would be to require some J-holomorphic convexity property at infinity insuring that a compact holomorphic curve with boundary in a compact set K is contained in a compact set K' depending only on K (see 4.4). For instance, if J is integrable one could require V to be Stein outside a compact set.

4.2. Tame symplectic manifolds

If (V, ω) is a symplectic manifold, we say that it is *tame* if there exists a tame adapted almost-complex structure.

Examples of tame symplectic manifolds.

1. Every closed symplectic manifold. Indeed, there exists J of class $\mathcal{C}^\infty$ calibrated par ω (see chapter II) : it suffices then to define $\langle X, Y\rangle = \omega(JX, Y)$.

2. Every manifold covering symplectically a compact manifold.

3. Every cotangent (T^*M, ω_M), even if M is non compact.

4. Every manifold which is isomorphic at infinity to the symplectisation of a compact contact manifold.

4.3. Isoperimetric inequalities and monotonicity

In this section we consider a tame almost complex manifold (V, J, μ), a compact Riemann surface with boundary S, and a J-holomorphic map $f : S \to V$. We assume that J is of class $\mathcal{C}^r$ for some $r > 0$ so that f is of class $\mathcal{C}^{r+1}$. We denote $A = \text{area}\,(f)$ and $L = \text{length}\,(f_{|\partial S})$.

PROPOSITION 4.3.1. — *There exists positive constants C_3, C_4, C_5 and a_0 depending only on C_1, C_2 and r_0, with the following properties.*

(i) *If $f(S)$ is contained in a ball $B(x, r_0)$ then $A \leq C_3 L^2$.*

(ii) *Monotonicity: assume that $f(S) \subset B = B(x,r)$, with $r \leq r_0$, and $f(\partial S) \subset \partial B$. Assume also that $f(S)$ contains x. Then $A \geq C_4 r^2$.*

(iii) *If $A \leq a_0$ then $A \leq C_5 L^2$. In particular, a_0 is a lower bound for the area of a closed holomorphic curve in V.*

One can take $C_3 = C_1 C_2$, $C_4 = 1/(4C_3)$, $a_0 = \min(C_3, C_4) r_0^2/4$ and $C_5 = \max(C_3, C_4)$.

Proof.

(i) By (T2), $f(\partial S)$ bounds a surface $\Sigma \subset B$ of area $\leq C_1 L^2$. Using the inequality $||\omega_x|| \leq 1$ and the contractibility of B, we have

$$\iint_S f^\star \omega_x = \iint_\Sigma \omega_x \leq C_1 L^2.$$

On the other hand, the taming property and the fact that f is J-holomorphic give

$$A \leq C_2 \iint_S f^\star \omega_x.$$

This implies (i).

(ii) Let $S_t = f^{-1}(B(x,t))$ and $a(t) = \text{area}(f|S_t)$. Since f is of class $\mathcal{C}^{r+1}$ for some $r > 0$, Sard's theorem implies that for almost all t, S_t is a subsurface (not necessarily connected) with $\mathcal{C}^1$-smooth boundary $\partial S_t = f^{-1}(\partial B(x,t))$. Denote by $\ell(t)$ the total length of $f(\partial S_t)$. Then

- $a(t)$ is an absolutely continuous function

- $a'(t) = \ell(t)$ a.e.

Inequality (i) gives $a(t) \leq C_3 \ell(t)^2$ for every $t \leq r$. Since f is not constant and $\operatorname{Im}(f)$ contains x_0, $a(t)$ is > 0 for $t > 0$. Thus for $t > 0$ we have

$$\left(\sqrt{a(t)}\right)' = \frac{a'(t)}{2\sqrt{a(t)}} \geq \frac{1}{2\sqrt{C_2}}.$$

This implies $\sqrt{a(r)} \geq r/(2\sqrt{C_3})$, i.e. $a(r) \geq r^2/(4C_3)$. □

(iii) If $L \geq r_0/2$ then clearly $A \leq C_5 L^2$. If $C = f(S)$ is contained in a ball $B(x, r_0)$ then we can apply (i) which gives $A \leq C_3 L^2 \leq C_5 L^2$.

Now assume that $L \leq r_0/2$ and that C is not contained in any ball $B(x, r_0)$. This implies that there exists $x \in C$ which is not in the $r_0/2$-neighbourhood of ∂C. We can apply (ii) to $B = B(x, r_0/2)$ and get

$$A > \operatorname{area}(C \cap B) \geq C_4 \frac{r_0^2}{4} \geq a_0,$$

which contradicts the hypothesis. □

4.4. Bound on the diameter in terms of the area

A crucial corollary of the monotonicity is the following result.

PROPOSITION 4.4.1. — *Let (V, J, μ) be tame. Let $C_6 = 1/C_4 r_0$. Then for any compact subset K of V we have the following property. Any compact and connected J-curve $C = f(S)$ such that $C \cap K \neq \emptyset$ and $\partial C = f(\partial S) \subset K$ is contained in the neighbourhood of K of size $C_6\operatorname{area}(C)$:*

$$C \subset U(K, C_6 \operatorname{area}(C)).$$

Proof. — Let $v_1, v_2, \ldots, v_N \in C$ in maximal number such that $v_1 \in K$, $d(v_i, v_j) > 2r_0$ if $i \neq j$ and $d(v_i, \partial C) > r_0$ for all i (we allow $N = +\infty$ a priori). Then C is contained in the Nr_0-neighbourhood of K since it is connected. Moreover one can apply the monotonicity to each $C_i = C \cap B(v_i, r_0)$: this gives $\operatorname{area}(C_i) \geq C_4 r_0^2$. Since the C_i are disjoined, one deduces $N \leq \operatorname{area}(C)/(C_4 r_0^2)$, which implies the desired result. □

4.5. Removable singularity

THEOREM 4.5.1. — *Assume that (V, J, μ) is tame and that J is of class C^r for some $r > 0$. Let $f : D \setminus \{0\} \to V$ be a J-holomorphic map with finite area. Then it can be extended to a J-holomorphic map on D.*

Proof. — Using the remark at the end of § 2.2, it suffices to prove that f has a continuous extension, i.e. (since the metric is complete) that for every $\varepsilon > 0$ there exists $\eta > 0$ such that $\operatorname{diam} f(D(\eta)) \leq \varepsilon$. Set

$$A(r) = \operatorname{area}(f|D(r) \setminus \{0\}), \quad L(r) = \operatorname{length} f(\partial D(r)).$$

Using the J-holomorphicity and property (T3), we have

$$\begin{aligned} A'(r) &= \int_0^{2\pi} \left| \frac{\partial f}{\partial \theta} \wedge J \frac{\partial f}{\partial \theta} \right| d\theta \geq C_2^{-1} \int_{\partial D(r)} \left| \frac{\partial f}{\partial \theta} \right|^2 d\theta \\ &\geq C_2^{-1} \frac{L(r)^2}{2\pi r} \text{ by the Schwarz inequality.} \end{aligned}$$

There exists $0 < r_0 \leq 1$ such that $A(r_0) \leq a_0$. Then for every $0 < \eta \leq r_0$ we have

$$a_0 \geq C_2^{-1} \int_\eta^{r_0} \frac{L(r)^2}{2\pi r} dr \geq C_2^{-1} \min_{\eta \leq r \leq r_0} \frac{L(r)^2}{2\pi} \log \frac{r_0}{\eta}.$$

Thus if

$$\eta = r_0 \exp\left(\frac{-8\pi C_2 a_0}{\varepsilon^2}\right),$$

we can find $r \in [\eta, r_0]$ such that $L(r) \leq \varepsilon/2$. Then proposition 3.2.1 (ii) gives $A(r) \leq C_5 \varepsilon^2/4$, and proposition 4.4.1 gives $\operatorname{diam} f(D(r)) \leq \operatorname{diam} f(\partial D(r)) + C_6 A(r)$. Thus

$$\operatorname{diam} f(D(\eta)) \leq \operatorname{diam} f(D(r)) \leq \frac{\varepsilon}{2} + C_6 C_r \frac{\varepsilon^2}{4}.$$

If ε is small enough this is $\leq \varepsilon$, which concludes the proof. □

4.6. Schwarz's lemma

THEOREM 4.6.1. — *Let (V, J, μ) be tame, J being of class C^r for some $r > 0$. Assume that a_0 satisfies* 3.2.1 (iii). *Then there exists a positive constant C with the following property: every J-holomorphic map $f : D \to V$ such that* area$(f) \leq a_0$ *satisfies $\|df(0)\| \leq C$.*

Proof. — Using (ii) of theorem 2.2.1 it suffices to prove that f has a uniform modulus of continuity at 0, i.e. that for every $\varepsilon > 0$ we can find $\eta > 0$ independent of f such that $\operatorname{diam} f(D(\eta)) \leq \varepsilon$. But this clearly follows from the proof of the result on the removable singularity. Indeed, we can take $r_0 = 1$ so that η is explicitly given by

$$\eta = \exp\left(\frac{-8\pi C_2 a_0}{\varepsilon^2}\right). \square$$

Remark. — The constant depends on $\|J\|_{C^r}$ if μ is fixed. Also, if V is closed it suffices that a_0 should be less than the area of any holomorphic sphere.

4.7. Curves with boundary

Definition 4.7.1. — *Let (V, J) be an almost complex manifold, μ a Riemannian metric and W be a totally real submanifold (properly embedded). We say that (V, J, W, μ) is tame if it satisfies the properties* (T1), (T2) *of* 4.1.1 *and if in addition there exists positive constants r_W and C_W such that*

(T'1) *If x, y are two points on W with distance in V less than r_W, then their distance in W (with respect to the induced metric) is less than $C_W . d_V(x, y)$.*

(T'2) *For all x in W, $W \cap B(x, r_W)$ is contractible and ω_x-Lagrangian.*

We shall assume $r_W \leq r_0$ which is no real restriction. Also, we say that (V, J, W) is *tame* if there exists a metric μ for which (V, J, W, μ) is tame.

Then we easily get versions with boundary of the inequalities of 3.2 (isoperimetric inequalities and monotonicity).

Proposition 4.7.2. — *Let (V, J, W, μ) be tame. Then there exists positive constants $C_3(W)$, $C_4(W)$, $C_5(W)$, $C_6(W)$ and $a_0(W)$ with the following properties. Let $C \subset V$ be a compact J-curve. Denote $A = \mathrm{area}\,(C)$ and $L_W = \mathrm{length}(\partial C \setminus W)$. Then*

(i) *If $C \subset B(x, r_W)$, then $A \leq C_3(W)L_W^2$.*

(ii) *If $(C, f(\partial C)) \subset (B(x, r), \partial B(x, r) \cup W)$ with $r \leq r_W$ and if $x \in C$, then $A \geq C_4(W)r^2$.*

(iii) *If $A \leq a_0(W)$ then $A \leq C_6(W)L_W^2$.*

(iv) *If $C \cap K \neq \emptyset$ and $\partial C \subset K \cup W$, then $C \subset N(K, C_6(W) \cdot A)$.*

Actually, one can take $C_3(W) = C_1(C_2 + 1 + C_W)$, $C_4 = 1/(4C_3(W))$, $C_5(W) = \max(C_3(W), C_4(W))$ and $a_0(W) = \min(C_3(W), C_4(W))r_W^2/4$.

Proof. — It is enough to explain the proof of (i). Now $\partial C \setminus W$ is the union of a closed curve γ and a union (possibly infinite) α of arcs α_j from W to W in B. We already know that γ bounds $\Sigma_1 \subset B$ of area $\leq C_1 \,\mathrm{length}(\gamma)^2$.

Complete α_j into a loop $\alpha_j \cup \alpha'_j \subset B$ using a path α'_j in $W \cap B$, of length $\leq C_W L(\alpha_j)$. Then $\alpha \cup \alpha'$ bounds a union of discs $\Sigma_2 \subset B$ of area $\leq C_1(1 + C_W)\,\mathrm{length}(\alpha)^2$. Putting $\Sigma = C \cup \Sigma_1 \cup \Sigma_2$ we get a surface with boundary and a map $(\Sigma, \partial\Sigma) \subset (B, W \cap B)$. The fact that $W \cap B$ is Lagrangian and contractible enables us to finish the proof as before. □

Removable singularity: the case with boundary. — Exactly as in the case without boundary, one proves (see also [9]):

Theorem 4.7.3. — *Assume that (V, J, μ, W) is tame, that J is of class C^r and W of class C^{r+1} for some $r > 0$, and that $f : (D^+ \setminus \{1\}, \partial D^+) \to (V, W)$ is J-holomorphic with finite area. Then f extends to a J-holomorphic map defined on D^+.*

5. Gromov's compactness theorem for holomorphic curves

In this § we consider an almost complex manifold (V, J) with a Riemannian metric μ and a totally real submanifold W (maybe empty). Assume that (V, J, W, μ) is tame, that J is of class C^r for some $r > 0$ not integer, and that W is of class C^{r+1}. In §§ 1,2 and 3 we consider a fixed compact Riemann surface S (with or without boundary) which we equip with a conformal metric.

5.1. Fixed Riemann surface: apparition of bubbles

Let S be a compact Riemann surface. We recall (see theorem 2.2.1 above) that on the space $\mathrm{Hol}_J(S, \partial S; V, W)$ the topologies of uniform convergence C^0 or C^{r+1} coincide.

DEFINITION 5.1.1. — *Let (f_n) be a sequence of J-holomorphic maps from $(S, \partial S)$ to (V, W).*

(i) *Let $g : \mathbf{CP}^1 \to V$ be a non constant J-holomorphic sphere. We say that it occurs as a bubble in the sequence (f_n) if there exists a sequence of holomorphic charts $\varphi_n : D(R_n) \to S$, with $R_n \to +\infty$, converging on $\mathbf{C}$ to a point $p \in S$, and such that*

$$f_n \circ \varphi_n \to g_{|\mathbf{C}},$$

where $\mathbf{C}$ is identified to $\mathbf{CP}^1 \setminus \{\infty\}$.

(ii) *Let $g : (D, \partial D) \to (V, W)$ be a non constant J-holomorphic disc. We say that it occurs as a bubble in the sequence (f_n) if there exists a sequence of holomorphic charts $\varphi_n : (D \setminus B(-1, \delta_n), \partial D) \to (S, \partial S)$ with $\delta_n \to 0$, converging on $D \setminus \{1\}$ to a point $p \in S$, and such that*

$$f_n \circ \varphi_n \to g_{|D \setminus \{-1\}}.$$

Notice that in both cases we have $\mathrm{area}(g) \leq \liminf \mathrm{area}(f_n)$.

The following result shows that noncompactness can only occur if there is some bubbling.

PROPOSITION 5.1.2. — *Let (f_n) be a sequence of J-holomorphic maps from $(S, \partial S)$ into (V, W) whose areas are bounded by a fixed constant A, and whose images meet a fixed compact set. Then, after passing to a subsequence (which we still name (f_n)), one of the following conditions is satisfied:*

(i) *(f_n) converges to a J-holomorphic map $f_\infty : S \to V$.*

(ii) *There is a bubbling of some J-holomorphic sphere.*

(iii) *There is a bubbling of some J-holomorphic disc.*

Proof. — We equip S with a conformal metric. Assume that (i) is not satisfied, i.e. (f_n) has no converging subsequence. Since $f_n(S)$ meets a fixed compact set, Ascoli's theorem implies that the f_n are not equicontinuous, so that $M_n = \max_{z \in S} \|df_n(z)\|$ tends to $+\infty$. Let $p_n \in S$ be a point such that $\|df_n(p_n)\| = M_n$. By passing to a subsequence, we may assume that p_n converges to a point $p \in S$. By proposition 4.5.1, $f_n(S)$ is contained in the fixed compact set $N(K, C_4A)$, so that we can also assume that $f_n(z_n)$ converges to a point $v \in V$.

(a) Assume first that $p \in \operatorname{Int} S$. Let φ be a holomorphic chart $(D,0) \to (S,p)$. Then $u_n = \varphi^{-1}(p_n)$ is defined and $|u_n| < \frac{1}{2}$ for n large enough. Set $R_n = \frac{1}{2}M_n$ and $\varphi_n(u) = \varphi(\frac{u}{M_n} + u_n)$ so that φ_n is defined over $D(R_n)$. Then $R_n \to +\infty$ and φ_n converges to p on $\mathbf{C}$.

Consider the J-holomorphic map $g_n = f_n \circ \varphi_n : D(R_n) \to V$. It satisfies $\|dg_n\| \le M_n . C . \frac{1}{M_n} = C < +\infty$ and $g_n(0) \to v$, thus by passing to a subsequence we may assume that g_n converges on $\mathbf{C}$ to a J-holomorphic map $g : S \to V$. Moreover $\|df(0)\| = \lim \|dg_n(0)\| = \|\varphi'(0)\| > 0$, so g is not constant. Finally Fatou's lemma implies area $(g) \le \liminf$ area $(g_n) \le \liminf$ area $(f_n) \le A$. Thus the theorem 4.5.1 of removable singularities implies the extension of g to $\mathbf{CP}^1$, so that we have proved that g is a J-holomorphic sphere satisfying (ii).

(b) Assume now that $p \in \partial S$. Let $\delta_n = dist\,(p_n, \partial S)$. Assume first that $M_n \delta_n = R_n$ is unbounded. By passing to a subsequence we may assume that $R_n \to +\infty$. We can find a holomorphic chart $\psi_n : (D, \delta_n) \to (S, z_n)$ with the property $|\psi_n'| \sim 1$, i.e. $C^{-1}\delta_n \le |\psi_n'| \le C\delta_n$ for some positive constant C. Set $\varphi_n(u) = \psi_n(\frac{u}{R_n})$. Then we can prove that (ii) holds exactly as in case (a).

(c) Finally, assume that $p \in \partial S$ and that $M_n \delta_n$ is bounded. Let φ be a holomorphic chart $(D^+, \partial D, 1) \to (S, \partial S, p)$. Then $u_n = \varphi^{-1}(p_n)$ is defined for i large enough and converges to 1, with $1 - |u_n| \sim \delta_n$. Set

$$v_n = \left(1 - \frac{1}{M_n}\right) \frac{u_n}{|u_n|} \text{ and } \varphi_n(u) = \varphi\left(\frac{u + v_n}{1 + \overline{v_n} u}\right) = \varphi(h_n(u)).$$

Let K be a compact subset of $D \setminus \{-1\}$. Then the domain of definition of φ_n contains K for n large enough and since $v_n \to 1$ we have $\varphi_n \to z$ on $D \setminus \{-1\}$. Moreover,

$$|\varphi_n'(u)| = |\varphi'(h_n(u))| . \frac{1 - |v_n|^2}{|1 + \overline{v_n} u|^2}.$$

For $u \in K$ this is equivalent to $1 - |v_n| = M_n^{-1}$. Thus for $u \in K$ we have

$$C_1(K) \cdot M_n^{-1} \le \|\varphi_n'(u)\| \le C_2(K) \cdot M_n^{-1}.$$

Now set $g_n = f_n \circ \varphi_n$ as before, so that g_n is a J-holomorphic map defined eventually on any compact subset of $D \setminus \{-1\}$ and sending the boundary to W. Then on each K we have $\|dg_n\| \le M_n . C(K) . M_n^{-1} = C_2(K)$. Moreover, set $u_n' = (u_n - v_n)/(1 - \overline{v_n} u_n)$ so that $u_n = (u_n' + v_n)/(1 + \overline{v_n} u_n')$ whence $\varphi_n(u_n') = p_n$. Then

$$u_n' = \frac{u_n}{M_n - (M_n - 1)|u_n|^2} \sim \frac{u_n}{1 + 2M_n\delta_n} \sim u_n$$

thus we can assume that u_n' converges to a point $u_\infty' \in D^+$.

On the other hand, $g_n(u'_n)$ converges to v, so by passing to a subsequence we can assume that g_n converges on $D \setminus \{-1\}$ to a J-holomorphic map $g : (D \setminus \{-1\}, \partial D \setminus \{-1\}) \to (V, W)$. Furthermore

$$\begin{aligned} \|dg(1)\| &= \lim \|dg_n(u'_\infty)\| \geq \liminf M_n \cdot \|\varphi'_n(u_n)\| \\ &\geq \liminf M_n \cdot C_1(D^+) \cdot M_n^{-1} = C_1(K) > 0, \end{aligned}$$

thus g is not constant. Finally, Fatou's lemma and the theorem 4.7.3 on removable singularity in the case with boundary allow us to prove that g extends to a J-holomorphic disc satisfying (iii). This finishes the proof of the proposition. □

5.2. The compactness theorem for closed curves

DEFINITION 5.2.1. — *Let (V, J) be an almost complex manifold. A cusp-curve is a (j, J)-holomorphic map $f : (\Sigma, j) \to (V, J)$ defined on a closed Riemann surface, not necessarily connected, with $2k$ distinct points identified by pairs. That is to say, we are given $p_1, \dots, p_k, q_1, \dots, q_k$ on Σ satisfying $f(p_j) = f(q_j)$.*

Thus f induces $\hat{f} : \hat{\Sigma} \to V$ where $\hat{\Sigma} = \Sigma/(p_j \sim q_j)$.

DEFINITION 5.2.2. — *Let (J_n) be a sequence of almost complex structures on V, and $f_n : (S, j_n) \to (V, J_n)$ be a sequence of (j_n, J_n)-holomorphic maps. Assume that J_n converges to J and that $f : (\Sigma, j) \to (V, J)$ is a cusp-curve. We say that f_n converges to f if there exists a continuous map $\sigma_n : S \to \hat{\Sigma}$ with the following properties:*

- *$\sigma_n^{-1}(p_j) = \gamma_{n,j}$ is an embedded loop for $j = 1, ..., k$*
- *the topological type of $(S, \tilde{\gamma}_n)$ is constant, where $\tilde{\gamma}_n = \cup_{j=1}^k \gamma_{n,j}$*
- *σ_n induces a diffeomorphism of $S \setminus \tilde{\gamma}_n$ on $\Sigma' = \Sigma \setminus \{p_1, ..., q_k\}$, and $\sigma_{n*}(j_n)$ converges to j on Σ'*
- *$f_n \circ \sigma_n^{-1}$ converges to f uniformly on Σ'.*

Remarks.

1. The uniform convergence implies that the the homotopy class $[f_n] \in [S, V]$ is eventually constant and satisfies $[f_n] = [f_\infty \circ \sigma_n]$, and also that

$$\lim \operatorname{area}(f_n) = \operatorname{area}(f_\infty).$$

2. If the complex structure j_n is fixed (or more generally if it converges to some j_∞), the proof will show that Σ is equal to S plus a finite number of spheres corresponding to bubbles as described in § 5.1.

3. In the case of curves with boundary, the compactness theorem becomes unwieldy to state (although there are essentially no new ideas in the proof). If the complex structure j_n is fixed, it implies that the homotopy class of f_n is for n large enough equal to $[f_\infty]+\sum_i[g_i]+\sum_j[h_j]$ where f_∞ is a map from $(S,\partial S)$ to (V,W), the g_i are J-holomorphic spheres and the h_j are J-holomorphic discs with boundary in W.

THEOREM 5.2.3. — *Let (V, J_n, μ_n) be uniformly tame and such that J_n converge to J_∞. Let S be a closed real surface (not necessarily connected) and (j_n) a sequence of complex structures on S. Let $f_n : (S, j_n) \to (V, J_n)$ be a sequence of J_n-holomorphic maps. Assume that* area(f_n) *is bounded by some constant A and that the image $f_n(S_n)$ meet a fixed compact set. Then there is a subsequence which converges to a cusp curve.*

Sketch of proof. — We define ν_n to be a conformal metric on (S, j_n) of constant curvature ± 1 on each component of genus $\neq 1$, of constant curvature 0 and total area 1 on each component of genus 1. Note that ν_n is unique on each component of genus > 1.

We shall prove a slightly more general result than the one asserted in the theorem. Instead of assuming that f_n is defined on all of S, we make the following hypotheses:

(H1) f_n is defined on a closed subset $S \setminus U_n$, where $U_n = \bigcup_{r=1}^{s} B(q_{n,r}, \varepsilon_{n,r})$ is a union of a fixed number of μ_n-balls with radii tending to zero and the centers $q_{i,r}$ converge to $q_1, ..., q_s$ (not necessarily distinct)

(H2) The total length of $f_n(\partial U_n)$ tends to zero.

The conclusion will be almost the same, with the following precisions:

- $\gamma_{n,j}$ is contained in $S \setminus U_n$,
- σ_n is defined on $S \setminus U_n$ and sends each compoment of ∂U_n to a point,
- the convergence of (f_n) to f_∞ is uniform on each compact subset of $S \setminus \{p_1, \dots, p_k, q_1, \dots, q_s\}$ and also near $\{q_1, \dots, q_s\} \setminus \{p_1, \dots, p_k\}$.

Case 1. — Assume first that, after passing to a subsequence, j_n converges modulo Diff S. There is no loss of generality in assuming that j_n converges to j. The proof will be by induction on $\left[\frac{A}{a_0'}\right]$, where a_0' is the smallest area of a J-holomorphic sphere.

If $N = 0$ then 5.1.2 implies that we have compactness. So we assume that $N \geq 1$ and that the result is already known when $\left[\frac{A}{a_0'}\right] = N - 1$.

Define $\tilde{M}_n(z) = d(z, U_n)\|df_n(z)\|$. If $\tilde{M}_n(z)$ is uniformly bounded then f_n converges on $S \setminus \{a_1, \dots, a_s\}$ to some J-holomorphic map f_∞. Because of the theorem on the removal of singularities, f_∞ can be extended to all of S, and the proof of that result also shows that the convergence is uniform also near $\{q_1, \dots, q_s\}$.

On the other hand, if $\tilde{M}_n(z)$ is not uniformly bounded, let p_n be a point where $\tilde{M}_n(z)$ is maximal. By passing to a subsequence, we can assume that $\tilde{M}_n(p_n) = \tilde{M}_n$ tends to $+\infty$.

Set $M_n = \|df_n(p_n)\|$ and consider $g_n(u) = f_n \circ \varphi\left(\frac{u}{M_n}\right)$, where π is a chart near p_n. This is defined for $|u| \leq C\tilde{M}_n$. Then, as in proposition 5.1.2, by passing to a subsequence we can assume that it converges to a holomorphic sphere g.

Using the metric $|du|/(1+|u|^2)$ on $\mathbf{CP}^1$, we see that $\|dg_n(u)\| \leq C/|u|^2$ for some constant C. Thus, after taking a subsequence we have $\|dg_n(u)\| \leq 2C/n^2$ for $|u| \leq n$. We can also assume $\operatorname{area}(g_n(D(n))) \to \operatorname{area}(g)$ and $\varepsilon_n = n/M_n \to 0$.

Thus the length of $g_n(\partial D(n)) = f_n(\partial B(p_n, \varepsilon_n))$ is $2\pi n \cdot 2C/n^2$ which tends to 0. Moreover the area of $f_n(U_n \cup B(p_n, \varepsilon_n))$ has diminished at least by a_0'. Therefore, we can apply the induction hypothesis to

$$\tilde{f}_n = f_n \mid S \setminus \partial B(p_n, \varepsilon_n).$$

Case 2. — Now assume that (j_n) goes to infinity in the space $\mathcal{J}(S)/\operatorname{Diff} S$ (moduli space). We shall use induction on the maximal genus of a component of S. The hypothesis on j_n implies that we can find, in a fixed nonzero homotopy class, a j_n-conformal annulus A_n of large module and whose generator curves have small μ_n-length (Mahler's compactness theorem, see [8] and chapter VIII). Since U_n is a union of s small balls, the annulus A_n can be found in $S \setminus U_n$.

By the length-area argument we can find a meridian γ_n of A_n such that the length of $f_n(\gamma_n)$ tends to 0 and separating A in two annuli of large modulus. Then we can apply the induction hypothesis to $S \setminus (U_n \cup \gamma_n)$.

Appendix: Stokes' theorem for forms with differentiable coefficients

A.1. Statement

THEOREM A.1.1. — *Let α be a differential form of degree $n-1$ defined on a compact domain $B \subset R^n$ with a piecewise C^1 boundary and differentiable except at a finite number of points. Assume that one of the two following properties hold:*

(i) *$d\alpha$ is Lebesgue integrable on B*

(ii) *$d\alpha \geq -C > -\infty$, where $d\alpha$ is identified with a function (this implies that the Lebesgue integral of $d\alpha$ is well defined as an element of $\mathbf{R} \cup \{+\infty\}$).*

Then we have $\int_B d\alpha = \int_{\partial B} \alpha$.

For C a positive constant, we say that a domain $U \subset \mathbf{R}^n$ is *C-fat* if $\operatorname{area} U \cdot \operatorname{diam} U \leq C \operatorname{vol} U$. The proof will use the following lemma.

LEMMA A.1.2. — *For every $x \in B$ there exists $\delta(C) > 0$ such that for every C-fat domain $U \subset B(x, \delta)$ one has*

$$\int_U h \leq \int_{\partial U} \alpha.$$

A.2. Proofs

The proof of A.1.1 generalises the one of [10] p.180 who proves the result in case (i) for $n = 1$. I am indebted for this proof to Bruno Sévennec.

It is easy to see that it is enough to prove it in the case where the coefficients are everywhere differentiable. Fix $\varepsilon > 0$. There exists an upper semicontinuous function h which satisfies $h < d\alpha$ and

$$\int_B h > \int_B d\alpha - \varepsilon \text{ (or } > \frac{1}{\varepsilon} \text{ if} \int_B d\alpha = +\infty). \tag{A.2.1}$$

For a suitable C we can find arbitrarily fine triangulations of B with C-fat n-simplices. In particular we can find one triangulation such that all n-simplices satisfy the conclusion of the lemma. By summing the inequalities $\int_{\sigma_i} h \leq \int_{\partial_{\sigma_i}} \alpha$ over all the n-simplices we have a cancellation of all the integrals on an interior $n-1$-simplex on the right, so we get

$$\int_B h \leq \int_{\partial B} \alpha.$$

Comparing this with (A.2.1) and since ε is arbitrary, we obtain

$$\int_B d\alpha \leq \int_{\partial B} \alpha$$

(notice that this inequality is all that we need for the regularity of holomorphic curves). Similarly, using a function g which is lower semicontinuous and $> d\alpha$ we can prove the reverse inequality $\int_B d\alpha \geq \int_{\partial B} \alpha$. This finishes the proof of Stokes theorem. □

Proof of the lemma. — Since h is upper semicontinuous and $h(x) < d\alpha(x)$, one has $h(y) < d\alpha(x) - \eta$ for some $\eta > 0$ and $|y - x| < \delta$ small enough. Thus

$$\int_U h \leq (d\alpha(x) - \eta)\,\mathrm{vol}(U). \tag{A.2.2}$$

On the other hand,

$$\alpha(y) = \alpha(x) + D\alpha(x).(y - x) + o(|y - x|).$$

Using the classical Stokes theorem and the fact that U is C-fat, we deduce

$$\int_{\partial U} \alpha = d\alpha(x)\,\mathrm{vol}(U) + o(\mathrm{vol}(U)).$$

Comparing this and (A.2.2) we obtain the lemma. □

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Chapter VI

Singularities and positivity of intersections of J-holomorphic curves

Dusa McDuff

with an appendix by Gang Liu

This chapter is devoted to proving some of the main technical results about J-holomorphic curves which make them such a powerful tool when studying the geometry of symplectic 4-manifolds. We begin by establishing some elementary local properties of these curves. Next, we develop enough of the theory of deformations of J-holomorphic curves to prove the following result in Gromov [2, 2.2.C_2] on the positivity of intersections of two curves in an almost complex 4-manifold.

THEOREM 2.1.1. — *Two closed distinct J-holomorphic curves C and C' in an almost complex 4-manifold (M, J) have only a finite number of intersection points. Each such point x contributes a number $k_x \geq 1$ to the algebraic intersection number $C \cdot C'$. Moreover, $k_x = 1$ iff the curves C and C' intersect transversally at x.*

Finally, we simplify and sharpen some of the work in [4] in order to prove that immersed curves are C^2-dense in the "big" moduli space $\mathcal{M}_s(A, \omega)$ of all curves which are J-holomorphic for some tame J.

COROLLARY 4.2.1. — *Immersed curves are dense in the "big" moduli space $\mathcal{M}_s(A, \omega) \subset \mathcal{X} \times \mathcal{J}$ with respect to the topology induced by the C^2-topology on $\mathcal{X}$ and the C^1-topology on $\mathcal{J}$.*

Although, as we shall see, the weaker density result proved in [4] is enough to establish the adjunction formula, the sharper form is needed in [6] in order to prove that evaluation maps for spaces of rational curves preserve orientation. Applications of these results to symplectic 4-manifolds are described, for example, in [5, 6].

The proofs given are adaptations of the arguments in Nijenhuis and Woolf's fundamental paper [9], which to my knowledge is the first paper to discuss J-holomorphic curves for non-integrable J. They are quite elementary and are complete, except that some estimates are quoted from [9] without proof.

I wish to thank Dietmar Salamon for his many useful comments on the first section, and Gang Liu for a very careful reading of the text.

1. Elementary properties

1.1. J-holomorphic curves

Let (M, J) be an almost complex manifold and (Σ, j) be a Riemann surface. A smooth map $u : \Sigma \to M$ is called *J-holomorphic* if the differential du is a complex linear map with respect to j and J:

$$J \circ du = du \circ j.$$

There are many ways of writing this equation. It will be most convenient for us to use the form derived by Nijenhuis and Woolf in [9], which writes it in terms of the usual delbar operator on $\mathbf{C}$.

By using appropriate charts on Σ and on M, we may suppose that the domain of u is a closed disc $D = D(R)$ centered at 0 and of radius R in $\mathbf{C}$, and that the range M is $\mathbf{C}^n$ equipped with a C^∞ almost complex structure J which equals the standard structure J_0 at the origin $\{0\} = (0, \dots, 0)$. We will write $u_i, \bar{u}_i, i = 1, \dots, n$, for the component functions of u together with their conjugates, and ∂ and $\bar{\partial}$ for the usual del and delbar operators on $\mathbf{C}^n$. We will consider maps $u : D \to \mathbf{C}^n$ such that $u(0) = 0$.

LEMMA 1.1.1. — *u is J-holomorphic if and only if*

$$\bar{\partial} u_i + A_{im}(u(z)) \bar{\partial} \bar{u}_m = 0, \tag{1.1.2}$$

where, for each $w \in \mathbf{C}^n$, $A_{im}(w)$ is a certain $n \times n$ complex-valued matrix, which is made from the entries of $J(w)$ and which vanishes when $J(w) = J_0$. Thus $A_{im}(u(0)) = A_{im}(0) = 0$ for all i, m.

Proof. — Consider the complexified cotangent bundle $V = T^\star \mathbf{C}^n \otimes \mathbf{C}$, where $\mathbf{C}^n$ has its usual complex structure J_0. Each fiber $V_w, w \in \mathbf{C}^n$ of this bundle has two commuting almost complex structures, one which we will call i induced by multiplication by $\sqrt{-1}$ in $\mathbf{C}$, and the other which we will call J_0 induced by the action of J_0 on $T(\mathbf{C}^n)$. We think of the former as the basic complex structure, and consider the latter to be a complex linear automorphism of the complex vector space (V_w, i). Then each V_w decomposes as ${V_w}^{1,0} \oplus {V_w}^{0,1}$, where ${V_w}^{1,0}$ is the $+i$ eigenspace of J_0 and ${V_w}^{0,1}$ is the $-i$ eigenspace. Correspondingly, V decomposes into a sum $V^{1,0} \oplus V^{0,1}$, where $V^{1,0}$, respectively $V^{0,1}$, is the space of complex valued forms with J_0-type $(1,0)$, respectively $(0,1)$.

Here is an explicit formula for this decomposition. If $w_j = x_j + iy_j, j = 1, \dots, n$ are the coordinates of $\mathbf{C}^n$, then J_0 acts by:

$$J_0\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \quad J_0\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j},$$

so that the transpose action on the cotangent space is

$$J_0(dx_j) = -dy_j, \quad J_0(dy_j) = dx_j.$$

Thus

$$J_0(dw_j) = J_0(dx_j + idy_j) = -dy_j + idx_j = idw_j,$$

which implies that $dw_j \in V^{1,0}$. Thus the summands $V^{1,0}$ and $V^{0,1}$ have bases $dw_1, \ldots, dw_n$, and $d\bar{w}_1, \ldots, d\bar{w}_n$. For short, we will denote the element $\sum_k a_k dw_k + b_k d\bar{w}_k$ of V as $adw + bd\bar{w}$ where $a, b \in \mathbf{C}^n$, and correspondingly will describe automorphisms X of V by means of 2×2 block matrices of the form:

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.$$

V has a similar decomposition $V = V_J^{1,0} \oplus V_J^{0,1}$ with respect to the almost complex structure J. It is easy to check that $P = \frac{1}{2}(\mathrm{Id} - iJ)$ and $\bar{P} = \frac{1}{2}(\mathrm{Id} + iJ)$ are the corresponding projections onto the $\pm i$ eigenspaces of J. Note that $P(\mathrm{Id} - P) = 0 = P\bar{P}$. Thus the space of complex valued 1-forms on $(\mathbf{C}^n, J)$ of J-type $(1,0)$ is spanned by the forms represented by the columns of P, namely:

$$Q_1 dw + Q_3 d\bar{w}, \qquad Q_2 dw + Q_4 d\bar{w},$$

where Q_k denotes the transpose $(P_k)^t$ of P_k. It is well-known that a map $u : D \to \mathbf{C}^n$ is J-holomorphic if and only if it pulls the 1-forms of J-type $(1,0)$ back to 1-forms of type $(1,0)$ on D. Hence we require that the 1-forms

$$\begin{aligned} u^\star(Q_1 dw + Q_3 d\bar{w}) &= Q_1(u)du + Q_3(u)d\bar{u} \\ u^\star(Q_2 dw + Q_4 d\bar{w}) &= Q_2(u)du + Q_4(u)d\bar{u} \end{aligned}$$

have type $(1,0)$ on D. Since the type $(0,1)$ part of the 1-form du on D is $\bar{\partial} u$ this is equivalent to the equations

$$Q_1 \bar{\partial} u + Q_3 \bar{\partial} \bar{u} = 0, \qquad Q_2 \bar{\partial} u + Q_4 \bar{\partial} \bar{u} = 0.$$

We claim that the second of these equations follows from the first. To see this, observe that the equation $P\bar{P} = 0$ implies:

$$P_1 \bar{P}_2 + P_2 \bar{P}_4 = 0, \quad P_3 \bar{P}_2 + P_4 \bar{P}_4 = 0.$$

By the definition of $\bar{P}$, $\bar{P}_4 = \mathrm{Id}$ at $w = 0$. Thus P_4 is invertible near $\{0\}$, and if $B = -\bar{P}_2(\bar{P}_4)^{-1}$,

$$P_2 = P_1 B, \qquad P_4 = P_3 B.$$

Thus

$$Q_2 = P_2^t = B^t \bar{Q}_1, \qquad Q_4 = B^t \bar{P}_3^t,$$

showing that the second equation is B^t times the first. Moreover, by multiplying the first equation by $(Q_1)^{-1}$, one gets an equation of the required form. Note that $A(0) = 0$ because $Q_2(0) = 0$. □

Remark. — Nijenhuis and Woolf show by a simple calculation that all the components P_k of P can be written in terms of A (they use the fact that J is real, which implies $J_1 = J_4, J_2 = J_3$). Conversely, given any matrix A of complex valued functions such that $(1 - A\bar{A})$ is invertible, they construct a corresponding P and hence J. Thus, for such A, every solution of equation (1.1.2) is J-holomorphic for an appropriate J.

COROLLARY 1.1.3. — *Any map u which satisfies equation* (1.1.2) *is a solution of a system of equations of the form:*

$$\partial\bar{\partial}u_i = \psi_i(u_m, \partial_j u_m, \bar{\partial}_j u_m) \quad i = 1, \dots n,$$

where $\psi_i = 0$ when all its arguments equal zero (here ∂_j denotes the partial derivative $\partial/\partial z_j$, etc).

Proof. — In shorthand notation we may write equation (1.1.2) as:

$$\bar{\partial}u + A\bar{\partial}\bar{u} = 0.$$

Thus $\partial\bar{\partial}u + A\partial\bar{\partial}\bar{u}$ is a function of u and its first derivatives. By taking conjugates, we see that $\partial\bar{\partial}\bar{u} + \bar{A}\partial\bar{\partial}u$ is too. Therefore, substituting for $\partial\bar{\partial}\bar{u}$ in the first equation, we find $(1 - A\bar{A})\partial\bar{\partial}u$ also has this property. The result now follows by multiplying by $(1 - A\bar{A})^{-1}$. □

Here is a useful consequence of this result.

LEMMA 1.1.4. — *If two J-holomorphic curves $u, u' : \Sigma \to (M, J)$ have the same ∞-jet at a point z of a connected Riemann surface Σ, then $u = u'$.*

Proof. — Since Σ is connected, it is enough to prove this locally. Hence, by the above corollary, we may assume that u and u' are solutions of the following system of equations on D:

$$\bar{\partial}f_i = \psi_i(f_m, \partial_j f_m, \bar{\partial}_j f_m) \quad i = 1, \dots n,$$

and that $g = u - u'$ vanishes to infinite order at $\{0\} \in D$. Because g and its derivatives are bounded on D, it is easy to check that g satisfies differential inequalities of the form:

$$|\partial\bar{\partial}g_i(z)|^2 \leq K \sum_m (|g_m|^2 + |\partial g_m|^2 + |\bar{\partial}g_m|^2),$$

for $i = 1, \dots, n$ and all $z \in D$. The desired conclusion now follows from Aronszajn's strong unique continuation theorem which we now quote:

THEOREM 1.1.5 (Aronszajn). — *Suppose that $g \in W^{2,2}_{\mathrm{loc}}(D, \mathbf{R}^m)$ satisfies the pointwise estimate*

$$|\Delta g(s,t)| \leq c\left(|g(s,t)| + \left|\frac{\partial g}{\partial s}(s,t)\right| + \left|\frac{\partial g}{\partial t}(s,t)\right|\right)$$

(almost everywhere) where Δ is the Laplace operator. Suppose further that g vanishes to infinite order at the point $z = 0$ in the sense that

$$\int_{|z|\leq r} |g(z)| = O(r^k)$$

for every $k > 0$. Then $g \equiv 0$.

Here $W^{2,2}_{\mathrm{loc}}(D, \mathbf{R}^m)$ is the Sobolev space of maps from D to $\mathbf{R}^m$ whose second derivative is L^2 on all precompact open subsets of D. Since all the maps which we consider here are smooth, a reader who is unfamiliar with Sobolev spaces can suppose that g is $\mathcal{C}^\infty$. This theorem can be viewed as a generalisation of the unique continuation theorem for analytic functions. It is proved by Aronszajn in [1] and by Hartman and Wintner in [3].

1.2. Critical points

A *critical point* of a J-holomorphic curve $u : \Sigma \to M$ is a point $z \in \Sigma$ such that $du(z) = 0$. If we think of the image $C = u(\Sigma) \subset M$ as an unparametrised J-holomorphic curve, then a point $x \in C$ is said to be *critical* if it is the image of a critical point of u. Points on C which are not critical are called *regular* or *non-singular*. In the integrable case critical points of nonconstant holomorphic curves are obviously isolated. The next proposition asserts this for arbitrary almost complex structures.

LEMMA 1.2.1. — *Let $u : \Sigma \to M$ be a nonconstant J-holomorphic curve, for some compact connected Riemann surface Σ. Then the set*

$$X = \{z \in \Sigma \mid du(z) = 0\}$$

of critical points is finite.

Proof. — It suffices to prove that critical points are isolated, and so we may work locally. Thus we may suppose that Ω is an open neighbourhood of the origin $\{0\}$ in $\mathbf{C}$, and that the map u from $(\Omega, \{0\})$ to $(\mathbf{C}^n, \{0\})$ is J-holomorphic for some $J : \mathbf{C}^n \to \mathrm{GL}(2n, \mathbf{R})$. Moreover, we may assume that J is standard, that is equal to J_0, at the point $\{0\}$. Thus

$$u(0) = 0, \qquad du(0) = 0, \qquad u \not\equiv 0, \qquad J(0) = J_0.$$

Write $z = s + it$. Since u is non-constant it follows from lemma 1.1.4 that the ∞-jet of $u(z)$ at $z = 0$ must be non-zero. Hence there exists an integer $k \geq 2$ such that $u(z) = O(|z|^k)$ and $u(z) \neq O(|z|^{k+1})$. This implies $A_{im}(u(z)) = O(|z|^k)$. Now examine the Taylor expansion of the equation

$$\bar{\partial} u_i + A_{im}(u(z)) \bar{\partial} \bar{u}_m = 0,$$

up to order $k-1$ to obtain

$$\bar{\partial} T_k(u) = 0,$$

where $T_k(u)$ denotes the Taylor expansion of u of order up to and including k. It follows that $T_k(u) : \mathbf{C} \to \mathbf{C}^n$ is a holomorphic function and there exists a nonzero vector $a \in \mathbf{C}^n$ such that

$$u(z) = az^k + O(|z|^{k+1}), \qquad \frac{\partial u}{\partial s}(z) = kaz^{k-1} + O(|z|^k).$$

Hence

$$0 < |z| \leq \varepsilon \qquad \Longrightarrow \qquad u(z) \neq 0, \quad du(z) \neq 0$$

with $\varepsilon > 0$ sufficiently small. This proves the lemma. □

We now show how to choose nice coordinates near a regular point of a J-holomorphic curve.

LEMMA 1.2.2. — *Let $\Omega \subset \mathbf{C}$ be an open neighbourhood of $\{0\}$ and $u : \Omega \to M$ be a local J-holomorphic curve such that $du(0) \neq 0$. Then there exists a chart $\alpha : U \to \mathbf{C}^n$ defined on a neighbourhood of $u(0)$ such that*

$$\alpha \circ u(z) = (z, 0, \ldots, 0), \qquad d\alpha(u(z))J(u(z)) = J_0 d\alpha(u(z))$$

for $z \in \Omega \cap u^{-1}(U)$.

Proof. — Write $z = s + it \in \Omega$ and $w = (w_1, w_2, \ldots, w_n) \in \mathbf{C}^n$ where $w_j = x_j + iy_j$. Shrink Ω if necessary and choose a complex frame of the bundle $u^\star TM$ such that

$$Z_1(z), \ldots, Z_n(z) \in T_{u(z)}M, \qquad Z_1 = \frac{\partial u}{\partial s}.$$

Define $\varphi : \Omega \times \mathbf{C}^{n-1} \to M$ by

$$\varphi(w_1, \ldots, w_n) = \exp_{u(w_1)} \left(\sum_{j=2}^{n} x_j Z_j(w_1) + \sum_{j=2}^{n} y_j J(u(w_1)) Z_j(w_1) \right)$$

where exp denotes the exponential map with respect to some metric on M. Then φ is a diffeomorphism of a neighbourhood V of zero in $\mathbf{C}^n$ onto a neighbourhood U of $u(0)$ in M. It satisfies $\varphi(z_1, 0, \ldots, 0) = u(z_1)$ and

$$\frac{\partial \varphi}{\partial x_j} + J(\varphi) \frac{\partial \varphi}{\partial y_j} = 0$$

at all points $z = (z_1, 0, \ldots, 0)$ and $j = 1, \ldots, n$. Hence the inverse $\alpha = \varphi^{-1} : U \to V$ is as required. □

1.3. Intersections

We next start investigating intersections of two distinct J-holomorphic curves. The most significant results in this connection occur in dimension 4, and will be discussed in the next section. For now, we prove a useful result which is valid in all dimensions asserting that intersection points of distinct J-holomorphic curves $u : \Sigma \to M$ and $u' : \Sigma' \to M$ can only accumulate at points which are critical on both curves $C = u(\Sigma)$ and $C' = u'(\Sigma')$. For local J-holomorphic curves this statement can be reformulated as follows.

LEMMA 1.3.1. — *Let $u, v : \Omega \to M$ be local non-constant J-holomorphic curves defined on an open neighbourhood Ω of 0 such that*

$$u(0) = v(0), \qquad du(0) \neq 0.$$

Moreover, assume that there exist sequences z_ν, $\zeta_\nu \in \Omega$ such that

$$u(z_\nu) = v(\zeta_\nu), \qquad \lim_{\nu\to\infty} z_\nu = \lim_{\nu\to\infty} \zeta_\nu = 0, \qquad \zeta_\nu \neq 0 \neq z_\nu.$$

Then there exists a holomorphic function $\varphi : \Omega' \to \Omega$ defined in some neighbourhood of zero such that $\varphi(0) = 0$ and $v = u \circ \varphi$.

Proof. — By lemma 1.2.2 we may assume without loss of generality that $M = \mathbf{C}^n$ and

$$u(z) = (z, 0), \qquad J(w_1, 0) = i$$

where $w = (w_1, \tilde{w})$ with $\tilde{w} \in \mathbf{C}^{n-1}$. Write $v(z) = (v_1(z), \tilde{v}(z))$.

We show first that the ∞-jet of $\tilde{v}$ at $z = 0$ must vanish. Otherwise there would exist an integer $\ell \geq 0$ such that $\tilde{v}(z) = O(|z|^\ell)$ and $\tilde{v}(z) \neq O(|z|^{\ell+1})$. The assumption of the lemma implies $\ell \geq 1$ and hence $J(v(z)) = J_0 + O(|z|^\ell)$. As in the proof of lemma 1.2.1, consider the Taylor expansion up to order $\ell - 1$ on the left hand side of the equation $\bar{\partial} v + A(v)\bar{\partial}\bar{v} = 0$ to obtain that $T_\ell(v)$ is holomorphic. Hence

$$v_1(z) = p(z) + O(|z|^{\ell+1}), \qquad \tilde{v}(z) = \tilde{a} z^\ell + O(|z|^{\ell+1})$$

where $p(z)$ is a polynomial of order ℓ and $a \in \mathbf{C}^{n-1}$ is nonzero. This implies that $\tilde{v}(z) \neq 0$ in some neighbourhood of 0 and hence $u(z) \neq v(z)$ in this neighbourhood, in contradiction to the assumption of the lemma. Thus we have proved that the ∞-jet of $\tilde{v}$ at $z = 0$ vanishes.

We prove that $\tilde{v}(z) \equiv 0$. To see this note that, because $J = J_0$ along the axis $\{\tilde{w} = 0\}$, $A(w) = 0$ there, and so

$$\frac{\partial A(w_1, 0)}{\partial x_1} = \frac{\partial A(w_1, 0)}{\partial y_1} = 0,$$

for all w_1 (here A is as in lemma 1.1.1). Hence

$$\left|\frac{\partial A(w)}{\partial x_1}\right| + \left|\frac{\partial A(w)}{\partial y_1}\right| \leq c\,|\tilde{w}|,$$

and hence

$$|\partial_\zeta A(w)| \le c\left(|\tilde{w}| + \left|\tilde{\zeta}\right|\right).$$

By arguing as in corollary 1.1.3, one easily sees that

$$|\partial\bar{\partial}\tilde{v}| \le c\left(|\tilde{v}| + |\partial_s \tilde{v}|\right).$$

Hence it follows from Aronszajn's theorem that $\tilde{v} \equiv 0$. The required function φ is now given by $\varphi(z) = v_1(z)$. □

2. Positivity of intersections

2.1. The main theorem

Here is the statement of the main theorem.

THEOREM 2.1.1. — *Two closed distinct J-holomorphic curves C and C' in an almost complex 4-manifold (M, J) have only a finite number of intersection points. Each such point x contributes a number $k_x \ge 1$ to the algebraic intersection number $C \cdot C'$. Moreover, $k_x = 1$ iff the curves C and C' intersect transversally at x.*

The proof of this theorem has two ingredients: some topology and a deformation result (proposition 2.1.2). The necessary topology is only sketched: interested readers can refer to [4] for more detail. The deformation result is proved in detail in §3 because the proof given in [4] can be simplified by using the approach of [7]. We also take the opportunity to correct some minor mistakes in [4, 7].

Step 1: Definition of k_x and k_U. — When x is an isolated point of intersection, the number k_x may be calculated as follows:

(i) choose closed neighbourhoods $U \subset C$ and $U' \subset C'$ of x which intersect only at x;

(ii) perturb U keeping ∂U disjoint from U' and $\partial U'$ disjoint from U to make U intersect U' in transverse double points; and

(iii) count the number of these points with sign according to the orientation.

It is not hard to check that this number k_x is well-defined, i.e. independent of all choices. Indeed, it is a basic topological fact that the algebraic intersection number of U and U' does not change under deformations which satisfy condition (ii) above.

When x is not isolated, one cannot define k_x in this way since it is impossible to satisfy condition (i). However, by the results in the previous section, we know that the only accumulation points of $C \cap C'$ are critical on both C and C' and that critical points are isolated. Therefore, there is a closed neighbourhood U of x in C whose boundary ∂U does not meet C'. We may also suppose that x is the only critical point on U and that U does not meet C' in any critical point other than x.

Choose a closed neighbourhood U' of x in C' such that $U \cap C' = U \cap \operatorname{Int} U'$. Then, if we perturb U in such a way that condition (ii) continues to hold, the intersection number of U with U' does not change, and so, given a perturbation which intersects U' in transverse double points, we may define k_U, as before, by counting these points according to orientation. Note that this number may depend on the choice of U but not on the choice of U'. Moreover, if x is regular on one of the curves, x is an isolated point of intersection and so $k_U = k_x$ for sufficiently small neighbourhoods U.

Step 2: Proof of the theorem when x is a regular point of C. — By lemma 1.3.1 we may identify a neighbourhood of x in M with a neighbourhood V of $\{0\}$ in $\mathbf{C}^2$ in such a way that $J = J_0$ at $\{0\}$, C is given by $z \mapsto (z, 0)$ and C' is parametrised by a function of the form $z \mapsto (f(z), az^k + O(k+1))$. Suppose further that $\{0\}$ is the only intersection point of C and C' in V. Then their boundary curves are disjoint, and so remain disjoint under small perturbations. It follows that if V and $\varepsilon > 0$ are both sufficiently small, we may perturb C' first to the image of the map $z \mapsto (f(z), az^k)$ and then to the image of the map $z \mapsto (f^1(z), \varepsilon + az^k)$ without changing the algebraic number of intersection points of this curve with C. An easy calculation now shows that this intersection number is k.

This proves the theorem when x is regular on one of the curves. We now have to make sure that if x is critical on both curves it is not possible to perturb C and C' near x so as to create lots of points which contribute negatively to $C \cdot C'$. This bad behavior is ruled out in the next step.

Step 3: Proof that $k_U \geq 0$. — Observe first that it suffices to prove this for any sufficiently small neighbourhood of x in C, where x is critical on both curves. For, by step 2, k_U increases when U is enlarged to include more points of intersection which are non-singular on at least one of the curves. Therefore we may suppose that x is the only singular point on C and C'. The argument is based on the following proposition.

PROPOSITION 2.1.2. — *If u is a J-holomorphic map $(D(R), 0) \to (\mathbf{C}^2, (0,0))$ and if $\gamma(t)$ is any arc beginning at $\gamma(0) = (0,0)$, there is, for some $R' < R$ and some $t_0 > 0$ a smooth family $u_t : (D(R'), 0) \to (\mathbf{C}^2, \gamma(t)), |t| \leq t_0$, of local J-holomorphic curves in $\mathbf{C}^2$ with $u_0 = u$. If u is critical at $z = 0$, we may assume that each u_t is also critical at $z = 0$. Further, given $u = u_0$, we may choose $\gamma(t)$ so that the disc $u_t(D(R'))$ does not go through $\gamma(0)$ for $t \neq 0$.*

Note that here we have written $D(R)$ for the closed disc of radius R centered at 0 in $\mathbf{C}$. This result is a deformation result: it says that local J-holomorphic curves through the point x persist as the point x moves, even if x is a singular point.

This proposition is proved in the next section. To apply it here, let us suppose that our two curves $C = \operatorname{Im} u$ and $C' = \operatorname{Im} u'$ meet at $(0,0) \in \mathbf{C}^2$. Choose the arc $\gamma(t)$ so that there is a deformation $u_t : D(R') \to \mathbf{C}^2$ of u which does not go through

$(0,0)$, when $t > 0$. Further, choose $U = u(D(R_1))$, for some $R_1 \leq R'$, so that its boundary is disjoint from C', and choose a suitable U'. Then, for small t, the deformation $U_t = u_t(D(R_1))$ satisfies condition (ii) above, and k_U is the intersection number of $u_t(D(R'))$ with C' for small $t \neq 0$. Since we have arranged that all of the points in this intersection are non-singular on C', this intersection number is ≥ 0 by step 2. Note that it may be 0 since $u_t(D(R'))$ might be disjoint from U'.

Step 4: Proof of the theorem. — We first show that $C \cap C'$ is finite. For suppose not, let N be the intersection number $C \cdot C'$ and let $\{x_1, \ldots, x_k\}$ be the finite set of points in $C \cap C'$ which are singular on both curves. (Observe that N is finite because it is the intersection number of the homology classes represented by C and C'.) Then there are infinitely many points in $C \cap C'$ which are regular on at least one of the curves, and so there is a neighbourhood U of $\{x_1, \ldots, x_k\}$ in C such that $(C-U) \cap C'$ contains more than N points. Since $k_x \geq 1$ for all $x \in (C-U) \cap C'$, we must have $(C-U) \cdot C' > N$. But then

$$N = C \cdot C' = k_U + (C-U) \cap C' \geq (C-U) \cap C' > N,$$

which is absurd.

Since $C \cap C'$ is finite, all points of intersection are isolated, and k_x is defined for all points $x \in C \cap C'$. Suppose that $x \in C \cap C'$ is singular on both curves. We must show that $k_x > 1$. Choose $R > 0$ so that $u(D(R))$ meets C' only at $x = u(0)$, choose an arc $\gamma \subset C'$ through x, and deform $C = \operatorname{Im} u$ along γ as in proposition 2.1.2. We may assume that, for small t, u_t is critical at $z = 0$. Then k_x may be calculated by counting up the intersection points of $\operatorname{Im} u_t|_{D(R)}$ with C'. By step 2, the contribution from $y = u_t(0) \in C'$ is > 1. Since all other points of intersection contribute a number which is ≥ 0 by step 3, the total intersection number is > 1 as required. □

It remains to prove proposition 2.1.2. This will follow (in § 3.3) by using the methods which Nijenhuis and Woolf developed in [9] to prove local existence for J-holomorphic curves and the integrability theorem for almost complex structures J. Our presentation will follow [7], and is simpler than that in [4] in that we do not use branched covers.

2.2. The adjunction formula

The other main result about singularities which is needed for applications to 4-dimensional symplectic geometry is the *adjunction formula*, which gives a useful numerical criterion for a curve to be embedded. Recall that if C is the image of a J-holomorphic map from a Riemann surface of genus g to V, then the *virtual genus* of C is defined to be the number

$$g(C) = 1 + \frac{1}{2}(C \cdot C - c_1(C))$$

where c_1 is the first Chern class of the complex vector bundle (TV, J).

THEOREM 2.2.1 (the adjunction formula). — *If C is the image of a somewhere injective J-holomorphic map $u : \Sigma \to M$, the virtual genus $g(C)$ is an integer. Moreover,*

$$g(C) \geq g = \text{genus}\,\Sigma,$$

with equality if and only if C is embedded.

A proof is sketched at the end of §4. Roughly speaking, this theorem says that each singular point x of a J-holomorphic curve contributes a positive number k_x to the self-intersection number. It is harder to prove than positivity of intersections because it involves only one curve. Indeed, it is even quite hard to measure k_x (this is, of course, the intersection number of a neighbourhood U of x with some perturbation of U, but one has to be careful about prescribing how the boundary of U is allowed to move since this affects the number of intersections). If U is a topologically embedded neighbourhood of x in C, we may define k_x to be the self-intersection number of a perturbation U' of U which is immersed and is equal to U near its boundary. However, this definition is not suitable in our context since the perturbations U' which we use are J-holomorphic and so never equal U near the boundary. A full discussion may be found in [4, §4].

In [7] the methods developed in the next section are pushed further to give a complete description of the topological type of a J-holomorphic singularity. It is shown that no new singularities occur in the non-integrable case. In particular, in dimension 4 a neighbourhood of a singular point is homeomorphic to a cone over an algebraic knot in S^3. (Another proof of this, one which is valid under weaker smoothness assumptions on J, may be found in [8].) It is easy to see that this result also implies the adjunction formula. However, it does not imply corollary 4.2.1 on the density of immersed curves, since the methods in [7] are purely local: they allow us to deform C to a curve C' which is non-singular at the given point x, but they give no control over the size of the neighbourhood on which C' is non-singular.

3. Local deformations

3.1. An integral equation

Nijenhuis and Woolf establish local existence of J-holomorphic curves by transforming equation (1.1.2) into an integral equation which involves the usual Cauchy kernel S'_R and the usual Green's operator T'_R for $\bar\partial$ on $D = D(R)$. Thus, given a complex valued function f, we define

$$S'_R(f) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

and

$$T'_R f(z) = \frac{1}{2\pi i} \iint_D \frac{f(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta}.$$

In order that our maps all preserve 0, it will be convenient also to define

$$\begin{aligned} S_R(f) &= S'_R(f) - S'_R f(0), \quad \text{and} \\ T_R(f) &= T'_R(f) - T'_R f(0). \end{aligned}$$

LEMMA 3.1.1.

(i) *If* $f : (D(R), 0) \to (\mathbf{C}, 0)$ *is* C^1*-smooth then:*

(a) $\bar{\partial} S_R(f) = 0$ *and* $\bar{\partial} T_R(f) = f$;

(b) $f = S_R f + T_R \bar{\partial} f$;

(c) $S_R S_R(f) = S_R(f)$ *and* $S_R T_R(f) = 0$.

(ii) *If* f *is* C^∞, *then for each* $k \geq 0$ *there is a constant* c, *which depends on* k, R_0 *and* f, *such that*

$$\|S_R f\|_{k:R} \leq c \|f\|_{k+1:R}, \quad \textit{for all} \quad R \leq R_0,$$

where $\|\cdot\|_{k:R}$ *denotes the* C^k*-norm on* $D(R)$.

Proof. — It is well-known that the properties in (i) hold for S'_R and T'_R: see, for example, [9]. Therefore, they also hold for S_R and T_R. (ii) is also elementary, and may be proved by applying the usual integral formula to evaluate the k-th derivative of $S_R(f)$, where f is expressed as a polynomial of degree k plus $(k+1)$st order remainder. □

Remark. — We will denote the ∞-jet of f at $z = 0$ by $j^\infty(f)$, and will consider it to be a power series (with complex coefficients) in z and $\bar{z}$. Its "holomorphic part" consists of all terms which involve powers of z alone and will be called $j^\infty_h(f)$. When f is real analytic in D, the formal power series $j^\infty(f)$ is convergent in D with sum $f(z)$. In this case, it is easy to describe S_R. Namely[1], if

$$f = \sum_{m,p \geq 0} a_{mp} z^m \bar{z}^p,$$

then

$$S_R f = \sum_{m \geq p \geq 0} a_{mp} R^{2p} z^{m-p}.$$

Thus S_R depends on R. Further, as $R \to 0$, the coefficient of each term in $j^\infty(S_R(f))$ tends to the coefficient of the corresponding term in $j^\infty_h(f)$. In the general case, we may think of $S_R(f)$ as the function on $D(R)$ given by the non-negative part of the Fourier series of $f|_{\partial D(R)}$.

[1]The following sentences correct Note 1.2 in [7]. I am grateful to Gang Liu for pointing out this error.

Following [9], we consider the norm $\|\cdot\|'_R$ which is defined as follows. For f in the space $\mathcal{C}_0^{1+\alpha}(D(R))$ of functions $(D(R),0) \to (\mathbf{C},0)$ which are $\mathcal{C}^{1+\alpha}$-smooth, set

$$\|f\|'_R = \max\{\|\partial f\|_R, \|\bar\partial f\|_R\}, \quad \|f\|_R = |f|_R + (2R)^\alpha H_\alpha[f].$$

Here $H_\alpha[f]$ is the usual Hölder norm of exponent $\alpha, 0 < \alpha < 1$, and $|f|_R = \sup\{|f(z)| : z \in D(R)\}$. Since $f(0) = 0$, the value of $\|f\|'_R$ does bound $|f|_R$. In fact, it is not hard to see that

$$|f|_R \leq \|f\|_R J \leq 6R\|f\|'_R.$$

(See [9] 7.1c.)

Remark. — If $u = (u_1, u_2)$ maps $(D,0)$ to $(\mathbf{C}^2,(0,0))$, we extend all these definitions componentwise. Further, to be consistent with [9], we define $\|u\|'_R = \sup\|u_i\|'_R$, etc. □

Given $u : (D(R),0) \to (\mathbf{C}^2,(0,0))$ set

$$\begin{aligned} \theta'_i(u) &= -T_R\left(\sum_m A_{im}(u(z))\bar\partial\bar u_m\right), \quad \text{and} \\ \theta_i(u) &= \theta'_i(u) - z\left(\frac{\partial}{\partial z}\theta'_i(u)(0)\right), \quad \text{for} \quad i = 1,2, \end{aligned}$$

and set

$$\theta(u) = (\theta_1(u), \theta_2(u)).$$

LEMMA 3.1.2. — *$d\theta(u)(0) = 0$ when $u(0) = (0,0)$.*

Proof. — Because $\bar\partial T_R(f) = f$ for all f and because $A_{im}((0,0)) = 0$, one easily checks that $\bar\partial T_R(A(u)\bar\partial\bar u)$ vanishes at $z = 0$. Thus the linear terms of $T_R(A(u)\bar\partial\bar u)$ are holomorphic. □

We took the linear terms out of θ in order to gain control over the first derivatives of the J-holomorphic curve which we will construct. Thus, here we are following [9, 4] rather than [7].

Consider the following equation on $D = D(R)$:

$$\text{(3.1.3)} \qquad u(z) = \varphi(z) + \theta(u)(z), \quad u(0) = (0,0),$$

where $\varphi : (D,0) \to (\mathbf{C}^2,(0,0))$ is any J_0-holomorphic map, i.e. $\bar\partial\varphi = 0$. The next lemma shows that this integral equation is equivalent to the generalised Cauchy-Riemann equation (1.1.2).

LEMMA 3.1.4. — *Assume that J is an almost complex structure on $\mathbf{C}^2$ such that $J((0,0)) = J_0$ and that φ is J_0-holomorphic. If u solves equation* (3.1.3) *on $D(R)$ for a given φ then u is J-holomorphic. Conversely, every J-holomorphic map $u : (D(R),0) \to (\mathbf{C}^2,(0,0))$ is a solution of equation* (3.1.3) *for $\varphi = \varphi_R(u) = S_R(u) + z\left(\frac{\partial}{\partial z}\theta'(u)(0)\right)$.*

Proof. — The first statement follows by applying $\bar\partial$ to equation (3.1.3) and using lemma 3.1.1(a), equation (1.1.2) and the definition of θ. The converse is proved by substituting $-A\bar\partial\bar u$ for $\bar\partial u$ in lemma 3.1.1(b). □

3.2. Local existence results

We aim to show that for every J-holomorphic map u there is a constant R_1 such that u is uniquely determined by $\varphi(u)$ for all $R \le R_1$. We begin with the following proposition.

PROPOSITION 3.2.1. — *Suppose that $\varphi : (D(R_0), 0) \to (\mathbf{C}^2, (0,0))$ is a holomorphic map with $\|\varphi\|'_{R_0} \le K$. Then, if the functions A_{im} are $\mathcal{C}^\infty$, there is $R_1 \le R_0$ such that, for each $R \le R_1$, equation (3.1.3) has a unique solution u_R for which $u_R(0) = (0,0)$ and $\|u_R\|'_R \le 2K$. Moreover, this solution is $\mathcal{C}^\infty$ and depends smoothly on φ and on the component functions A_{im}.*

Proof. — Here is a sketch of the proof. Let $\mathbf{B} = \mathbf{B_R}$ be the Banach space consisting of all functions $h : (D(R), 0) \to (\mathbf{C}^2, (0,0))$ such that $\|h\|'_R < \infty$, with norm $\|\cdot\|'_R$, and let

$$\mathbf{A} = \{h \in \mathbf{B} : \|h\|'_R \le 2K\}.$$

By [9, (6.1.2,4)] there is a constant c_1 which is independent of R such that

$$\|T_R(u)\|'_R \le c_1 \|u\|_R \quad \text{for all} \quad u \in \mathcal{C}_0^{1+\alpha}(D(R)).$$

Using this, we find for $h \in \mathbf{B}$ that

$$\text{(3.2.2)} \qquad \begin{aligned} \|\theta'(h)\|'_R &= \|T_R(A_{im}(h)\bar\partial\bar h_m)\|'_R \le c_1 \|A_{im}(h)\bar\partial\bar h_m\|_R \\ &\le nc_1 \|A_{im}(h)\|_R \|h\|'_R. \end{aligned}$$

If $\|h\|'_R \le 2K$, then ∂h and $\bar\partial h$ are uniformly bounded by $2K$ and so h is uniformly bounded by $2KR$ on $D(R)$. Thus

$$\text{(3.2.3)} \quad \|A_{im}(h)\|_R = |A_{im}(h)|_R + 2R^\alpha H_\alpha[A_{im}(h)] \le c_2 |h|_R + c_3 R^\alpha \le c_4 R^\alpha,$$

for some constant c_4 which is independent of $R \le R_0$. Therefore

$$\|\theta'(h)\|'_R \le c_5 R^\alpha \|h\|'_R \quad \text{where} \quad c_5 = nc_1c_4,$$

which implies that

$$\text{(3.2.4)} \qquad \|\theta(h)\|'_R \le 2c_5 R^\alpha \|h\|'_R.$$

It is also shown in [9] that when $A_{im} \in \mathcal{C}^{1+\alpha}$ there is a constant c_6 which is independent of R such that

$$\text{(3.2.5)} \qquad \|A_{im}(h') - A_{im}(h)\|_R \le c_6 R \|h' - h\|_R$$

It is now not hard to show that

$$\|\theta(h') - \theta(h)\|'_R \le c_7 R^\alpha \|h' - h\|'_R,$$

where c_7 is independent of $R \le R_0$. (These last two steps are discussed further in the next section.) Therefore, if R_1 is chosen so that both $2c_5 R_1^\alpha$ and $c_7 R_1^\alpha$ are $\le \frac{1}{2}$, the map $h \mapsto \varphi + \theta(h)$ is a contraction mapping of $\mathbf{A}$. The first statement in the the proposition now follows easily. The claim that u_R is $\mathcal{C}^\infty$ holds by standard elliptic regularity (and is proved in [9]). The proof of the statement about the smoothness of the dependence on parameters may be found in [9, (5.4)]. See also the appendix, which treats the question of $\mathcal{C}^2$-dependence on parameters in the slightly more general situation considered in §4. □

Given φ, we will write $H_R(\varphi)$ for the solution u_R of equation (3.1.3) which is constructed above. Note that this solution will depend on R in general and exists only when $R \le R_1 = R_1(K, A_{im})$.

COROLLARY 3.2.6. — *If u is a J-holomorphic map $(D(R_0), 0) \to (\mathbf{C}^2, (0,0))$, then there is $R_1 \le R_0$ such that $H_R(\varphi_R(u))$ exists for all $R \le R_1$ and equals u.*

Proof. — This is almost obvious. We just have to check that given u, there is $R_0 > 0$ and $K > 0$ such that $\|\varphi_R(u)\|'_R \le K$ for all $R \le R_0$. For if this is so, we may increase K if necessary so that $\|u\|'_R \le 2K$ for all $R \le R_0$ and then conclude that $u = H_R(\varphi_R(u))$ for all $R \le R_0$ by the uniqueness statement in the above proposition. But $\varphi_R(u) = S_R(u) + z\left(\frac{\partial}{\partial z}\theta'(u)(0)\right)$, and both parts of this are uniformly bounded (for fixed u) as $R \to 0$: this holds for S_R by part (ii) of lemma 3.1.1, and for θ' by inequality (3.2.4). □

3.3. Local perturbations

The following result is the key to the proof of proposition 2.1.2.

THEOREM 3.3.1. — *Suppose that $J_\lambda, \lambda \in \Lambda$, is any family of $\mathcal{C}^\infty$-smooth almost complex structures on M which depends $\mathcal{C}^k$-smoothly on the parameter $\lambda \in \Lambda$. If, for some $\lambda_0 \in \Lambda$, $u_{\lambda_0} : D \to M$ is a J_{λ_0}-holomorphic curve through x, then one can find a neighbourhood N of λ_0 in Λ and an $R > 0$ such that u_{λ_0} extends to a family*

$$u_\lambda : (D(R), 0) \to (M, x)$$

of J_λ-holomorphic curves through x which depend $\mathcal{C}^k$-smoothly on $\lambda \in N$. If u_{λ_0} is singular at x, we may assume that the u_λ are also singular at x.

Moreover, there is a constant $c > 0$ which is independent of R such that

$$\|u_\lambda - u_{\lambda_0}\|'_R \le c\|J_\lambda - J_{\lambda_0}\|_{C^1}.$$

Proof. — Clearly, we can choose coordinates near x so that $u(0) = (0,0)$ and $J((0,0)) = J_0$. Further, by replacing J_λ by $(g_\lambda)_*(J_\lambda)$, for a suitable family of local diffeomorphisms g_λ, we may suppose that $J_\lambda(0) = J_0$. By corollary 3.2.6, $u_{\lambda_0} = H_R(\varphi_R(u_{\lambda_0}))$ for sufficiently small R. Thus, by proposition 3.2.1, we can get a smooth family of solutions u_λ by taking $\varphi = \varphi_R(u_{\lambda_0})$ and perturbing $J = J_{\lambda_0}$. Lemma 3.1.2 implies that, when u is J-holomorphic, $du(0) = d\varphi_R(u)(0)$. Therefore, if u_{λ_0} is singular at x, so is $\varphi_R(u_{\lambda_0})$ and the u_λ.

To prove the last statement, let θ_λ be the operator θ with coefficients A^λ_{im} coming from J_λ and let K_λ be the corresponding contraction mapping $h \mapsto \varphi + \theta_\lambda(h)$. Then,

$$\begin{aligned}
\|u_\lambda - u_{\lambda_0}\|'_R &\leq \|K_\lambda(u_{\lambda_0}) - u_{\lambda_0}\|'_R \\
&= \|\theta_\lambda(u_{\lambda_0}) - \theta_{\lambda_0}(u_{\lambda_0})\|'_R \\
&= \|T_R\left(\left(A^\lambda_{im}(u_{\lambda_0}) - A^{\lambda_0}_{im}(u_{\lambda_0})\right)\bar\partial(\bar u_{\lambda_0})_m\right)\|'_R \\
&\leq c_1 \sup_{im} \|A^\lambda_{im}(u_{\lambda_0}) - A^{\lambda_0}_{im}(u_{\lambda_0})\|_R \|u_{\lambda_0}\|'_R \\
&\leq c_2 \sup_{im} \|A^\lambda_{im} - A^{\lambda_0}_{im}\|_{C^1}
\end{aligned}$$

as required. □

Proof of proposition 2.1.2. — To prove the first two statements we apply theorem 3.3.1 with $J_t = (g_t)_*(J)$, where g_t is a family of local diffeomorphisms of M such that $g_t(\gamma(t)) = u(0)$ to get a family of J_t-holomorphic maps u'_t, and then put $u_t = (g_t)^{-1} \circ u'_t$.

To prove the last statement we need to choose $\gamma(t)$ so that, for small t, the images $u'_t(D(R))$ of the J_t-holomorphic maps u'_t do not go through $g_t(x)$. (Note that R must be independent of t here.) It is easy to see that we may choose $\gamma(t)$ and g_t so that (i) $J_t = J$ at $u(0)$ for all t;
(ii) $\mathrm{dist}(g_t(x), u_0(D(R))) > k_1 t$ for all small t and some $k_1 > 0$; and
(iii) $\|J_t - J_0\|_{C^1} \leq k_2 t$ for some k_2.

Applying the inequality

$$|h|_R = \sup\{|h(z)| : z \in D(R)\} \leq 6R\|h\|'_R$$

to $h = u'_t - u$, we find that

$$|u'_t(z) - u(z)| \leq 6Rck_2 t \leq \frac{k_1}{2}t, \quad \text{for small} \quad t, \quad \text{and all} \quad z \in D(R)$$

provided that $R \leq \frac{k_1}{12ck_2}$. Hence $g_t(x) \notin u'_t(D(R))$, as required. □

4. Perturbing away singularities

4.1. Approximation by immersions

In this section we show that singular local J-holomorphic curves may be perturbed to be immersed. We will prove a slightly sharper result than in [4].

THEOREM 4.1.1. — *Suppose that* $u_0 : (D(R), 0) \to (\mathbf{C}^2, (0,0))$ *is a* J*-holomorphic curve which is singular at* $z = 0$*. Then, for some* $R_1 \leq R$, $\varepsilon_0 > 0$ *there is a family of* J*-holomorphic maps* $u_\varepsilon : (D(R_1), 0) \to (\mathbf{C}^2, (0,0))$, $|\varepsilon| < \varepsilon_0$, *whose second derivatives are continuous functions of* ε*. Moreover, the* u_ε *are immersions for* $\varepsilon \neq 0$.

Proof. — The idea of the proof is to lift u_0 over a suitable branched cover to get an immersion $\tilde{u}_0$. If $\tilde{J}$ is the corresponding lift of J, then one can perturb $\tilde{u}_0$ to a family of $\tilde{J}$-holomorphic immersed curves $\tilde{u}_\varepsilon$ which project downstairs to immersions. Thus, the outline of the proof is the same as in [4, §3]. However, we begin with a different choice of coordinates near $(0,0) \in \mathbf{C}^2$, and so consider a slightly different branched cover[2]. This makes the lift $\tilde{J}$ of J smoother than before, and so enables us to simplify the argument.

We first claim that we may choose J_0-holomorphic coordinates (w_1, w_2) near $(0,0)$ in $\mathbf{C}^2$ so that u has the form

$$u(z) = (z^k, \alpha z^{k+1}) + O(|z|^{k+2}),$$

where $k \geq 2$ and $\alpha \in \mathbf{C}$. To see this, observe that the argument of lemma 1.2.1 shows in fact that u is holomorphic through order $2k-1$, that is

$$\bar{\partial} T_{2k-1}(u) = 0.$$

Since $k \geq 2$, this means in particular that the terms in u of order k and $k+1$ are holomorphic. Thus u has the form

$$u(z) = (a,b)z^k + (c,d)z^{k+1} + O(|z|^{k+2}),$$

where $a, b, c, d \in \mathbf{C}$ and $a \neq 0$. We may make $b = 0$ by the coordinate change $(w_1, w_2) \mapsto (w_1, w_2 - b w_1/a)$ and then make $(a,c) = (1,0)$ by replacing the coordinate w_1 by $w_1(a + c w_1)^{1/k}$. This puts u into the stated form.

We now choose $\mathcal{C}^\infty$-smooth coordinates (w_1', w_2') near $(0,0)$ so that in addition the axis $\{w_2' = 0\}$ and all the vertical curves $w_1' =$ const. are J-holomorphic. (To arrange this, use proposition 2.1.2 to find a J-holomorphic disc tangent to $\{w_2 = 0\}$ to serve as the new axis $\{w_2' = 0\}$, and then find a family of J-holomorphic discs perpendicular to $\{w_2' = 0\}$ which will serve as the verticals $w_1' =$ const.) Clearly, by reparametrising these discs (using the fact that every almost complex structure on a 2-manifold is integrable), we may further assume that $J = J_0$ on the tangent bundles to these verticals.

[2] In fact, Micallef pointed out that [4, Lemma 2.5] is not quite correct: one cannot always choose coordinates so that an arbitrary almost complex structure J on $\mathbf{C}^2$ equals J_0 at all points on the axes $\{w_1 = 0\}$ and $\{w_2 = 0\}$, though one can make them equal along one axis as we saw in lemma 1.2.2. Thus something has to be done to this proof, and it is fortunate that we can simplify it at the same time.

From now on, we will call our coordinates w_1, w_2 and assume that they satisfy all these conditions. Consider the branched cover:

$$\Psi : \mathbf{C}^2 \to \mathbf{C}^2 \quad (w_1, w_2) \mapsto (w_1^k, w_2).$$

If we write $kw_1^{k-1} = a + ib$, then a, b are real homogeneous polynomials of order $k-1$ in x_1, y_1, where $w_j = x_j + iy_j$, and it is easy to check that

$$d\Psi|_{(w_1,w_2)} = \begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$d\Psi^{-1}|_{\Psi(w_1,w_2)} = \begin{pmatrix} a/k^2|w_1|^{2k-2} & b/k^2|w_1|^{2k-2} & 0 & 0 \\ -b/k^2|w_1|^{2k-2} & a/k^2|w_1|^{2k-2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Next, consider the lift $\tilde{J} = d\Psi^{-1} \circ J \circ d\Psi$ of J. Our hypotheses imply that

$$J = J_0 + B = J_0 + \begin{pmatrix} B_{11} & B_{12} & 0 & 0 \\ B_{21} & B_{22} & 0 & 0 \\ B_{31} & B_{32} & 0 & 0 \\ B_{41} & B_{42} & 0 & 0 \end{pmatrix}$$

where the functions $B_{pq} = 0$ when $w_2 = 0$. Thus their Taylor expansion has order ≥ 1 in x_2, y_2. It follows easily that the lift $\tilde{J} = J_0 + \tilde{B}$ is well-defined except on the branching axis $\{(0, w_2) : w_2 \neq 0\}$. Further, the functions $\tilde{B}_{pq}(w_1, w_2)$ are either smooth or have the form

$$x_2 h(x_1, y_1)/|w_1|^{2k-2} + y_2 h'(x_1, y_1)/|w_1|^{2k-2} + \text{higher order terms}$$

where h, h' are homogeneous polynomials of order $2k-2$. Thus the first derivatives of the functions $\tilde{B}_{pq}$ are bounded on the cone

$$\Gamma = \{(w_1, w_2) \mid |w_2| < |w_1| \leq 1\}.$$

The reason for considering Ψ is that the J-holomorphic map u lifts over Ψ to an immersion $\tilde{u}$ of the form

$$\tilde{u}(z) = (\tilde{u}_1(z), \tilde{u}_2(z)) = (z(1 + h(z)), \tilde{u}_2(z))$$

where h is at least C^1. To see this, recall that we chose coordinates so that

$$u(z) = (u_1, u_2) = (z^k, \alpha z^{k+1}) + O(|z|^{k+2}).$$

Because the Taylor expansion of u_1 involves $\bar{z}$ as well as z, it is not divisible by z^k. But, the worst terms of $z^{-k}u_1$ have the form $\bar{z}^\ell/z^p$ where $p \leq k$ and $\ell - p \geq$

$k+2-k=2$, and it is easy to check that these are $\mathcal{C}^1$. Thus $z^{-k}u_1 = 1+f$, where f is $\mathcal{C}^1$. Since $1+h$ is the k-th root of $1+f$, the result follows.

Consider the $\mathcal{C}^2$ change of coordinates:

$$\Phi(w_1, w_2) = \left(\tilde{u}_1^{-1}(w_1), w_2 - \tilde{u}_2(\tilde{u}_1^{-1}(w_1))\right).$$

Then $\Phi = \mathrm{Id}$ on the axis $\{w_1 = 0\}$ and the map $\hat{u}$, given by

$$\hat{u}(z) = \Phi \circ \tilde{u}(z) = (z, 0),$$

is $\hat{J}$-holomorphic, where $\hat{J} = \Phi_\star(\tilde{J})$. Note that $\hat{J}$ is as smooth as $\tilde{J}$. In particular, it is easy to check that its first derivatives are bounded on the deleted cone $\Gamma - \{0\}$, and that its second derivatives grow no faster than $1/\|w\|$ as $\|w\| \to 0$. Clearly, the corresponding functions $\hat{A}_{im}$ also have these growth properties.

We now want to apply proposition 3.2.1 with the almost complex structure $\hat{J}$ to the family of maps $\varphi_\varepsilon = z(1, \varepsilon)$. Since φ_0 is $\hat{J}$-holomorphic, the solution $H_R(\varphi_0)$ corresponding to φ_0 is just $\varphi_0 = \hat{u}$. We claim that, for some $R_1, \varepsilon_0 > 0$, there is a family of solutions $H_R(\varphi_\varepsilon)$ for $|\varepsilon| < \varepsilon_0$. To see this, note first instead of working in $\mathbf{A}$, we can set up our contraction map on the space:

$$\mathbf{L} = \left\{h : h(z) = z(1,0) + \hat{h},\ \|\hat{h}\|'_R \le \frac{1}{2}\right\}.$$

Note that all the elements of $\mathbf{L}$ are embeddings. Further, it is easy to check that, if $h \in \mathbf{L}$, then $h(D(R)) \subset \Gamma$ for $R < 1/2$.

Next observe that all the estimates in the proof of proposition 3.2.1 except possibly for (3.2.5) apply with our $\hat{A}_{im}$, which are of class $\mathcal{C}^\alpha$ rather than of class $\mathcal{C}^\infty$. To check that (3.2.5) holds we will need the following lemmas.

LEMMA 4.1.2. — *Let $\psi : D(R) - \{0\} \to \mathbf{C}$ be a bounded function whose first derivative is such that*

$$|d\psi(z)| \le \frac{c_8}{|z|},$$

and let $\varphi \in \mathbf{B}$. Then, $\|\varphi \cdot \psi\|_R \le c_9 R \|\varphi\|'_R$, where c_9 depends only on c_8 and $m = \sup|\psi|$.

Proof. — Note first that

$$\begin{aligned} H_\alpha[\varphi \cdot \psi] &\le \sup_{|z| \le |z'|}\left(\frac{|\varphi(z) - \varphi(z')||\psi(z')|}{|z-z'|^\alpha} + \frac{|\varphi(z)||\psi(z) - \psi(z')|}{|z-z'|^\alpha}\right) \\ &\le m\, H_\alpha[\varphi] + \sup_{|z|\le|z'|}|z|\,\|\varphi\|'_R \frac{c_8}{|z|}|z-z'|^{1-\alpha} \\ &\le m\, H_\alpha[\varphi] + \|\varphi\|'_R c_8 R^{1-\alpha}. \end{aligned}$$

Here we used the fact that $|\varphi(z)| \le |z| \max\{|\partial\varphi|_R, |\bar{\partial}\varphi|_R\} \le |z|\,\|\varphi\|'_R$, since $\varphi(0) = 0$. Thus

$$\begin{aligned} \|\varphi \cdot \psi\|_R &\le |\varphi|_R \sup|\psi| + R^\alpha H_\alpha[\varphi \cdot \psi] \\ &\le R m\, \|\varphi\|'_R + R^\alpha \left(H_\alpha(\varphi) m + c_8 R^{1-\alpha}\|\varphi\|'_R\right) \end{aligned}$$

which has the required form because

$$H_\alpha(\varphi) \le R^{1-\alpha}\max\{|\partial\varphi|_R, |\bar\partial\varphi|_R\} \le R^{1-\alpha}\|\varphi\|'_R.$$

□

Here is the needed replacement for equation (3.2.5).

LEMMA 4.1.3. — *Let A be one of the functions $\hat{A}_{im}$ so that its first derivatives satisfy the assumptions of the previous lemma on the cone Γ, and let $h, h' \in \mathbf{L}$. Then*

$$\|A(h') - A(h)\|_R \le c_6 R\,\|h' - h\|'_R.$$

Proof. — As in [9, 7.1h] we may write

$$\begin{aligned} A(h'(z)) - A(h(z)) &= \sum_{i=1}^{2} (h'_i(z) - h_i(z)) \int_0^1 \partial_i A\,(h(z) + t(h'(z) - h(z)))\,dt \\ &+ \sum_{i=1}^{2} \left(\bar h'_i(z) - \bar h_i(z)\right) \int_0^1 \bar\partial_i A\,(h(z) + t(h'(z) - h(z)))\,dt \end{aligned}$$

Each term in the sums on the right hand side is an integral over t of a product $\varphi\psi$, where $\varphi = h'_i(z) - h_i(z)$ and $\psi = \partial_i A$ or $\bar\partial_i A$ evaluated at $(h(z) + t(h'(z) - h(z)))$. Thus the conditions of the previous lemma are satisfied and the result follows. □

As in proposition 3.2.1, one can now derive the inequality:

$$\|\theta(h') - \theta(h)\|'_R \le c_7 R^\alpha \|h' - h\|'_R,$$

for $h, h' \in \mathbf{L}$. To see this, note

$$\begin{aligned} \|\theta(h') - \theta(h)\|'_R &\le \|T_R\left(A_{im}(h') - A_{im}(h))\bar\partial\bar h'_m\right)\|'_R \\ &\quad + \|T_R\left(A_{im}(h)(\bar\partial\bar h'_m - \bar\partial\bar h_m)\right)\|'_R \\ &\le c_1 \sup_{im}\left(\|A_{im}(h') - A_{im}(h)\|_R \|h\|_R \right. \\ &\quad \left. + \|A_{im}(h)\|_R \|h' - h\|'_R\right), \end{aligned}$$

which clearly has the right form for some constant c_7.

It follows easily that, when $c_7 R_1^\alpha \le \frac{1}{2}$, the mapping $h \mapsto \varphi_\varepsilon + \theta(h)$ is a contraction mapping M_ε on $\mathbf{L}$ for small ε. Hence, it has a unique fixed point $H_R(\varphi_\varepsilon)$ on $\mathbf{L}$.

Finally, let $\tilde u_\varepsilon = \Phi^{-1} \circ H_R(\varphi_\varepsilon)$. As remarked above, this map is an embedding. Moreover, since it intersects the branching axis only at the point $(0,0)$, where it is tangent to the plane $\varepsilon w_1 - w_2 = 0$, it projects down to an immersion $u_\varepsilon = \Psi \circ \tilde u_\varepsilon$.

The statement about smoothness cannot quite be quoted from Nijenhuis and Woolf [9] (note that in their language, we are claiming that the functions u_ε are $\mathcal{C}^{2,0}$, that is, they vary continuously in the $\mathcal{C}^2$-topology when ε varies continuously in the $\mathcal{C}^0$-topology). They show that when J is $\mathcal{C}^\infty$, the solution u_ε varies as smoothly as does its holomorphic part $S(u_\varepsilon)$. Thus one cannot work with $\tilde u_\varepsilon$ since the almost complex structure $\tilde J$ is not sufficiently smooth. It is also not immediate that the variation of $S(u_\varepsilon)$ is $\mathcal{C}^2$-smooth. However, Gang Liu realised that Nijenhuis and Woolf's methods do suffice to prove what we need. The details are in the appendix.

□

4.2. J-spheres in symplectic manifolds

Here is a corollary which is important for some applications. We assume that (M, ω) is a symplectic manifold, and write $\mathcal{J}$ for the space of all $\mathcal{C}^\infty$ ω-tame almost complex structures on M and $\mathcal{X}$ for the space of all $\mathcal{C}^\infty$-maps $S^2 \to M$. Further, given $A \in H_2(M, \mathbf{Z})$, we denote by $\mathcal{M}_s(A, \omega)$ the "big" moduli space of A-curves. This consists of all pairs $(u, J) \in \mathcal{X} \times \mathcal{J}$ such that u is J-holomorphic and represents the class A.

COROLLARY 4.2.1. — *Immersed curves are dense in the "big" moduli space $\mathcal{M}_s(A, \omega) \subset \mathcal{X} \times \mathcal{J}$ with respect to the topology induced by the $\mathcal{C}^2$-topology on $\mathcal{X}$ and the $\mathcal{C}^1$-topology on $\mathcal{J}$.*

Proof. — Given a J-holomorphic curve $C = \operatorname{Im} u$, the above theorem implies that there is a neighbourhood $U \subset C$ of its singular points which has a J-holomorphic perturbation U_ε. Then, one can patch U_ε to $C - U$ by an annulus A which is a subset of U_ε at one end and a subset of $C - U$ at the other to get an immersed curve $C' \subset U_\varepsilon \cup A \cup (C - U)$. Observe that C' is J-holomorphic except over the patch A. Because U_ε is $\mathcal{C}^2$-close to U, the tangent spaces of A are $\mathcal{C}^1$ close to those of C, and hence are invariant under some almost complex structure J' which is $\mathcal{C}^1$-close to J. Thus we may extend J', setting it equal to J except near A, so that it is $\mathcal{C}^1$-close to J, and so that C' is J'-holomorphic. The pair (C', J') is the desired immersed curve near (C, J) (for more details of the patching procedure, see [4, Lemma 4.3]). □

Remark. — This is exactly what is needed in [6, Proposition 3.4] in order to show that the evaluation map from a suitably defined 4-dimensional space of immersed spheres in a symplectic 4-manifold M to M is locally orientation preserving. In this argument we have to work in a topology on the moduli space which is sufficiently fine for the set of regular points of the Fredholm projection operator $P_A : \mathcal{M}_s(A, \omega) \to \mathcal{J}$ to be open, and yet sufficiently coarse for the immersed curves to be dense. One could pursue the methods of the appendix to prove density of the immersed curves in the $\mathcal{C}^\infty$-topology on $\mathcal{X} \times \mathcal{J}$, but at present there is no application of this result.

4.3. Sketch proof of the adjunction formula

It is easy to see that if the curve C is immersed with k double points, then each double point contributes 2 to the self-intersection number and

$$g(C) = 1 + \frac{1}{2}(C \cdot C - c_1(C)) - 2k.$$

Thus the result holds in this case. Further, since, by corollary 4.2.1, every curve has a perturbation which is immersed, and since $g(C)$ is invariant under perturbation,

$g(C)$ must be an integer which is at least as large as the genus. Thus the only difficulty is to show that a singular closed curve cannot perturb to an embedded curve.

It is shown in [4] that it is enough to show this locally. In fact, one can show that the contribution k_x of a singular point to the self-intersection number $C \cdot C$ is twice the number of double points of any immersed J-holomorphic perturbation U' of a neighbourhood U of x in C. Therefore, the essential step is to see that k_x is always positive.

This follows, of course, from the analysis of singularities in [7] or [8]. But such a powerful result is not necessary. To calculate k_x, first suppose that C has the local parametrisation:

$$u(z) = (z^k, z^m) + O(m+1),$$

where $k < m$ and $(k, m) = 1$. It is not hard to see that k_x does not change when u is perturbed first to the topological embedding $u' : z \mapsto (z^k, z^m)$ and then to $u'' : z \mapsto (z^k, \varepsilon z + z^m)$. It is now easy to calculate that $k_x = (k-1)(m-1)$ which is > 0 as required.

However, when k and m have a common factor, the map u' above is not a topological embedding, and so this argument breaks down. In this case, one makes a preliminary perturbation of $u = H_R(\varphi_R)$ by making a quadratic perturbation $\varphi_R + z^2(\varepsilon, 0)$ of φ_R. By lemma 3.1.2 this gives us a map u' with a singularity of order 2 at 0, i.e. in suitable local coordinates u' has the form

$$u'(z) = (z^2, z^m) + O(m+1).$$

In addition, we can arrange that m is odd: see [4, Proposition 2.6]. Therefore, by the previous case, this singularity contributes the even number $m - 1 \geq 2$ to k_x. Because the contribution of the other singular points is non-negative, we must have $k_x > 0$. □

Appendix: The smoothness of the dependence on ε, by Gang Liu

A.1. Statements

In this appendix, we show how to adapt Nijenhuis and Woolf's argument to our situation. We give all details, modulo quoting some estimates, and thus this appendix can serve as an introduction to the process of "elliptic bootstrapping".

Given any $k \geq 0$, we will say that the family of functions u_ε is $\mathcal{C}^{k,0}$ if u_ε varies continuously in the $\mathcal{C}^k$-topology as ε varies continuously in the $\mathcal{C}^0$-topology. Observe that k need not be an integer here. We will mostly be concerned with the case when $1 \leq k \leq 2$. When $1 < k = 1 + \beta < 2$, we will take the $\mathcal{C}^{1+\beta}$-norm to be defined in the same way as the norm $\|.\|'_R$ in § 3, so that all the estimates of §§ 3,4 apply. In particular, the $\mathcal{C}^{1+\alpha}$-norm for functions on $D(R)$ is exactly $\|.\|'_R$.

We aim to prove that, for any $R_2 < R_1$, the family $u_\varepsilon : D(R_1) \to \mathbf{C}^2$ constructed in theorem 4.1.1 is $\mathcal{C}^{2,0}$ when restricted to $D(R_2)$. Recall that $\varphi_\varepsilon(z) = (1, \varepsilon)z$. We know from the definition of $\mathbf{L}$ that:

(i) *there is* $M > 0$ *such that* $\|H(\varphi_\varepsilon)\|' \leq M$ *for all* ε; and
(ii) $H(\varphi_\varepsilon)$ *is the unique solution in* $\mathbf{L}$ *of the equation*

$$u(z) = \varphi_\varepsilon(z) + \theta\,(u)(z).$$

Here, for simplicity, we have written H instead of H_R. Recall also that the norm $\|\cdot\|_R$ incorporates a Hölder norm of some arbitrary but fixed exponent α, where $0 < \alpha < 1$.

LEMMA A.1.1. — *For* $0 < \beta < \alpha$, $H(\varphi_\varepsilon)$ *is* $\mathcal{C}^{1+\beta,0}$.

Proof. — We must show that, for any sequence $\delta_j \to \delta$,

$$\|H(\varphi_{\delta_j}) - H(\varphi_\delta)\,\|_{\mathcal{C}^{1+\beta}} \to 0.$$

But, by (i) above, the sequence $\{H(\varphi_{\delta_j}\}$ is uniformly bounded and equicontinuous in the $\mathcal{C}^{1+\alpha}$-norm. Therefore it has a subsequence which converges in the $\mathcal{C}^{1+\beta}$-norm to some limit, say $H'(\varphi_\delta)$. Now, by construction,

$$H(\varphi_{\delta_j}) = \varphi_{\delta_j} + \theta\,(H(\varphi_{\delta_j})).$$

We showed in §4 above that θ is continuous with respect to the $\mathcal{C}^{1+\beta}$-norm, for any $\beta < 1$. (The only limitation on β here is that we need the functions A_{im} to be of class $\mathcal{C}^\beta$.) Therefore, we may let j go to ∞ in the above equation. This gives

$$H'(\varphi_\delta) = \varphi_\delta + \theta\,(H'(\varphi_\delta)).$$

By the uniqueness statement (ii), $H'(\varphi_\delta) = H(\varphi_\delta)$. This shows that every subsequence of $\{H(\varphi_{\delta_j}\}$ converges in $\mathcal{C}^{1+\beta}$ to $H(\varphi_\delta)$. □

Now $\tilde{u}_\varepsilon = \Phi^{-1} \circ H(\varphi_\varepsilon)$, where Φ is $\mathcal{C}^2$. This implies that the functions $u_\varepsilon = \Psi \circ \tilde{u}_\varepsilon$ still satisfy:
(i)$'$ *for some constant* M', $\|u_\varepsilon\|'_{R_1} \leq M'$; and
(ii)$'$ $u_\varepsilon \in \mathcal{C}^{1+\beta}$.

Recall further that J and hence the functions A_{im} are $\mathcal{C}^\infty$.

LEMMA A.1.2. — $\{u_\varepsilon\} \in \mathcal{C}^{2,0}$

Proof. — Suppose we can prove that for any $R_2 < R_1$,

$$\sum_{i+j\leq 2} \|\partial^i\bar{\partial}^j u_\varepsilon\|_{R_2} \leq F(\|u_\varepsilon\|'_{R_1}), \tag{A.1.3}$$

for some continuous function F. Then, this will imply that u_ε is uniformly bounded and equi-continuous under the $\mathcal{C}^{2+\beta}$-norm for any $\beta < \alpha$. Therefore, if $\delta_j \to \delta$, the sequence $\{u_{\delta_j}\}$ has a subsequence which converges in $\mathcal{C}^{2+\beta}$ with limit u'_δ, say. But, by (ii)$'$, we know that $\{u_{\delta_j}\}$ converges $\mathcal{C}^{1+\beta}$ to u_δ. Thus $u'_\delta = u_\delta$. This proves that $\{u_\varepsilon\} \in \mathcal{C}^{2,0}$, as required. □

A.2. Proof of estimation A.1.3

Estimate (A.1.3) is of a standard elliptic type. Given a solution u of a first order elliptic equation, it estimates the $\mathcal{C}^{2+\alpha}$-norm of u in terms of its $\mathcal{C}^{1+\alpha}$-norm. (In general, if the equation has order ℓ, one can estimate a $\mathcal{C}^{k+\ell+\alpha}$- norm in terms of a $\mathcal{C}^{k+\alpha}$-norm.) Its proof is based on the following fundamental elliptic estimate.

(iii) *If $f, w \in \mathcal{C}^{\alpha}(D(R_1))$, and $f = S_{R_1}f + T_{R_1}w$, then for any $R_2 < R_1$, $f \in \mathcal{C}^{1+\alpha}(D(R_2))$ and*

$$\|f\|'_{R_2} \leq c(R_1, R_2)\,(\|f\|_{R_1} + R_1\|w\|_{R_1}).$$

See [9] 4.3d. We will also use the following formulae.

(iv) [9, (6.3b)] *If $f, g \in \mathcal{C}^{1+\alpha}(D)$, then $T(f\bar\partial g) \in \mathcal{C}^{1+\alpha}(D)$ and*

$$\partial T(f\bar\partial g) = T(\partial f \bar\partial g - \bar\partial f \partial g) + f\partial g - S(f\partial g) - \bar S(f\bar\partial g);$$

(v) [9, (6.2c)] *If $f \in \mathcal{C}^1(D)$ then $\partial S(f) = S(\partial f) + \bar S(\bar\partial f)$.*

Here $\bar S$ is defined by:

$$\bar S(f)(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)\,d\bar\zeta}{\zeta - z}.$$

Proof of (A.1.3). — For simplicity, we write u instead of u_ε. By construction, we know that

$$\partial u + A(u)\bar\partial\bar u = 0,$$

or equivalently

(vi) $$u = S(u) - T(A(u)\bar\partial\bar u).$$

Therefore, using (iv) and (v), we find:

$$\begin{aligned}\partial u &= \partial S(u) - \partial T(A(u)\bar\partial\bar u)\\ &= S(\partial u) + \bar S(\bar\partial u) - T(\partial A(u).\bar\partial\bar u - \bar\partial A(u).\partial\bar u)\\ &\qquad -A(u)\partial\bar u + S(A(u)\partial\bar u) + \bar S(A(u)\bar\partial\bar u)\\ &= S(\partial u) - T(\partial A(u).\bar\partial\bar u - \bar\partial A(u).\partial\bar u) - A(u)\partial\bar u + S(A(u)\partial\bar u).\end{aligned}$$

Thus, because $\partial\bar u = -\bar A(u)\partial u$, we have

$$[1 - A(u)\bar A(u)].\partial u = S([1 - A(u)\bar A(u)].\partial u) - T(\partial A(u).\bar\partial\bar u - \bar\partial A(u).\partial\bar u).$$

Applying (iii), we find that, when $R_2 < R_1$,

$$\begin{aligned}\|[1 - A(u)\bar A(u)].\partial u\|'_{R_2} &\leq c(R_1, R_2)(\|[1 - A(u)\bar A(u)].\partial u\|_{R_1}\\ &\qquad + R_1\|\partial A(u).\bar\partial\bar u - \bar\partial A(u).\partial\bar u\|_{R_1})\\ &\leq c_1(R_1, R_2)\,\|u\|'_{R_1} + c_2(R_1, R_2)\,(\|u\|'_{R_1})^2.\end{aligned}$$

Here we use the fact that

$$\partial A(u) = \partial A(w)\big|_{w=u} \cdot \frac{\partial u}{\partial z} + \bar{\partial} A(w)\Big|_{w=u} \cdot \frac{\partial \bar{u}}{\partial z},$$

and the inequality $\|a.b\| \leq \|a\|.\|b\|$ from [9, (7.1b)]. Thus

$$\begin{aligned} \|\partial u\|'_{R_2} &= \|[1 - A(u)\bar{A}(u)]^{-1}[1 - A(u)\bar{A}(u)]u\|'_{R_2} \\ &\leq c_3(R_1, R_2)\,\|u\|'_{R_1} + c_4\,(R_1, R_2)(\|u\|'_{R_1})^2. \end{aligned}$$

This is an estimate for $\|\partial u\|'_{R_2}$ of the required form. To get a similar estimate for $\|\bar{\partial} u\|'_{R_2}$, we note

$$\begin{aligned} \|\bar{\partial} u\|'_{R_2} &= \|A(u).\bar{\partial}\bar{u}\|'_{R_2} = \|\bar{A}(u).\partial u\|'_{R_2} \\ &\leq c_5\,\|u\|'_{R_2} + c_6\,\|\partial u\|'_{R_2} \\ &\leq c_7\,\|u\|'_{R_1} + c_8\,(\|u\|'_{R_1})^2. \end{aligned}$$

This completes the proof.

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Chapter VII

Gromov's Schwarz lemma as an estimate of the gradient for holomorphic curves

Marie-Paule Muller

1. Introduction

In a symplectic manifold (M^{2n}, ω) with a tamed almost complex structure J, the 2-form ω induces an area form on (immersed) J-curves $f : (\Sigma, i) \to (M, J)$, where (Σ, i) is a Riemann surface which will be closed, open or compact with boundary, according to the context. The image $f(\Sigma)$ will be denoted by S. Throughout this chapter, we shall assume that S is contained in a fixed compact subset of M.

Roughly speaking, the final goal is to ensure the existence of "enough" J-curves verifying a given homological condition. This will be a consequence of an Ascoli type "compactness theorem", explained in the next chapter.

Our immediate task, in this chapter, is to present an essential ingredient in the proof of the compactness theorem: a Schwarz lemma which is "ad hoc" for J-curves, that is an estimate for the derivative of J-holomorphic maps.

In order to simplify the presentation, we assume that the various data are of class C^∞. Nevertheless, the minimal regularity class for each statement can easily be made clear by the context.

2. A review of some classical Schwarz lemmas

2.1. Holomorphic functions

Let us begin with the well-known standard Schwarz lemma, which is a consequence of the maximum principle: denoting by D_1 the (Euclidean) unit disc $|z| < 1$ in $\mathbf{C}$, for every holomorphic function $f : D_1 \to D_1$ with $f(0) = 0$, we have $|f'(0)| \leq 1$.

The following equivalent formulation, where the assumption $f(0) = 0$ is dropped, can easily be obtained using two homographies of the disc:

2.1.1. Exercise. — Show that, for every holomorphic function $f : D_1 \to D_1$,

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

Now, in terms of the Poincaré metric m_{-1} (with constant curvature -1) on the disc D_1, the inequality in 2.1.1 says that f is a contraction when D_1 (as source and as target) is endowed with m_{-1} (instead of the Euclidean metric m_0):

$$f^\star m_{-1} \leq m_{-1}$$

(note that $f^\star m_{-1}$ is conformal).

2.2. Negatively curved ranges

The preceeding Schwarz lemma (which is now a lemma in hyperbolic geometry) was generalized to the case of non-constant negative curvature by L. Ahlfors (1938), then by H. Grauert and H. Reckziegel (1965) (see [1], [7]).

LEMMA 2.2.1. — *Let M be a Hermitian manifold whose metric μ is such that the holomorphic sectional curvature is less than -1. Then $f^\star \mu \leq m_{-1}$ for every holomorphic curve $f : D_1 \to M$.*

Let us mention the existence of many other references concerning Schwarz lemmas, see for example chapter 2 of [12]. In all of them, the assumption that the relevant sectional curvature in the target manifold M has a (strictly) negative upper bound is made. The next exercise will explain where exactly the curvature is involved, and why it is assumed to be negative.

2.2.2. Exercise. — Consider two Hermitian metrics $m = e^g\, dzd\overline{z}$ and $\mu = e^h\, dzd\overline{z}$ on D_1 (the first one is supposed to be the reference metric to which we want to compare the second one). Assume that the respective Gaussian curvatures verify $k_\mu \leq k_m$, that there is a point z_0 in D_1 where the ratio μ/m achieves its maximum and that at this point, $k_m(z_0) < 0$. Prove that $\mu \leq m$ everywhere on D_1 (Hint: see for example chapter 2 in volume 2 of [6]).

Now, there are many interesting cases where the condition that μ/m achieves its maximum at some point in the disc is fulfilled. For example, prove it when m is the Poincaré metric, or even more generally, a metric with infinite area which is invariant under rotations around 0 (using if necessary a slightly smaller disc D_r in order to have the function h (corresponding to μ) bounded on that disc, then let r tend to 1).

3. Isoperimetric inequalities for J-curves

Let us choose a Hermitian metric μ on the manifold (M, J). For each J-curve $f : (D_1, i) \to (M, J)$, we consider the induced metric on D_1, which will also be denoted by μ from now on. The goal is to construct on D_1 a "reference metric" m which will be suitable for all these J-curves. Clearly, we must make the assumption that M is compact (or, at least, that we consider only J-curves whose images $S = f(D_1)$ remain in a compact set).

The first step is to obtain an upper bound K for the Gaussian curvature k of J-curves. The trouble is that this constant K can be positive. Therefore, it will not be convenient to use a spherical metric m_K, whose area is at most $4\pi/K$, as a reference metric on the disc. Moreover, as the parametrisation is involved, it is evident that this local property $k \leq K$ for J-curves cannot be sufficient for an estimate of $|f'(0)|$. This explains why we shall need further (global) information about the geometry of J-curves. An isoperimetric inequality, valid for large values of the area, will complete the analysis, allowing us to draw a "profile" which corresponds to the sought after "reference metric".

3.1. A bound on the mean curvature

Let ∇' be the ambient (ie. on (M, μ)) Levi-Civita connection. The connection ∇ for the metric induced on S is the tangential part of ∇', and the second fundamental form II is its normal part:

$$\nabla'_X Y = \nabla_X Y + \mathrm{II}(X, Y).$$

Now we estimate the trace of II and thus the mean curvature $1/2\,\mathrm{trace}\,(\mathrm{II})$. As is appearent from the following calculations, we assume here that J is at least of class C^1. Let (X, JX) be a (local) orthonormal frame for the tangent bundle TS. We have

$$\begin{aligned} \nabla'_{JX} JX &= (\nabla'_{JX} J)X + J\,\nabla'_{JX} X \\ &= (\nabla'_{JX} J)X + J\,(\nabla'_X JX + [JX, X]) \\ &= (\nabla'_{JX} J)X + J\,(\nabla'_X J)X - \nabla'_X X + \text{tangential component.} \end{aligned}$$

Finally, trace (II), which is the normal part of $\nabla'_{JX} JX + \nabla'_X X$, is such that

$$\text{(3.1.1)} \qquad \begin{aligned} \|\mathrm{trace}\,(\mathrm{II})\| &\leq \|\nabla'_{JX} JX + \nabla'_X X\| \\ &\leq 2\,\|\nabla' J\| \end{aligned}$$

which is bounded on compact subsets of M.

3.2. An upper bound for the Gaussian curvature

The Gauss equation relates the curvature R on S to the ambient curvature R':

$$R'(W, Z, X, Y) = R(W, Z, X, Y) + \langle \mathrm{II}(X, Z), \mathrm{II}(Y, W)\rangle - \langle \mathrm{II}(Y, Z), \mathrm{II}(X, W)\rangle.$$

With an orthonormal frame (X, JX) as before, we obtain the sectional curvature on TS:

$$R'(X, JX, X, JX) = k + \|\mathrm{II}(X, JX)\|^2 - \langle \mathrm{II}(JX, JX), \mathrm{II}(X, X)\rangle.$$

Let C_{sec} be an upper bound for the (J-holomorphic) sectional curvature on the compact manifold M. The preceeding relation yields

$$k \leq C_{\mathrm{sec}} + 1/2\,\|\mathrm{trace}\,(\mathrm{II})\|^2$$

which is bounded by a constant K, by (3.1.1).

3.3. Two isoperimetric inequalities

Let D_1 be the closed unit disc in $\mathbf{C}$. For a J-curve with boundary $f : D_1 \to (M, J, \mu)$, let a (resp. l) be the area (resp. the length of the boundary) of the J-curve, for the induced metric. The upper bound on Gaussian curvature $k \leq K$ implies an isoperimetric inequality.

LEMMA 3.3.1. — *The inequality $l^2 \geq 4\pi a - K a^2$ holds.*

This isoperimetric inequality, which is well-known when the curvature k is constant, is established by G. Bol in [4] (see a proof and references in the appendix to chapter III).

From now on, we shall make use of a symplectic form ω which tames J (see chapter II). The compactness of M allows one to assume that $\omega(X, JX) \leq \|X\|^2$ for every tangent vector X (by a convenient normalization of the metric).

LEMMA 3.3.2. — *Let $f : \Sigma \to (M, J, \mu)$ be a compact J-curve with boundary whose image S is contained in a region of M where ω is exact: $\omega = d\lambda$. Then*

$$a \leq \text{const} \cdot l$$

where the constant depends only on λ.

Proof. — We have $a \leq \int_\Sigma f^\star \omega \leq \int_{\partial\Sigma} f^\star \lambda \leq \|\lambda\|_{L^\infty} l$. □

Remarks.

- Notice that the boundary $\partial\Sigma$ must be non empty when the symplectic structure is exact.
- Think of the Darboux theorem. By compactness of M, there is a radius ε such that ω is exact on a neighbourhood of all the closed balls $B(x, \varepsilon)$ in M: $\omega = d\lambda_x$. Moreover, we can assume that $\omega(X, JX) \geq \|X\|^2$ and $|\lambda_x(X)| \leq \|X\|$ on each of these balls. We then obtain $l \geq a$ for the J-curves corresponding to the intersections $S \cap B(x, \varepsilon)$.

4. The Schwarz and monotonicity lemmas

4.1. Gromov's Schwarz lemma

Given a J-curve $f : D_1 \to (M, J, \mu)$, let $a(r)$ (*resp.* $l(r)$) denote the area (*resp.* the length of the boundary) of the Euclidean disc D_r with respect to the metric $f^\star\mu$.

PROPOSITION 4.1.1 (Schwarz lemma). — *Assume that for some constants K and C, the functions $a(r)$ and $l(r)$ corresponding to a J-curve $f : D_1 \to (M, J, \mu)$ fulfil the isoperimetric conditions $l^2 \geq 4\pi a - K a^2$ and $l \geq C a$.*

Then there exists a constant $c(K, C)$ (depending only on K and C) such that

$$\frac{f^\star \mu(0)}{m_0(0)} \leq c(K, C).$$

Equivalently,

$$|f'(0)|^{\mu}_{m_0} \leq \sqrt{c(K, C)}.$$

Proof. — The idea is to proceed as if we had on D_1 a "reference metric" m, conformal and invariant under rotation. In fact, we shall never use its metric tensor, but only its corresponding functions $l_m(r)$ and $a_m(r)$. More precisely, we shall exploit its *profile*, i.e., the function which expresses l_m^2 as a function of a_m.

Let P be the function defined on $[0, +\infty[$ by the formula $P(a) = \max\{4\pi a - K\,a^2, C^2 a^2\}$.

Our first task is to obtain two functions $l_m(r)$ and $a_m(r)$, defined for $0 < r < 1$, linked together by the relation $l_m^2 = P(a_m)$, behaving near 0 as the length and area functions of a true metric: $a_m(r) \sim \text{Const.}\, r^2$, and with enough area to cover all possibilities: $a_m(r) \to +\infty$ as $r \to 1$. Moreover, we require that $l_m(r)$ and $a_m(r)$ correspond to a metric which is conformal and invariant by rotation.

Let us first write down these two last conditions. Let $m = h^2(dx^2 + dy^2)$ be a conformal metric on the disc. By the Schwarz inequality, the corresponding functions $l(r)$ and $a(r)$ verify:

$$\begin{aligned} a'(r)\, 2\pi r &= \left(\int_{\partial D_r} h^2\right)\left(\int_{\partial D_r} 1\right) \\ &\geq \left(\int_{\partial D_r} |h|\right)^2 \\ &= l(r)^2 \end{aligned}$$

with equality if and only if h is constant on the circles ∂D_r, ie. m is invariant by rotation. This yields the condition $2\pi r\, a_m'(r) = l_m(r)^2$. Adding the "profile condition" $l_m^2 = P(a_m)$, we obtain the differential equation

$$\frac{a_m'(r)}{P\left(a_m(r)\right)} = \frac{1}{2\pi r}. \tag{4.1.2}$$

Let $Q(a)$ be a primitive of the function $1/P(a)$ defined when $a > 0$. According to (4.1.2),

$$Q(a_m(r)) = \frac{1}{2\pi} \log r + \text{constant} \tag{4.1.3}$$

Integrability conditions.

1. We still have the choice of the additive constant in 4.1.3: as the function $1/P(a)$ is integrable near $+\infty$, taking $Q(+\infty) = 0$ enables us to have $a_m(1) = +\infty$, as we want.

2. Let us examine the behaviour near 0 of the resulting $a_m(r) = Q^{-1}(\frac{1}{2\pi}\log r)$. As
$$\frac{1}{P(a)} - \frac{1}{4\pi a}$$
is integrable near 0, $Q(a) = \frac{1}{4\pi}\log a + g(a)$ for a function $g(a)$ which is continuous at 0. With (4.1.3), this gives us a constant C_1 such that $a_m(r) \sim C_1 r^2$.

We shall now exploit the preceding result in order to deduce the majoration of the metric μ at 0 from the inequality concerning its profile. As μ is conformal and satisfies the isoperimetric inequalities in the statement of 4.1.1, its area function $a(r)$ is such that $2\pi\, a'(r) \geq l(r)^2 \geq P(a(r))$.

For $0 \leq r \leq 1$, let $s(r)$ be the radius such that $a(r) = a_m(s(r))$. We have
$$\begin{aligned} 2\pi r\, a_m'(s)\, s'(r) &= 2\pi r\, a'(r) \\ &\geq P(a(r)) \\ &= P(a_m(s)) \\ &= 2\pi s\, a_m'(s) \end{aligned}$$
thus
$$\frac{s}{s'} \geq \frac{1}{r}.$$
As $s(1) < 1$, we obtain $s(r) \leq r$ for all r. Thus $a_m(r) \geq a(r)$. In particular for small r, $C_1 r^2 \sim a_m(r) \geq a(r) \sim |f'(0)|^2 \pi r^2$, which completes the proof. □

Remarks.

1. Since the only properties of $P(a)$ used above were the two integrability properties, the proof works for other functions verifying these conditions, as for example $P(a) = 4\pi a + a^2$, which corresponds to the classical case, where $K = -1$ and $m = m_{-1}$, the Poincaré metric.

2. As in lemma 2.2.1, we can drop the specialisation to the point 0 using m_{-1} instead of m_0 for the estimation of f': we obtain the equivalent conclusion
$$|f'(z)|^{\mu}_{m_{-1}} \leq \sqrt{c(K,C)}$$
for every z in the disc, whenever the profile condition is verified (e.g. in the framework of § 3.3).

3. Here is another way to drop the point 0: keeping m_0 on D_1, consider for each point z the restriction of f to the disc with centre z and radius $1 - |z|$. We have immediately
$$|f'(z)|^{\mu}_{m_0} \leq \frac{\sqrt{c(K,C)}}{1 - |z|}.$$

It is an easy exercise to generalise this last remark in order to obtain the

COROLLARY 4.1.4. — *Let Σ be a compact Riemann surface with boundary, with a metric μ_Σ. For every J-curve $f : \Sigma \to (M, J, \mu)$ whose image is contained in a fixed compact set where J is tamed by an exact symplectic form $\omega = d\lambda$, the following estimate holds*

$$|f'(z)| \leq \frac{\text{Const}}{\text{dist}(z, \partial\Sigma)}$$

where the constant depends on everything but f.

4.2. A monotonicity lemma for J-curves

For a complex analytic manifold S defined by an equation $f(z_1, \ldots, z_p) = 0$, where f is a holomorphic function on a domain of $\mathbf{C}^p$, metric properties were established by P. Lelong [8] as a consequence of general results concerning plurisubharmonic functions. In particular, Lelong proved that the mean value of the function $\log|f|$ on a sphere with variable radius R and fixed centre is a convex function of $\log R$. An immediate "monotonicity" corollary can be stated as follows: a point M of S being fixed, let S_R be the intersection of S with the ball of radius R and centre M; the monotonicity property asserts that the ratio between the volume of S_R and the corresponding volume of the tangent space at M (i.e. of the ball of radius R in $\mathbf{C}^{p-1}$, when M is regular), is an increasing function of R whose value at $R = 0$ is the order of multiplicity of the point M. A monotonicity lemma for minimal surfaces is presented in chapter III.

M. Gromov establishes an analogous lemma for J-curves in a compact M, which subsequently gives an upper bound for the number of components which can be cut out of a J-curve by little balls of a given radius ε, when the global area is bounded by a given constant. Throughout this paragraph, the compacity assumption for M means that all the constants which are mentionned do not depend on the considered points x, y... For a J-curve S and a point x on S, let us consider the intersection of S with the ball $B(x, \varepsilon)$ in M.

LEMMA 4.2.1 (Monotonicity lemma). — *There are constants ε_0 and c_0 such that for every $\varepsilon \leq \varepsilon_0$ and for every J-curve S, if $x \in S$ is such that $S \cap B(x, \varepsilon)$ is compact with its boundary contained in the boundary $\partial B(x, \varepsilon)$, then*

$$\text{Area}\,(S \cap B(x, \varepsilon)) \geq \frac{\pi}{1 + c_0\varepsilon}\varepsilon^2.$$

This result will be an easy consequence of the

LEMMA 4.2.2. — *There are constants ε_1 and c_1 such that for every compact J-curve S (with boundary) contained in a ball $B(x, \varepsilon)$ with $\varepsilon < \varepsilon_1$, the area a and the length l of the boundary verify*

$$l^2 \geq \frac{4\pi}{1 + c_1\varepsilon}a.$$

4.2.3. Exercise. — In the following detailed proofs, we give a short summary of what is done at each stage; as an exercise, the reader can anticipate the text, and complete the proofs starting from these indications.

Proof of lemma 4.2.2. — Starting with $J(x)$ and $\mu(x)$ on the tangent space T_xM, let J_x and μ_x be the (constant) complex structure and Hermitian metric obtained on T_xM by translation. We consider also the standard calibrated symplectic structure ω_x associated with J_x, μ_x. Choosing ε'_1 small enough, the exponential map allows to carry all these structures onto $B(x, \varepsilon'_1)$. We shall indicate by means of a lower index x the use of the Riemannian metric μ_x.

As μ and J are of class C^1, we have at every point $y \in B(x, \varepsilon'_1)$:

1. $\|\mu(y) - \mu_x(y)\| \leq c_\mu \mathrm{dist}(x, y)$,

2. $\|J(y) - J_x(y)\| \leq c_J \mathrm{dist}(x, y)$

for some constants c_μ and c_J (independent of x and y).

With the first inequality, we easily compare the lengths and areas corresponding to μ_x with those corresponding to μ. In particular, there is a constant c such that for every curve contained in a ball $B(x, \varepsilon)$ with $\varepsilon \leq \varepsilon'_1$, the respective lengths l_x and l verify:

$$(1 + c\varepsilon)\, l \geq l_x \tag{4.2.4}$$

The next step yields a relation between the area a of S and the integral of ω_x on S. Consider an orthonormal (for μ) frame (U, JU) of T_yS. When we use μ_x, the complex structure J_x is calibrated by ω_x and we have

$$\omega_x(U, J_xU) = \|U\|_x^2$$

and

$$|\omega_x(U, JU - J_xU)| \leq \|J - J_x\|_x \|U\|_x^2 .$$

Applying the first inequality repeatedly and the second one, we obtain a constant c as before, verifying moreover:

$$\omega_x(U, JU) \geq 1 - c \cdot \mathrm{dist}(x, y)$$

which implies that for a J-curve S as in the lemma, contained in some $B(x, \varepsilon)$ with $\varepsilon < \varepsilon'_1$:

$$\int_S \omega_x \geq (1 - c\varepsilon)\, a. \tag{4.2.5}$$

Finally, a homological argument and Wirtinger's inequality allow us to relate the preceeding integral to the length l, as follows. Let us go back to T_xM and consider there the image of S (still denoted by S). The Riemannian metric we shall use is μ_x. Choose a surface S' (connected or not) which has the same boundary as S and whose area a'_x verifies the isoperimetric inequality $l_x^2 \geq 4\pi a'_x$.

Any surface with non positive curvature would do (e.g. a minimal or ruled surface), thanks to a result of A. Weil [11], see also [2]. As a hand-made solution, we can also consider, for each connected component $\partial_i S$ of the boundary, a cone S'_i whose vertex is a convenient point on the loop $\partial_i S$ itself; its planar development verifies the required isoperimetric inequality, even if there are lines of singular points, and even if the total angular variation is more than 2π; and for the union S' of these cones we obtain

$$l_x^2 = \left(\sum l_{i,x}\right)^2 \geq \sum l_{i,x}^2 \geq 4\pi a'_{ix} = 4\pi a'_x$$

as required.

Applying Wirtinger's inequality to S' and using (4.2.5) we obtain

$$l_x^2 \geq 4\pi a'_x \geq 4\pi \int_{S'} \omega_x = 4\pi \int_S \omega_x \geq 4\pi\,(1 - c\varepsilon)\,a.$$

The relation (4.2.4) between l and l_x completes the proof, for a radius $\varepsilon_1 \leq \varepsilon'_1$ taking also into account the value of the constant c: for example $2c\varepsilon_1 \leq 1$.□

Notice that the image of the μ-ball $B(x, \varepsilon)$ may be non-convex in T_xM, thus S' is not necessarily contained in it; nevertheless, μ_x and ω_x are everywhere available in T_xM. This is the reason for going back to T_xM (an alternative technique would be to observe that the image of $B(x, \varepsilon)$ contains a μ_x-ball with radius $(1 - \text{cons.}\varepsilon)\,\varepsilon$, by the first inequality above, and to work with S contained in this smaller ball).

Remark. — If we assume J and μ only of class C^α $(0 < \alpha < 1)$, it is easy to obtain the corresponding formulation of lemma 4.2.2, with ε^α instead of ε (in the inequalities, $\text{dist}(x, y)$ is replaced by $\text{dist}(x, y)^\alpha$).

Proof of the monotonicity lemma 4.2.1. — Let $A(\varepsilon)$ be the area of the intersection $S \cap B(x, \varepsilon)$. Its derivative with respect to ε is related to the length $L(\varepsilon)$ of the boundary $\partial\,(S \cap B(x, \varepsilon))$ by $A'(\varepsilon) \geq L(\varepsilon)$. The lemma yields for every $\varepsilon \leq \varepsilon_1$

$$\frac{A'(\varepsilon)}{\sqrt{A(\varepsilon)}} \geq \sqrt{\frac{4\pi}{1 + c_1\varepsilon}}.$$

Integration from 0 to ε together with Taylor's formula gives immediately

$$A(\varepsilon) \geq \pi\varepsilon^2 \left(1 - \frac{c_1}{2}\varepsilon\right)$$

and hence the result (with, for example, $c_0 = c_1$ and $c_1\varepsilon_0 \leq 1$, $\varepsilon_0 \leq \varepsilon_1$).□

Remarks.

1. In the following, we shall assume that $\varepsilon_0 \leq \varepsilon_1$.

2. Notice that in lemma 4.2.2, the point x needs not belong to the J-curve.

5. Continuous Lipschitz extension across a puncture

5.1. Continuous extension

In the next chapter VIII, J-curves "with missing points" will occur (as limits). The monotonicity lemma allows to eliminate these singularities in the case where the area is finite.

THEOREM 5.1.1. — *Let $f : D_1 - 0 \to (M, J, \mu)$ be a J-curve with finite area. Then f extends to a continuous map defined on D_1.*

Proof. — Assume that the image $S = f(D_1 - 0)$ has two different accumulation points p, q. Consider two closed disjoint balls $B(p, \varepsilon)$ and $B(q, \varepsilon)$, with $\varepsilon \leq \varepsilon_0$ (cf. lemma 4.2.1). We can assume that $f(\partial D_1)$ does not intersect these balls.

In $f^{-1}(B(p, \varepsilon))$, consider only the connected components which are compact and whose images in $B(p, \varepsilon)$ meet the smaller ball $B(p, \varepsilon/2)$. For each of them, the monotonicity lemma 4.2.1 ensures that its μ-area is at least $\mathrm{cst}(\varepsilon/2)^2$. As the total area of S is finite, the number of these components is finite too. Thus, there exists a connected component in $f^{-1}(B(p, \varepsilon))$ which has the point 0 in its closure; consequently, this component meets all the circles ∂D_r for r small enough. As the situation is the same near q, we conclude that for every r small enough, the loop $f(\partial D_r)$ meets both $B(p, \varepsilon)$ and $B(q, \varepsilon)$. Thus its length satisfies $l(r) \geq d > 0$, where $d = \mathrm{dist}\,(B(p, \varepsilon), B(q, \varepsilon))$. Now, as μ is conformal (see § 4.1)

$$a'(r) \geq \frac{l^2(r)}{2\pi r} \geq \frac{d^2}{2\pi r}$$

for $r > 0$ small enough. The last function being non integrable near 0, this contradicts $a(r) < \infty$.

5.2. Isoperimetric inequality for holomorphic curves with punctures

The preceeding proof shows that there is, at least, a sequence (r_n) decreasing to 0 such that the corresponding lengths $l(r_n)$ tend to 0 (of course, $a(r_n)$ decreases also to 0, as the total area is finite). This is enough for the following extension of lemma 4.2.2. Here, $\Sigma^\star$ is a punctured compact Riemann surface with boundary (finitely many interior points can be removed).

LEMMA 5.2.1. — *Lemma* 4.2.2 *holds for every J-curve $f : \Sigma^\star \to B(x, \varepsilon)$ with finite area.* □

In fact we have much more than the remark above:

LEMMA 5.2.2. — *Let $f : D_1 - 0 \to (M, J, \mu)$ be a J-curve with finite area. Then* $\lim_{r \to 0} l(r) = 0$.

Proof. — By theorem 5.1.1, the J-curve f has a continuous extension at 0. Let r_0 be such that $f(D_{r_0})$ is contained in some $B(x, \varepsilon_0)$. We can assume that $r_0 = 1$. For the moment, let us consider on $D_1 - 0$ the usual complete hyperbolic conformal metric $m^\star$ with finite area at the end 0; in polar coordinates

$$m^\star = \frac{dr^2 + r^2 d\theta^2}{r^2 (\log r)^2}.$$

For every point z in $D_1 - 0$, we have a covering $p : (D_1, m_{-1}) \to (D_1 - 0, m^\star)$ by the Poincaré disc, which is a local isometry and sends 0 onto z. Applying § 4.1 (Schwarz lemma and the remark) to $f \circ p$, we obtain $|f'(z)|_{m^\star} \leq$cst, for every z. Now, the length of the circle ∂D_r is easy to compute for $m^\star$; it tends to 0 as $r \to 0$. We conclude that the length of the image $f(\partial D_r)$ also tends to 0. □

LEMMA 5.2.3. — *There is a constant c_2 such that for every J-curve $f : D_1 - 0 \to B(x, \varepsilon_0)$ with finite area, the diameter Δ of $S = f(D_1 - 0)$ and the length l of the boundary $f(\partial D_1)$ verify*

$$l \geq c_2 \Delta.$$

Proof. — Let us assume that $l \leq \Delta/4$. Then, we certainly have a point x on S such that the ball $B(x, \Delta/8)$ neither meets $f(\partial D_1)$, nor contains the point $f(0)$ given by theorem 5.1.1 (take two points x', x'' with $\mathrm{dist}(x', x'') = \Delta$; if neither x' nor x'' is convenient, for example if $f(0)$ is near x' and ∂S near x'', notice that $\mathrm{dist}(B(x', \Delta/4), B(x'', \Delta/2)) = \Delta/4$). Applying first lemma 5.2.1 to S, then the monotonicity lemma 4.2.1 to $S \cap B(x, \varepsilon)$ with $\varepsilon = \Delta/8$

$$l^2 \geq \frac{4\pi a}{1 + c_1 \varepsilon_0} \geq \frac{4\pi}{1 + c_1 \varepsilon_0} \cdot \frac{\pi \varepsilon^2}{1 + c_0 \varepsilon_0}$$

thus $l^2 \geq \mathrm{const} \cdot \Delta$. □

Notice that lemma 5.2.3 is also valid for every J-curve $f : \Sigma^\star \to B(x, \varepsilon_0)$ as in lemma 5.2.2 (with a constant c_2 depending on the number of punctures and components in the boundary).

As a consequence of lemma 5.2.3, we can rewrite lemma 5.2.1 for $\Sigma^\star = D_1 - 0$, replacing ε by l (given S, relate the smallest possible value for ε to the diameter):

LEMMA 5.2.4. — *There is a constant c_3 such that for every J-curve $f : D_1 - 0 \to B(x, \varepsilon_0)$ with finite area a, the length l of the boundary verifies*

$$l^2 \geq \frac{4\pi a}{1 + c_3 l}. \square$$

5.3. Lipschitz extension

Lemma 5.2.4, together with lemma 3.3.2 (which remains valid, see the remark before lemma 5.2.1), gives the following complement to theorem 5.1.1:

PROPOSITION 5.3.1. — *Let $f : D_1 - 0 \to B(x, \varepsilon_0)$ be a J-curve with finite area. Then*

$$|f'(z)| \leq \frac{\text{cons.}}{1 - |z|}$$

for every z in $D_1 - 0$.

Proof. — Fixing a point $z \neq 0$, let us consider the J-curve obtained by restriction of f to the disc with centre z and radius $1 - |z|$, punctured at 0. We try to apply the Schwarz lemma. By lemma 5.2.4, for every smaller disc, the area a and boundary length l verify $l^2 \geq P(a)$ for a profile function $P(a) = \max\{h^{-1}(a), C^2a^2\}$ which is defined for $a \geq 0$ by

$$h(u) = u\frac{1 + c_3u^{1/2}}{4\pi}$$

and the constant C is given on the ball by lemma 3.3.2. It is easy to see that the integrability conditions in the proof of Schwarz lemma are satisfied, and this is enough (see the first remark in § 4.1). □

In conclusion, a J-curve $f : D_1 - 0 \to (M, J, \mu)$ with finite area has a continuous extension at 0 and its derivative f' is bounded in a neighbourhood of 0, so f is Lipschitz.

Remark. — In the next §, it will be shown that the extension is smooth and holomorphic at 0.

6. Higher derivatives

6.1. A Schwarz lemma for higher derivatives

Following a classical procedure [9], one can view the derivative Df of a holomorphic map as a holomorphic map itself, apply the Schwarz lemma and get bounds on higher derivatives.

PROPOSITION 6.1.1. — *Let (V, μ) be a compact Riemannian manifold. Let J be an almost complex structure of class $C^{k+\alpha}$ on V. There exist constants ε_0 (depending only on the C^0 norm of μ and on the C^α norm of J) and C (depending only on the C^0 norm of μ and on the $C^{k+\alpha}$ norm of J) such that every J-holomorphic map f of the unit disc to an ε_0-ball of V has its derivatives up to order $k + 1 + \alpha$ near the origin bounded by C.*

Proof. — Let G be the bundle over $D_1 \times M$ whose fibre at (z, m) consists of all $\mathbf{C}$-linear maps $T_z D_1 \to T_m M$. The first jet of a holomorphic map $f : D_1 \to M$ is a lift to G of the graph of f, viewed as a section $D_1 \to D_1 \times M$. There is a unique complex structure $\mathcal{J}$ on G such that first jets of J-holomorphic maps $D_1 \to (M, J)$ are $\mathcal{J}$-holomorphic maps $D_1 \to (G, \mathcal{J})$. An elegant construction of $\mathcal{J}$ is given in the appendix to chapter II. A more down to earth description of it is suggested in exercise 6.1.2 below. The structure $\mathcal{J}$ depends on first derivatives of J only so $\mathcal{J}$ is of class $\mathcal{C}^{\alpha}$ provided J is of class $\mathcal{C}^{1+\alpha}$.

Let $f : D_1 \to M$ be J-holomorphic. Assume that the image $f(D_1)$ lies in an ε_0-ball B of M. We check that its first jet Df satisfies the assumptions of corollary 4.1.4. Proposition 4.1.1 implies that Df takes its values in a compact subset K of G. This subset is a Euclidean ball bundle over B, thus the restriction of $\mathcal{J}$ to K is close to the product structure on $B \times \mathbf{C}^m$, which is tamed by the product symplectic form ω. On K, ω still tames $\mathcal{J}$ (shrink the ball B if necessary), and $\omega = d\lambda$ with λ bounded. Thus Corollary 4.1.4 applies and an estimate on the derivative of Df follows. Repeating the argument leads to a $\mathcal{C}^{k+1}$-estimate when J is of class $\mathcal{C}^{k+\alpha}$.

It remains to climb from $\mathcal{C}^{k+1}$ to $\mathcal{C}^{k+1+\alpha}$. Let g be the k-th derivative of f. Since g is Lipschitz, the nonlinear holomorphic curve equation

$$\bar{\partial} g - q(g)\partial g = 0$$

can be treated as a linear equation with Hölder continuous coefficients. Elliptic regularity (see [5]) implies a $\mathcal{C}^{1+\alpha}$ bound on g. □

We conclude this § by an exercise in which the reader is asked to calculate the complex structure $\mathcal{J}$ in coordinates.

6.1.2. Exercise. — Choosing local coordinates, we assume that $M = \mathbf{R}^{2n}$ with coordinates $y^1, \ldots, y^{2n}$ and that $\Sigma = \mathbf{C}$ with $z = x^1 + ix^2$. Defining an element p in $\pi^{-1}(x, y)$ by the first column of its matrix in the frames $(\partial/\partial x^i), (\partial/\partial y^r)$, gives a local coordinate system (x^i, y^r, p^r) in G; check that the 1-jet Df of a J-curve $y = f(x)$ is given by $p^r = \partial y^r/\partial x^1$.

Consider the isomorphism of the tangent space $T_{(x,y,p)}G$ whose matrix in the frame $(\partial/\partial x^i, \partial/\partial y^r, \partial/p^r)$ is the following

$$\mathcal{J} = \begin{pmatrix} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & 0 & 0 \\ 0 & J(y) & 0 \\ 0 & \frac{\partial J}{\partial y^u}(y) \cdot p^u & J(y) \end{pmatrix}$$

and prove that

$$J \cdot \frac{\partial (Df)}{\partial x^1} = \frac{\partial (Df)}{\partial x^2}$$

(Hint: let $J = (J_r^s)$ denote the matrix of the almost complex structure on M. Write the equations expressing the fact that f is J-holomorphic and differentiate with respect to x^1).

For the germ of a J-curve $y = f(x)$ show that

$$\left(\frac{\partial J}{\partial y^u} \cdot p^u\right) \cdot J + J \cdot \left(\frac{\partial J}{\partial y^u} \cdot p^u\right) = \frac{\partial}{\partial x^1}\left(J^2(y)\right) = 0.$$

Now every element (x_0, y_0, p_0) of G is a $Df(x_0)$ for a suitable germ f [9]. Deduce that $\mathcal{J}^2 = -Id$ and that, for any J-curve f, the 1-jet Df is $\mathcal{J}$-holomorphic.

6.2. Holomorphic extension across a puncture

PROPOSITION 6.2.1. — *Let (M, μ) be a compact Riemannian manifold. Let J be an almost complex structure of class $C^{1+\alpha}$ on M. Every J-holomorphic map $f : D^* \to M$ of finite area extends as a J-holomorphic map of the disc to M.*

Proof. — Let $f : D_1 - 0 \to M$ be a holomorphic map with finite area. According to theorem 5.1.1, f extends continously at 0, thus one can assume that the image of f lies in an ε_0-ball of M. Then proposition 5.3.1 says that the first jet Df takes its values in a compact set K of the bundle of 1-jets G. As in the proof of 6.1.1, $\mathcal{J}$ is tamed by an exact form on K, and corollary 4.1.4 applies : Df is a Lipschitz map $D_1 - 0 \to M$. In particular, it extends continuously at 0, f is of class C^1 and thus holomorphic at 0. □

Remark. — Extra work is needed for boundary punctures, see [10].

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Chapter VIII

Compactness

Pierre Pansu

In this chapter, we will be concerned with the space of J-holomorphic maps of compact Riemann surfaces (S, J_S) into a fixed compact almost complex manifold (V, J). Even if an upper bound on area is imposed, this space is not compact in general.

Example. — The conics $x \mapsto (x, \varepsilon/x)$, $\mathbf{CP}^1 \to \mathbf{CP}^2$ all have the same area. As ε tends to 0, they converge smoothly, except at $x = 0$.

The graph of a dilation $x \mapsto (x, \varepsilon x)$, $\mathbf{CP}^1 \to \mathbf{CP}^1 \times \mathbf{CP}^1$ converges smoothly, as ε tends to 0, except at $x = \infty$.

In both cases, the images $f_\varepsilon(S)$ converge, as subsets in the range V, to a union of two holomorphic curves

In his paper [3], M. Gromov states a compactness property for sets of holomorphic curves, which lies between the convergence of images (as subsets in the range) and the convergence of maps (parameter included): given a sequence $f_j : (S, J_S) \to (V, J)$ of holomorphic maps with bounded areas, there exists a subsequence that converges smoothly away from a finite set of points, at which "bubbles" develop.

This kind of result, which is classical in analytic geometry (E. Bishop's compactness theorem for analytic submanifolds in Kähler manifolds [1]), appeared more recently in analysis on manifolds. In this context, the bubbling off phenomenon was first discovered by J. Sacks and K. Uhlenbeck in their work on harmonic maps of a Riemann surface to a Riemannian manifold [6]. Since then, it has shown up in other variational problems where a noncompact symmetry group arises (see the report by J.P. Bourguignon [2]).

In these notes, which follow [3] closely, a proof of Gromov's compactness theorem for closed holomorphic curves is given. Holomorphic curves with boundary are covered only in an easy special case.

The first step in the proof is the compactness of "cusp-curves", i.e., convergence up to a change of parameter. In the second step, convergence of parametrised curves is obtained as a consequence of the convergence of graphs in $S \times V$.

There are other approaches to compactness theorems, due to T. Parker and J. Wolfson [5] and Rugang Ye [7].

I express hearty thanks to Dusa McDuff and François Labourie for their help in completing these notes.

1. Riemann surfaces with nodes

Let us view a holomorphic curve in a fixed almost complex manifold V as the following set of data:

- an oriented differentiable 2-manifold S;
- a conformal or complex structure J_S on S;
- a holomorphic map $f : S \to V$.

In a degenerating family of holomorphic curves, it may happen that the complex structures J_S themselves degenerate. Thus a first step in understanding holomorphic curves is to understand noncompactness in the moduli space of complex structures on surfaces. This is a classical subject, for which a good reference is [4].

As we have seen in the above examples, topology can change in a degenerating family of holomorphic curves. It turns out a similar phenomenon is already present in the moduli space of complex structures on a surface. It is easy to visualise in terms of metrics of constant negative curvature.

1.1. Compact surfaces with constant curvature

Let S be a compact, orientable surface of genus greater than 1. In 2 dimensions, Riemannian metrics and complex structures are interrelated objects. Given a Riemannian metric and an orientation, rotation by 90^0 in each tangent space is well-defined and defines an almost-complex structure, which is automatically integrable (see chapter II for references). Two Riemannian metrics define the same complex structure if and only if they are conformal. Conversely, in a conformal class of Riemannian metrics, the curvature -1 condition singles out a unique metric. This is the content of the *uniformisation theorem.*

As a consequence, there is a 1-1 correspondence between complex structures on S and Riemannian metrics on S with curvature -1 (this correspondence breaks down for spheres and tori).

Noncompactness in the space of metrics with curvature -1 is easy to understand. The key notion is that of *injectivity radius* (see chapter III). Locally, any metric of curvature -1 is isometric to the hyperbolic plane (i.e., the disc with its hyperbolic metric). The injectivity radius at a point p is the largest r such that the geodesic disc centreed at p is isometric to a geodesic disc of the hyperbolic plane. The injectivity radius of the Riemannian surface is the infimum of the injectivity radii of points of S.

PROPOSITION 1.1.1. — *For every $\varepsilon > 0$, the space of Riemannian metrics on S with injectivity radius greater than ε is compact.*

Indeed, the Gauss-Bonnet formula implies that all metrics with curvature -1 on S have the same area. This means that S can be covered by a number of hyperbolic discs of radius ε that depends only on ε. The gluing data then vary in a compact set. □

Here, S_j converges to S means that there exist diffeomorphisms $\varphi_j : S_j \to S$ which pull back the metric of S_j to metrics on S which converge uniformly in C^∞-topology.

1.2. The modulus of an annulus

It is useful to translate the compactness criterion of proposition 1.1.1 into purely conformal terms. A small injectivity radius means that there exists a short closed geodesic γ, which is not null homotopic. Then γ has a large embedded tubular neighbourhood. In an orientable surface, such a neighbourhood is an incompressible annulus with large modulus.

DEFINITION 1.2.1. — *Let A be an annulus equipped with a smooth complex structure. Then A is either conformal to the punctured Euclidean plane, the punctured disc or a unique cylinder $S^1 \times [0, L]$ where the S^1 factor has unit length. The number L is called the* modulus *of A. Here is an alternative definition.*

$$\frac{1}{L} = \inf \int_A |du|^2 \tag{1.2.2}$$

over smooth functions u on A which take the value 0 *on one boundary component and* 1 *on the other.*

Conversely, if a compact Riemann surface S contains an incompressible annulus A with large modulus, then the conformal metric with curvature -1 has a small injectivity radius. A compactness criterion for sets of compact Riemann surfaces follows, where the following topology is used: the sequence (S_j, J_{S_j}) converges to (S, J_S) if there exist diffeomorphisms $\varphi_j : S \to S_j$ such that the pushed forward complex structures $(\varphi_j)_\star(J_{S_j})$ on S converge uniformly and smoothly to J_S.

PROPOSITION 1.2.3. — *Fix a number L. Consider all compact Riemann surfaces S of fixed genus greater than* 1, *with the following property: the modulus of every incompressible annulus in S is less than L. This space is compact.*

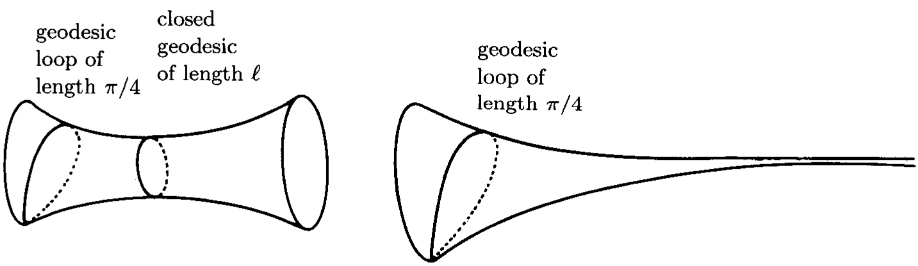
1.3. Degeneration

Given a complete Riemannian surface S with curvature -1, and $\varepsilon > 0$, the ε-*thick* part of S is the set of points where the injectivity radius is greater than ε. Note that the number of connected components of the ε-thick part of S is bounded in terms of the genus and ε.

DEFINITION 1.3.1. — *A sequence of complete surfaces S_j with curvature -1 converges to S if for every $\varepsilon > 0$, the ε-thick part of S_j converges to the ε-thick part of S.*

An obvious extension of Proposition 1.1.1 says that every sequence of complete surfaces with curvature -1 and uniformly bounded area has a convergent subsequence (in the C^∞-topology). The next point is to describe how area and topology can change in the limit.

The part of a compact surface with curvature -1 where the injectivity radius is small is also easy to describe. The model is the quotient of the hyperbolic plane by an isometry which translates a geodesic γ by a distance $\ell < \pi/4$. In this surface, the injectivity radius is an increasing function of the distance from γ. Let us *denote* by $A(\ell)$ the $\pi/8$-thin part of this surface, i.e., the set of points where the injectivity radius is less than $\pi/8$. When ℓ tends to 0, the annulus $A(\ell)$ splits into two isometric parts called *standard cusps*, and denoted by C. The standard cusp C is a complete Riemannian surface with boundary, conformal to a punctured disc. It can be viewed as the quotient of a *horodisc*, an open subset in the hyperbolic plane, by the parabolic rotation which translates boundary points by a distance $\pi/8$. Observe that the area of the ε-thin part of $A(\ell)$ tends to 0 as ε tends to 0 uniformly in ℓ. As a consequence, in the convergence $A(\ell) \to C \cup C$, the areas converge. Furthermore, as ℓ varies from 0 to $\pi/8$, the area of $A(\ell)$ varies between two positive constants.



the annulus $A(\ell)$ the cusp C

Figure 10

The next proposition describes the decomposition of the compact oriented constant curvature surface S into thick and thin parts.

PROPOSITION 1.3.2 (Thick and thin decomposition). — *The set of points in S where the injectivity radius is less than $\pi/8$ is a disjoint union of annuli, each of them being isometric to one of the $A(\ell)$.*

Thus, if a sequence S_j of surfaces with curvature -1 degenerates but converges to S in the sense of definition 1.3.1, a finite (bounded) number of disjoint closed geodesics have lengths that tend to zero, and S is diffeomorphic to the complement of the union of these curves. Furthermore, S contains two isometrically embedded copies of the standard cusp C for each removed curve. As a consequence, S is conformal to a disjoint union Σ of compact Riemann surfaces with points removed (exactly 2 for each removed curve). The total genus of these surfaces is no more than the genus of the S_j. For j large enough, S_j is obtained from Σ by performing a series of connected sums, thus to reconstruct the manifold S_j, its is sufficient to remember which pairs of punctures have to be glued together.

It should be clear by now that a description of degeneration in families of Riemannian surfaces with curvature -1 should include disconnected or non compact surfaces with finite area as well.

1.4. Noncompact surfaces with constant curvature

In general, a complete Riemannian surface with curvature -1 and finite area contains finitely many disjoint isometric copies of the standard cusp, whose union has a compact complement. As a consequence, it is conformal to the complement of a finite subset in a compact surface equipped with a smooth complex structure. Conversely, given a compact Riemann surface Σ and a finite subset $F \subset \Sigma$, the uniformisation theorem applies provided the Euler characteristic of $\Sigma \setminus F$ is negative: there exists a unique complete conformal metric on $\Sigma \setminus F$ with curvature -1 and finite area.

As a consequence, given a compact oriented surface Σ, there is a 1-1 correspondence between the data of a complex structure on Σ together with a finite subset $F \subset \Sigma$ and complete Riemannian metrics on $\Sigma \setminus F$ with curvature -1 (this correspondence now includes spheres with at least three punctures and tori with at least one puncture).

Let us call a *Riemann surface with nodes* the data of a disjoint union of compact surfaces equipped with complex structures, a finite set of marked points called *nodes*, and an equivalence relation identifying certain marked points in pairs. Its genus is defined to be the sum of the genus of all components.

There is a natural topology on the set of Riemann surfaces with nodes. One says that Σ_j converges to Σ if Σ_j is almost obtained from Σ by a connected sum at pairs of identified marked points. This means that there is a disjoint collection A_j of annuli in Σ_j of large modulus, a map φ_j from Σ_j to Σ which is a diffeomorphism away from A_j and collapses each annulus in A_j to a node of Σ, such that the pushed forward complex structures $(\varphi_j)_\star J_{\Sigma_j}$ converge on compact subsets of $\Sigma \setminus \{\text{marked points}\}$ to J_Σ.

By construction, uniformisation yields a homeomorphism between the space of Riemann surfaces with nodes and the space of Riemannian surfaces with nodes with curvature -1 and finite area.

PROPOSITION 1.4.1. — *The space of Riemann surfaces with nodes of genus less than g is compact.*

2. Cusp-curves

2.1. Definitions

DEFINITION 2.1.1. — *A* cusp-curve *in an almost complex manifold* (V, J) *is a* J*-holomorphic map from a Riemann surface with nodes to* V, *i.e., the data of a disjoint union of compact surfaces* Σ_ℓ *equipped with complex structures, a finite set of nodes, and an equivalence relation identifying certain nodes in pairs, together with a* J*-holomorphic map* $f : \cup_\ell \Sigma_\ell \to V$ *compatible with the identifications.*

DEFINITION 2.1.2 (C^k-topology on cusp-curves). — *Let* $f : \cup_\ell \Sigma_\ell \to V$ *be a cusp-curve. Given* $\varepsilon > 0$, *a Hermitian metric* ν *on* Σ_ℓ *and a neighbourhood* U *of the nodes, a neighbourhood of the cusp-curve* f *is defined as follows. It is the set of cusp-curves* $\tilde{f} : \cup_\ell \tilde{\Sigma}_\ell \to V$ *such that there exists a continuous map*

$$\sigma : \cup_\ell \tilde{\Sigma}_\ell \to \cup_\ell \Sigma_\ell$$

with the following properties: σ *is a diffeomorphism except above the nodes; the pull-back of a node is an annulus of modulus* $\geq 1/\varepsilon$ *or a node;*

$$\left\| f - \tilde{f} \circ \sigma^{-1} \right\|_{C^k} < \varepsilon$$

away from U, *where the metric* ν *on* Σ_ℓ *and a fixed metric* μ *on* V *are used when measuring norms;*

$$\left\| J_\Sigma - \sigma_\star^{-1} \tilde{J}_{\tilde{\Sigma}} \right\|_{C^k} < \varepsilon$$

away from U, *and*

$$\left| \text{area}\,(f) - \text{area}\,(\tilde{f}) \right| < \varepsilon.$$

By construction, the examples in the introduction illustrate the convergence of a sequence of holomorphic spheres to the join of two holomorphic spheres.

Figure 11

Since constant maps are holomorphic, the number of distinct curves in the image $f(\Sigma_\ell) \subset V$ can decrease in the limit. The above topology is non Hausdorff, but this should not be taken too seriously. It is the space of non parametrised curves which is Hausdorff.

Cusp-curves with boundary. — Let T be a compact complex manifold with boundary of dimension 1 (i.e., it has an atlas of holomorphic charts onto open subsets of $\mathbf{C}$ or of a closed half plane). Its *double* is a compact Riemann surface S with a natural antiholomorphic involution τ which exchanges T and $S \setminus T$ while fixing the boundary ∂T. If $f : T \to V$ is a continuous map, holomorphic in the interior of T, it is convenient to extend f to S by

$$f = f \circ \tau.$$

DEFINITION 2.1.3. — *A cusp-curve with boundary in (V, J), is the data of finitely many compact Riemann surfaces S_ℓ obtained by doubling surfaces with boundary T_ℓ and a continuous map $f : \cup_\ell S_\ell \to V$, holomorphic in the interior of each T_ℓ, and such that $f \circ \tau = f$. A finite set of "nodes" is given, together with identifications in pairs compatible with τ and f.*

The topology is the same as for closed surfaces. As in the case of closed surfaces, in a convergent sequence of cusp-curves, a finite number of simple closed curves, but also, of simple arcs with endpoints on ∂T_ℓ, may collapse to a node.

2.2. Compactness theorems

THEOREM 2.2.1 (Compactness for closed cusp-curves). — *Let V be a closed Riemannian manifold. Let J_j be a convergent (in $C^{k+\alpha}$) sequence of almost complex structures on V, and $f_j : S \to V$ a sequence of J_j-holomorphic curves with bounded areas. There exists a subsequence which converges (in $C^{k+1+\alpha}$) to a cusp-curve $f : \cup_\ell \Sigma_\ell \to V$, where, topologically, $\cup_\ell \Sigma_\ell$ is obtained from S by collapsing a finite number of disjoint simple closed curves. In particular, the genus cannot increase in the limit*

$$\sum_\ell g(\Sigma_\ell) \leq g(S).$$

THEOREM 2.2.2 (Compactness for cusp-curves with boundary). — *Let V be a closed Riemannian manifold, W a real analytic submanifold of V. Let J_j be a convergent (in $C^{k+\alpha}$) sequence of almost complex structures on V which are integrable in a neighbourhood of W, and for which W is totally real. For every sequence $f_j : (T, \partial T) \to (V, W)$ of J_j-holomorphic curves with boundary in W, and bounded areas, has a $C^{k+1+\alpha}$ convergent subsequence.*

3. Proof of the compactness theorem 2.2.1

3.1. Scheme of the proof

Constant curvature -1 metrics μ_j on S will be chosen so that the maps f_j become uniformly Lipschitz. Gromov's Schwarz Lemma provides us with a Lipschitz bound for maps of large hyperbolic discs immersed in S into small balls of V. By removing a controlled number of points on S and choosing for μ_j the conformal metric with cusps at these points, one can ensure that each (non removed) point of S is the centre of such an immersed disc. One can then choose convergent subsequences of metrics and maps. In the limit, a holomorphic map of finite area of a punctured surface is obtained. It needs be extended across the punctures.

The main tools, Gromov's Schwarz lemma and the removable singularity theorem, are needed in the following form (see chapter VII).

PROPOSITION 3.1.1. — *Let (V, μ) be a compact Riemannian manifold. Let J be an almost complex structure of class $C^{k+\alpha}$ on V. There exist constants ε_0 (depending only on the C^0 norm of μ and on the C^α norm of J) and C (depending only on the C^0 norm of μ and on the $C^{k+\alpha}$ norm of J) such that every J-holomorphic map of the unit disc to an ε_0-ball of V has its derivatives up to order $k+1+\alpha$ near the origin bounded by C.*

PROPOSITION 3.1.2. — *Let (V, μ) be a compact Riemannian manifold. Let J be an almost complex structure of class $C^{1+\alpha}$ on V. Every J-holomorphic map $f : D^\star \to V$ of finite area extends to a J-holomorphic map of the disc to V.*

A coarse form of monotonicity will also be needed. The constant ε_0 in this statement is the one in proposition 3.1.1.

PROPOSITION 3.1.3. — *Let (V, μ) be a compact Riemannian manifold. Let J be an almost complex structure of class C^α on V. Let $x \in V$. If S is a J-holomorphic curve in V with boundary contained in the sphere $\partial B(x, \varepsilon_0)$, then*

$$\text{area}\,(S) \geq \varepsilon_0^2.$$

3.2. Choice of metric on the domain

Let $f : S \to V$ be a holomorphic map. In this paragraph, a metric with curvature -1 will be constructed on S (perhaps with finitely many points removed) in such a way that the image by f of every unit geodesic disc in this metric has diameter less than ε_0, the constant that enters the Schwarz lemma 3.1.1.

Removal of a net. — In V, let us choose a maximal system of disjoint ε_0-balls with centres on $f(S)$. Let F be the set of centres. According to the monotonicity property 3.1.3, $f(S)$ leaves a definite quantum of area inside each of these balls, thus the number of points in F is bounded by a constant N that depends only on the area and ε_0.

Since F is maximal, every point of $f(S)$ lies at a distance at most $2\varepsilon_0$ from a point of F. Next we want to bound the diameter of discs contained in $f(S) \setminus F$. The trick is to construct annuli with large modulus.

Annuli. — Let A be an annulus contained in $f(S) \setminus F$, whose boundary components have length at most ε_0. Then the diameter of A is less than $12\varepsilon_0$. Indeed, every point of A lies at a distance at most $2\varepsilon_0$ of the boundary (by construction of F), and, for the same reason, the boundary components lie at most $4\varepsilon_0$ apart.

If a Riemannian annulus has a large modulus, then a slightly smaller isotopic annulus has a short boundary. Indeed, let $\Phi : S^1 \times [0, L] \to A$ be a conformal mapping. Denote by

$$S_t^1 = \Phi(S^1 \times \{t\}).$$

Then

$$\begin{aligned} \text{area}\,(A) &\geq \int_0^L \left(\int_{S^1 \times \{t\}} |\Phi'|^2 \right) dt \\ &\geq \int_0^L \left(\int_{S^1 \times \{t\}} |\Phi'| \right)^2 dt \\ &\geq \int_0^L \text{length}\,(S_t^1)^2\, dt. \end{aligned}$$

Thus there exists a t such that

$$\text{length}\,(S_t^1) \leq (\text{area}\,(A)/L)^{1/2}.$$

Splitting A into three *adjacent* annuli (i.e., they share boundary components like plumbing fixtures), one finds short curves in the extreme annuli. We sum up the discussion in a

LEMMA 3.2.1. — *Let A_0, A, A_1 be adjacent annuli in $S \setminus f^{-1}(F)$. If the moduli of A_0 et A_1 are larger than L, a constant which depends only on the area of $f(S)$ and on ε_0, then*

$$\text{diameter}\, f(A) \leq 12\varepsilon_0.$$

Bound on the diameter of discs in the $\mu^\star$-metric. — We choose to give $S \setminus f^{-1}(F)$ the unique complete conformal metric $\mu^\star$ with curvature -1.

LEMMA 3.2.2. — *There exists a constant ρ depending only on ε_0 and on the area of $f(S)$, such that every geodesic disc of radius ρ in the metric $\mu^\star$ is contained in an annulus A admitting adjacent annuli with moduli greater than L as in lemma* 3.2.1.

Proof. — Fix $r = \min\{\pi/8, \exp(-2L)\}$ and set $\rho = r^2$. The thick and thin decomposition of surfaces with constant curvature, (proposition 1.1.1 and § 1.4) tells us what a geodesic disc of radius r looks like. Either it is isometric to a hyperbolic geodesic disc (when the injectivity radius is larger than r, the thick case), or it is contained in a tube around a closed geodesic $A(\ell)$ or in a standard cusp C (when the injectivity radius is less than r, the thin case).

- The thick case: Let x be a point of $S \setminus f^{-1}(F)$ where the injectivity radius is larger than $\pi/8$. Choose a point $y \in \partial B(x, 2r)$. The shells

$$\begin{aligned} A_0 &= B(y,r^2) - y \\ A &= B(y,3r^2) - B(y,r^2) \\ A_1 &= B(y,r) - B(y,3r^2) \end{aligned}$$

are topological annuli. Also $B(x,\rho) \subset A$, modulus $(A_0) = +\infty$ and

$$\text{modulus}\,(A_1) \approx \frac{1}{2}\log\frac{1}{r} \geq L$$

as required.

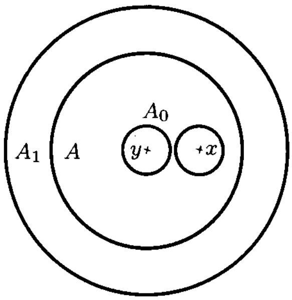


Figure 12

- The thin case: Then x is either close to a closed geodesic or in a cusp.

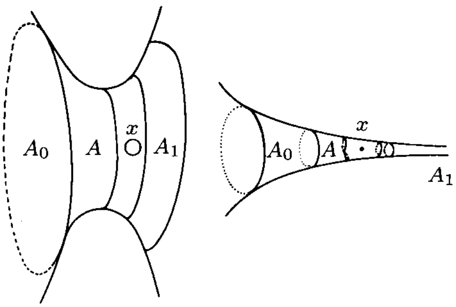


Figure 13

In both cases, the function

$$u = d(\cdot, \text{geodesic}) - d(x, \text{geodesic})$$

or

$$u = d(\cdot, \text{cusp}) - d(x, \text{cusp})$$

is smooth on $B(x, \pi/8)$, and $|du| = 1$. The sets

$$\begin{aligned} A_0 &= u^{-1}[-\pi/8, -r^2] \\ A &= u^{-1}[-r^2, +r^2] \supset B(x,\rho) \\ A_1 &= u^{-1}[+r^2, \pi/8]. \end{aligned}$$

are topological annuli. Using formula (1.2.2) one gets

$$\text{modulus}\,(A_i) \geq \frac{(\pi/8)^2}{r\text{sh}\,(\pi/8)}$$

of the order of $1/r$, which is much larger than L. □

Conclusion. — For all $x \in S \setminus f^{-1}(F)$, one has

$$|f'|^{\mu}_{\mu^\star} \leq \text{const}\ (V, \mu, \|J\|_{C^\alpha}\,,\ \text{area})\,.$$

Indeed, if Φ denotes the conformal immersion of the unit Euclidean disc onto the $\mu^\star$ geodesic ball $B(x, \rho)$, then

$$|f'|^{\mu}_{\mu^\star}(x) \leq \text{const} \cdot |(f \circ \Phi)'(0)|^{\mu}_{m_0}$$

where the constant only depends on the radius ρ. Since the image of $f \circ \Phi$ has a small diameter, the Schwarz lemma 3.1.1 applies and a bound on the derivatives of $f \circ \Phi$ follows.

Notice that this bound does not depend on the injectivity radius of the metric $\mu^\star$.

3.3. End of proof

Convergence of metrics. — Again, we are given a sequence $f_j : S_j \to (V, \mu)$ of holomorphic curves with bounded area and genus. On each of them, a finite set F_j is removed and a metric $\mu_j^\star$ is chosen in order that the maps f_j are uniformly Lipschitz.

Since the genus of S_j and the number of points in F_j are bounded, the area of $\mu_j^\star$ stays bounded, and the compactness criterion 1.3.1 for surfaces with curvature -1 applies. Up to taking a subsequence, the metrics $\mu_j^\star$ can be viewed as smoothly convergent metrics on larger and larger subsets of a fixed surface Σ which is diffeomorphic to the complement of finitely many "nodes" in a compact surface $\overline{\Sigma}$. The maps f_j are then uniformly Lipschitz maps on these subsets. A subsequence converges uniformly to a holomorphic map f defined on Σ. On compact subsets of Σ, the convergence is as smooth as the almost complex structure on V, thanks to the estimate on higher derivatives included in the Schwarz lemma and to elliptic regularity.

Convergence of areas. — According to 1.3.1, the metrics $\mu_j^\star$ converge smoothly away from sets of smaller and smaller area. Since the f_j are uniformly Lipschitz, their Jacobians are bounded and so

$$\text{area}\,(f(\Sigma)) = \lim_{j\to\infty} \text{area}(f_j(S_j)).$$

Bubbles intersect. — According to the removable singularity theorem 3.1.2, the map f extends across the nodes. It remains to prove that f is compatible with the identifications, i.e., it takes equal values at nodes that arise from the degeneration of the same annulus $A(\ell)$. As in the proof of the removable singularity theorem, one shows that if A_ε is the ε-thin part of $A(\ell)$, then the diameter of $f(A_\varepsilon)$ tends to 0 with ε. If not, the holomorphic annulus $f(A_\varepsilon)$ which has a short boundary and small area (Lipschitz estimate) would intersect a large ball, contradicting monotonicity. This ends the proof of theorem 2.2.1.

Remark. — The convergence of areas and monotonicity imply that the images $f_j(S_j)$ converge as subsets of V.

3.4. Compactness for cusp-curves with boundary

Under our strong assumption (the totally real data W is real analytic), only minor changes are needed to adapt the proof. The idea is that in the proof of the estimate for the first derivative of a conformal map

$$f : (S, \mu^\star) \to (V, \mu)$$

only *intrinsic* properties of the metric $\mu_{|f(S)}$ are used, like isoperimetric inequalities. In this respect, maps which are a mixture of holomorphic and anti-holomorphic are just as good as holomorphic maps.

LEMMA 3.4.1. — *Let W be a real analytic totally real submanifold in (V, J). Assume that J is integrable in a neighbourhood of W. Then there exists a unique anti-holomorphic involution τ_V defined in a neighbourhood of W whose fixed point set is W.*

It is sufficient to produce local holomorphic charts of V that take W to $\mathbf{R}^m \subset \mathbf{C}^m$. We can assume that $V = \mathbf{C}^m$. Write $\mathbf{C}^m = \mathbf{R}^m \oplus i\mathbf{R}^m$. Locally, W is the graph of a real analytic map φ of $\mathbf{R}^m$ to $\mathbf{R}^m$, i.e.,

$$W = \{x + i\varphi(x) \in \mathbf{C}^m;\ x \in \mathbf{R}^m\}.$$

The map φ extends holomorphically to a neighbourhood of $\mathbf{R}^m$ in $\mathbf{C}^m$. The map

$$z \mapsto z + i\varphi(z)$$

is a local diffeomorphism, as follows from the assumption $T_wW \cap iT_wW = \emptyset$. □

LEMMA 3.4.2. — *Assume the triple (V, J, W) satisfies the hypotheses of the previous lemma. Choose on V a Hermitian metric μ which is invariant under the involution τ_V. Let $f : S \to V$ be a cusp-curve with boundary (remember this means that $f \circ \tau = f$) such that $f(\partial T) \subset W$. Then the pull-back metric $f^\star \mu$ sur S is smooth and satisfies the isoperimetric inequalities and monotonicity holds.*

Remark. — These isoperimetric inequalities are the standard ones involved in the statement of the Schwarz lemma in chapter VII.

The proof follows from the *Schwarz reflection principle*.

THEOREM 3.4.3 (Reflection principle). — *Let T be a Riemann surface with boundary, let τ be the natural antiholomorphic involution on the doubled surface S. Let $f : (T, \partial T) \to (V, W)$ be holomorphic. The formula*

$$\tilde{f}(x) = \tau_W \circ f \circ \tau(x)$$

defines a holomorphic extension of f to a neighbourhood of T in S.

The identity $f^\star \mu = \tilde{f}^\star \mu$ proves lemma 3.4.2. □

Choice of a metric on the domain. — Keep the construction τ-invariant. Arrange things so that F does not intersect W. Since the metric $\mu^\star$ is τ-invariant, the boundary ∂T is totally geodesic.

Convergence of metrics on the domain. — Observe that in the limit, the geodesics which collapse are of three types:

- simple closed curves;
- simple arcs joining two boundary points
- boundary components.

The presence of nodes on ∂T causes no extra difficulty.

Non compact ranges. — All of these results persist when, while W is compact, V has a *coercivity* property that prevents holomorphic curves with boundary on W from escaping to infinity. In the case when $W = \mathbf{C}^m$, monotonicity, which holds globally, implies coercivity.

4. Convergence of parametrised curves

Finally, we are concerned with maps $f_j : S \to V$ where the complex structure on S is fixed. In this case, the "bubbles", i.e., the extra components in a limiting cusp-curve, are always spheres. This remains true if the complex structure is allowed to vary in a compact subset of the moduli space of S.

4.1. Graphs

The trick is to apply cusp-curve compactness to the graphs $\mathrm{gr}(f_j) : S \to S \times V$ or more generally to sections of bundles $X \to S$.

THEOREM 4.1.1. — *Let X be a compact manifold and $\pi : X \to S$ a fibration over a Riemann surface S. Let J_j be a $C^{k+\alpha}$-convergent sequence of almost complex structures on X, such that for each j the fibration $\pi : (X, J_j) \to S$ is holomorphic. Let $f_j : S \to X$ be a sequence of sections with bounded area. Then there exist a finite subset $\Gamma \subset S$ and a J_∞-holomorphic section $f_\infty : S \to X$ such that*

(i) a subsequence of the f_j converges in $C^{k+1+\alpha}$ to f_∞ away from Γ;

(ii) if $\gamma \in \Gamma$ and $f_j(\gamma)$ do not converge to $f_\infty(\gamma)$, then the fiber X_γ contains a non trivial rational curve

$$\varphi_\gamma : S^2 \to X_\gamma$$

which passes through $f_\infty(\gamma)$;

(iii) for large j, the homotopy class of f_j in $[S, X]$ is the same as

$$f_\infty + \sum_\gamma \varphi_\gamma$$

(care has to be taken if S is not a sphere, but if S is a sphere, the formula holds in $\pi_2(X)$).

Proof. — Let $F : \cup_\ell \Sigma_\ell \to X$ be a limiting cusp-cuve for some subsequence of f_j. Let $\tilde{F} = \pi \circ F$. Since $\tilde{F}$ is holomorphic, the domain splits as

$$\cup_\ell \Sigma^\ell = \Sigma_1 \cup \Sigma_2$$

where $\tilde{F}$ is constant on each component of Σ_1, and $\tilde{F}$ is a ramified covering on each component of Σ_2. This implies that

$$\mathrm{genus}\,(\Sigma_2) \geq \mathrm{genus}\,(S).$$

According to theorem 2.2.1,

$$\mathrm{genus}\,(\Sigma_1 \cup \Sigma_2) \leq \mathrm{genus}\,(S).$$

Thus the surface Σ_1, whose genus is 0, is a union of spheres.

Let x be a point of S which is neither in $\Gamma = \tilde{F}(\Sigma_1)$ nor a branch point of $\tilde{F}$. One can assume that the $f_j(x)$ converge. As the $f_j(S) \cap X_x$ converge in the Hausdorff sense to $F(\cup_\ell \Sigma^\ell) \cap X_x$, one concludes that $F(\cup_\ell \Sigma^\ell)$ intersects X_x in a single point. As a consequence, the ramified covering $F : \Sigma_2 \to S$ is an isomorphism, and $F(\Sigma_2)$ is the image of a section f_∞. As the $f_j(S)$ Hausdorff converge to the image of a section, the maps f_j converge uniformly to f_∞ in a neighbourhood of x. Since all of the f_j send this neighbourhood into a small ball of X, the Schwarz lemma applies, and the convergence is in fact $C^{k+1+\alpha}$. □

4.2. The corresponding statement for curves with boundary

THEOREM 4.2.1. — *Let $\pi : X \to T$ be a fibration over a Riemann surface with boundary T. Let W be a submanifold of $\pi^{-1}(\partial T)$. Let J_j be a $C^{k+\alpha}$-convergent sequence of almost complex structures on X. Assume that for each j, $\pi : (X, J_j) \to S$ is holomophic, that W is totally real with respect to J_j, and that J_j is integrable in a neighbourhood of W. Let $f_j : (T, \partial T) \to (X, W)$ be a sequence of J_j-holomorphic sections of π, with bounded areas. Then there exists a finite subset Γ in T and a J_∞-holomorphic section $f_\infty : T \to X$ such that*

(i) *a subsequence of the f_j $C^{k+1+\alpha}$-converges to f_∞ away from Γ;*

(ii) *if $\gamma \in \Gamma$ and $f_j(\gamma)$ does not converge to $f_\infty(\gamma)$, then the fiber X_γ contains a non trivial rational curve $\varphi_\gamma : S^2 \to X_\gamma$ (resp. a holomorphic disc $\varphi_\gamma : (D, \partial D) \to (X_\gamma, W_\gamma)$ if $\gamma \in \partial T$) which passes through $f_\infty(\gamma)$;*

(iii) *for large j, the homotopy class of f_j in $[(T, \partial T, (X, W))]$ is the same as*

$$f_\infty + \sum_\gamma \varphi_\gamma$$

(care has to be taken if T is not a disc, but if T is a disc, the formula holds in $\pi_2(X, W)$).

4.3. Simple homotopy classes

Under certain extra assumptions which forbid bubbles, a compactness theorem for parametrised curves will follow. Here are two typical assumptions.

- There are no nontrivial rational curves in the fibres. This appears in the proof that a compact embedded Lagrangian submanifold in $\mathbf{C}^m$ cannot be exact (one argues by contradiction).
- The homotopy class of f_j in $[S, X]$ is simple and, in the case when $S = S^2$, normalise the parametrisation in three points.

DEFINITION 4.3.1. — *Let J be an almost complex structure on V. A homotopy class*

$$\beta \in \pi_2(V)$$

is J-simple if in any decomposition $\beta = \sum_j f_j$ where $f_j : S^2 \to V$ is J-holomorphic, at most one of the f_j is nonconstant.

Example. — Let $V = S^2 \times V_1$, equipped with an almost complex structure J tamed by $\omega = \omega_1 \oplus \omega_2$. Let β be the homotopy class of the first factor. Assume that, for every J-holomorphic curve c in V_1

$$\int_c \omega_2 \quad \text{is an integral multiple of} \quad \int_\beta \omega_1.$$

then the class β is J-simple.

COROLLARY 4.3.2. — *Let $\beta \in \pi_2(V)$ be a J-simple class. Fix three distinct points s_1, s_2, s_3 in S^2. For all A, all $\delta > 0$, there exists an $\varepsilon > 0$ such that the set of J-holomorphic maps $f : S^2 \to V$ which satisfy*

$$\begin{aligned} \operatorname{area}(f) &\leq A, \\ \operatorname{dist}(f(s_j), f(s_k)) &\geq \delta, \\ \left\|\overline{\partial} f\right\|_{C^\alpha} &\leq \varepsilon \end{aligned}$$

is compact.

The operator $\overline{\partial} f$ is explained in chapter V. It takes its values in the space of sections of a certain bundle $\overline{X}$. Every such section g determines an almost complex structure J_g on $X = S^2 \times V$, and the equation $\overline{\partial} f = g$ means that the graph of f is J_g-holomorphic. Furthermore, as g tends to 0, the almost complex structure J_g tends to the product structure on X.

Proof. — We prove the corollary by contradiction. Let f_j be a noncompact sequence of J_{g_j}-holomorphic curves with bounded areas, which satisfy the normalisation condition in the statement, and where the g_j tends to 0. Let $F : \cup_\ell \Sigma^\ell \to X$ be a limiting cusp-curve of the graphs of the f_j (it is holomorphic with respect to the product almost complex structure on X). Since β is J-simple, at most one of the maps

$$\Sigma^\ell \xrightarrow{F} X \longrightarrow V$$

is nonconstant. If the limiting section f_∞ is non constant in V, then there are no bubbles and compactness holds. Otherwise f_∞ is constant and there is exactly one bubble over some point $\gamma \in S^2$. Away from γ, the f_j converge uniformly to a constant map but this contradicts the assumption $\operatorname{dist}(f(s_j), f(s_k)) \geq \delta$. □

Example. — Fix three disjoint submanifolds Σ_1, Σ_2, Σ_3 in $S^2 \times V$ and require that $f(s_i) \in \Sigma_i$. In this case the conclusion is that compactness holds for J_g-holomorphic curves with g small enough (this is theorem 2.3.C of [3]).

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Chapitre IX

Exemples de courbes pseudo-holomorphes en géométrie riemannienne

François Labourie

Rappelons qu'une *structure presque complexe* sur une variété est la donnée d'un champ d'endomorphismes de carré -1 du fibré tangent. Une application *pseudo-holomorphe* entre deux variétés presque complexes est une application dont la différentielle commute avec les structures complexes. Enfin une *courbe pseudo-holomorphe* dans une variété presque complexe est une surface réelle dont le plan tangent est stable par rapport à la structure complexe (voir les chapitres précédents et notamment II et III).

Notre but dans ce chapitre est, dans un premier temps, de traduire dans le langage des courbes pseudo-holomorphes un certain nombre de problèmes classiques de géométrie riemannienne sur les surfaces. Nous utiliserons en particulier cette interprétation pour étudier les dégénérescences possibles des solutions de ces problèmes. Nous étudierons ainsi

1. les immersions isométriques elliptiques (problème de Weyl), d'une surface S dans une variété M de dimension 3,

2. les surfaces à courbure de Gauss prescrite (problème de Minkowski, Monge-Ampère elliptique) dans une variété M de dimension 3,

3. les surfaces à courbure moyenne prescrite, et en particulier les surfaces minimales, dans une variété M,

4. les applications harmoniques dont la source est une surface,

5. les équations elliptiques du deuxième ordre sur les surfaces.

Pour chacun de ces cas, nous expliciterons la construction permettant de transcrire ces exemples dans le langage pseudo-holomorphe. En général, l'interprétation est la suivante : les solutions des problèmes considérés sont *a priori* des surfaces dans une variété, $S \times M$ dans le premier exemple (en considérant le graphe de l'immersion) et M dans les exemples 2 et 3. Ces surfaces se relèvent alors comme des surfaces tangentes à un sous-fibré d'holonomie — au sens de [5] — dans un certain espace de jets E se fibrant sur notre espace de base. Dans le cas des exemples 2 et 3, le fibré que nous considèrerons est le fibré unitaire tangent et le relevé se fait par l'application de Gauss. Chacune des fibres du sous-fibré d'holonomie sera alors munie d'une structure presque complexe qui stabilise l'espace tangent du relevé de notre surface. Réciproquement, si une courbe pseudo-holomorphe tangente à ce sous-fibré se projette sur une surface dans notre espace de base, cette surface est solution du problème.

Ces données suffisent alors pour appliquer la théorie des courbes pseudo-holomorphes. En particulier, nous nous sommes intéressés dans les deux premiers exemples au problème d'existence. Les résultats obtenus l'ont été en utilisant (comme dans le cas des courbes pseudo-holomorphes) la méthode de continuité. Cette méthode consiste schématiquement à partir d'une solution pour un problème proche, à la déformer et obtenir notre solution comme limite de cette déformation. L'un des points cruciaux est alors bien entendu d'obtenir un résultat de compacité pour une suite de solutions, et c'est à ce moment que les résultats de Gromov et en particulier le lemme de Schwarz s'avèrent particulièrement efficaces pour contrôler la convergence et décrire les dégénérescences.

Essayons de dérire les dégénérescences de solutions, à nouveau schématiquement, dans notre cas. Si on étudie une suite de solutions, deux types de divergences peuvent apparaître. D'une part, celles liées à l'apparition de bulles dans la suite de courbes pseudo-holomorphes qui se traduisent elles mêmes par des bulles pour les solutions (l'exemple des surfaces minimales est bien connu). D'autre part, et c'est le nouveau phénomène, ceux provenant de l'existence de courbes pseudo-holomorphes parasites non associées à des solutions de notre problème. En effet, une courbe pseudo-holomorphe dans notre espace de jets ne se projette pas nécessairement sur une surface dans l'espace de base. Si la projection est de rang 0, elle est incluse dans la fibre, ce qui dans le cas des surfaces minimales correspond à l'apparition de singularités différentiables. Si la projection est de rang 1, on obtient une dégénérescence *filiforme* comme dans le cas des immmersions isométriques elliptiques (voir le théorème 1.1.2). Ce dernier type de dégénéresecence apparaît lorsque l'on ne borne plus l'aire, comme nous allons l'expliquer maintenant.

Imaginons pour simplifier que notre espace E soit un fibré au dessus du disque D et que les courbes pseudo-holomorphes soient des graphes au dessus de D. A aire bornée, la théorie de Gromov — qu'il faut adapter très soigneusement dans le cas des variétés à bord — montre que toute suite de courbes pseudo-holomorphes converge vers une limite elle-même pseudo-holomorphe, modulo le détachement de bulles. Pour des raisons très simples, dans le cas d'une suite de courbes graphes, les bulles apparaissent dans la fibre. Dans le cas des immersions isométriques elliptiques par exemple, la fibre ne peut contenir de courbes pseudo-holomorphes pour la raison très forte que son espace tangent ne contient pas de sous-espaces complexes. Si maintenant l'on ne borne plus l'aire et que les bulles sont exclues, la suite des courbes pseudo-holomorphes va converger vers une *lamination* (imaginons une spirale s'enlaçant sur un cercle). Les points récurrents de cette lamination ne peuvent être tracés sur des courbes graphes dans un de leurs voisinages. Dans le cas des immersions isométriques, ces points récurrents se trouvent sur ces courbes pseudo-holomorphes *parasites* qui se projettent sur le disque en des géodésiques. Une description précise de l'apparition de ces points récurrents permet d'obtenir de nouveaux résultats.

En ce qui concerne les deux premiers exemples, nous présenterons également les résultats géométriques que cette interprétation nous a permis d'obtenir.

Dans une deuxième partie, nous rappellerons et énoncerons quelques lemmes concernant les courbes pseudo-holomorphes. Les deux premiers, et en particulier le lemme de Schwarz, permettent de faire converger une suite d'applications pseudo-holomorphes. Les deux suivants permettent de décrire la limite, en particulier de contrôler l'apparition de singularités et le comportement du bord d'une suite de courbes pseudo-holomorphes.

A la fin du paragraphe sur les immersions isométriques elliptiques, nous esquisserons une démonstration utilisant ces lemmes.

Une partie de ce qui est rédigé ici est déjà paru ailleurs de manière dispersée, nous avons choisi de réunir les passages utilisant les courbes pseudo-holomorphes (§§ 1 et 2), de présenter de nouveaux exemples (§§ 3.1, 3.2 et 3.3) d'isoler des résultats pseudo-holomorphes (§ 3.3) et de donner dans l'introduction un guide de lecture, afin d'essayer de montrer l'utilité de l'application de la théorie de Gromov à l'analyse globale.

Signalons encore que le codage pseudo-holomorphe permet également d'obtenir, en utilisant la positivité des intersections (chapitre VI), des résultats d'unicité de solutions, en particulier pour les immersions elliptiques. Nous ne développons pas ce point de vue ici et renvoyons à [6].

Je tiens à remercier ici Pierre Pansu, qui m'a patiemment expliqué le lemme de Schwarz, et Misha Gromov qui m'a suggéré ces problèmes.

1. Immersions isométriques elliptiques

Ce paragraphe est un résumé de la première partie de [6].

1.1. Définitions et résultats

Une immersion isométrique *elliptique* (ou localement convexe) (resp. *ε-elliptique*) d'une surface S dans une variété M de dimension 3, est une immersion isométrique dont le discriminant de la deuxième forme fondamentale (par rapport à la prmière forme fondamentale) est positif en tout point (resp. supérieur à un ε positif). Ceci est automatiquement le cas si, par exemple, la courbure sectionnelle de M est inférieure à un K_0, et la courbure de S est en tout point strictement supérieure à K_0.

Nous nous sommes intéressés à la divergence d'une suite de telles immersions. On se donne donc une suite (f_n) d'immersions ε-elliptiques d'une surface S dans une variété riemannienne (M, g) de dimension 3, telle que les métriques induites convergent de façon C^∞ vers une métrique g_0. On suppose également que (f_n) converge de façon C^0 vers une application f_0 ; il suffit pour cela par exemple que M soit compacte. Notre premier résultat exhibe un critère permettant d'affirmer sous une hypothèse assez faible que notre suite converge de façon C^∞. Remarquons que les théorèmes de ce type exigent habituellement une borne uniforme sur la courbure moyenne.

THÉORÈME 1.1.1. — *Si l'intégrale de la courbure moyenne des $f_n(S)$ est majorée, la suite (f_n) converge C^∞ sur tout compact vers une immersion isométrique f_0 de (S, g_0) dans (M, g).*

Notre deuxième résultat nous permet de décrire la façon dont dégénère notre suite d'immersions. Un point x de S est dit *singulier* si la suite f_n ne converge pas (pour la topologie C^∞) au voisinage de x.

THÉORÈME 1.1.2. — *Soit (f_n) une suite d'immersions ε-elliptiques de (S, g_0) dans (M, g) convergeant de manière C^0 vers une application f_0. Si la suite (f_n) ne converge pas de manière C^∞ au voisinage d'un point x de S, dont nous dirons alors que ce point est singulier. Il existe alors une unique géodésique γ de S, passant par x, telle que f_0 soit une isométrie de γ dans une géodésique de M. De plus, tous les points de γ sont singuliers.*

Donnons un exemple du phénomène décrit par ce théorème. On peut construire (voir [11]) une famille (f_n) d'immersions isométriques de la sphère moins deux points antipodaux dans $\mathbf{R}^3$. Chaque immersion f_n s'enroule alors n fois autour de l'axe des pôles, la limite étant un segment (figure 14).

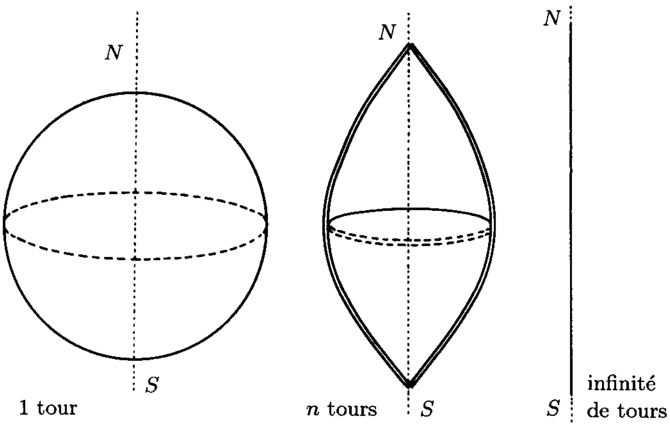

Figure 14

Nous nous proposons maintenant de décrire le 1-jet d'une immersion isométrique d'une surface S dans une variété M de dimension 3. Ce jet est à valeur dans l'ensemble $E = \mathrm{Isom}(TS, TM)$ des isométries linéaires de TS dans TM. Notre but est de munir un ouvert O de E d'une structure presque complexe telle que le 1-jet d'une immersion isométrique elliptique (ou localement convexe) définisse une courbe pseudo-holomorphe de O. L'aire induite sur ces courbes pseudo-holomorphes sera alors l'intégrale de la courbure moyenne.

Ceci est la première étape de la démonstration du théorème 1.1.1.

Nous verrons aussi que d'autres courbes pseudo-holomorphes "parasites", les *surfaces de pli*, s'introduisent naturellement (proposition 1.6.2). Démontrer le théorème 1.1.2 revient alors à démontrer que le graphe du 1-jet d'une suite d'immersions isométriques elliptiques converge soit vers un graphe, soit vers une telle surface de pli.

1.2. Espace des jets d'immersions isométriques

Notre première étape va être de décrire E et plus particulièrement son espace tangent. L'espace E est naturellement un fibré sur $S \times M$. On notera π_S et π_M, les projections sur S et M respectivement et pour simplifier on notera leur différentielle de la même manière.

La fibre au point (s, m) est $\mathrm{Isom}(T_sS, T_mM)$, l'ensemble des isométries de T_sS dans T_mM. L'espace tangent à la fibre en un point (s, m, g) s'identifie à l'ensemble des applications linéaires f de T_sS dans T_mM qui vérifient

$$\langle f(u) \mid g(u) \rangle = 0$$

où u appartient à T_sS et $\langle \mid \rangle$ désigne le produit scalaire de T_mM. Une telle f s'écrit nécessairement $f = g \wedge v$, où v appartient à T_mM, et $\wedge$ désigne le produit vectoriel dans T_mM (nous supposerons M orientée). Cet espace tangent s'identifie donc canoniquement à T_mM.

On munit ce fibré de la connexion induite par les connexions de Levi-Civita sur S et M, on note alors π_F la projection sur l'espace tangent à la fibre qui s'en déduit.

Nous avons donc décomposé l'espace tangent à E en

$$T_{(s,m,g)}E = T_sS \oplus T_mM \oplus T_mM$$

à l'aide des projections π_S, π_M et π_F.

1.3. 1-jets d'immersions isométriques

Soit maintenant f une immersion isométrique de S dans M. Nous noterons $f_\star$ la différentielle de f, j^1f son 1-jet, ∇ la connexion de M, n le vecteur normal extérieur à la surface $f(S)$ et J_0 la structure complexe naturelle sur $f_\star(TS)$ induite par la métrique et l'orientation, donnée par $J_0(u) = n \wedge u$.

Nous allons maintenant décrire l'espace des vecteurs tangents à la surface $j^1f(S)$ dans E.

PROPOSITION 1.3.1. — *A l'aide de la décomposition précédente, l'espace tangent à $j^1 f(S)$ est l'ensemble des vecteurs de la forme*

$$v = (u, f_\star(u), -J_0 \nabla_{f_\star(u)} n),$$

où u désigne un vecteur de TS.

Démonstration. — Seul le dernier terme est à identifier. Considérons d'abord l'espace tangent à la fibre comme un espace d'applications linéaires. Le dernier terme est alors l'application $\pi_F(v)$ qui à w associe $\pi_F(v)(w)$ où

$$\pi_F(v)(w) = \nabla_{f_\star(u)} f_\star(w) - f_\star(\nabla_u w) = \mathrm{II}(u, w) n$$

où II(,) désigne la deuxième forme fondamentale de la surface immergée. Or

$$\begin{aligned} \mathrm{II}(u, w) n &= -\langle \nabla_{f_\star(u)} n \mid f_\star(w) \rangle n \\ &= (n \wedge \nabla_{f_\star(u)} n) \wedge f_\star(w) \\ &= f_\star(w) \wedge -J_0 \nabla_{f_\star(u)} n. \square \end{aligned}$$

1.4. Cas elliptique

Dans le cas elliptique, la deuxième forme fondamentale est une métrique. On a alors

$$\nabla_{f_\star(u)} n = k J_0 J$$

où k^2 est le produit des courbures principales de la surface (autrement dit la courbure gaussienne extrinsèque) et J désigne la structure complexe associée à la deuxième forme fondamentale qui respecte l'orientation, c'est-à-dire la rotation d'angle $\frac{\pi}{2}$ pour cette métrique. En particulier, nous avons :

PROPOSITION 1.4.1. — *L'espace tangent à $j^1 f(S)$ dans E est constitué des vecteurs de la forme*

$$v = (u, f_\star(u), k f_\star(Ju)),$$

où u désigne un vecteur de TS.

L'équation de Gauss s'écrit

$$k = K(TS) - K(f_\star(TS))$$

(la courbure gaussienne extrinsèque est la différence entre la courbure gaussienne et la courbure sectionnelle du plan tangent pour la métrique ambiante). Il en résulte que la courbure extrinsèque est une fonction parfaitement définie sur E.

1.5. Structure presque complexe

Soit O l'ouvert de E constitué des points où la courbure extrinsèque k est strictement positive. Nous allons munir O d'une structure presque complexe telle que $j^1 f(S)$ soit une courbe pseudo-holomorphe lorsque f est une immersion isométrique elliptique.

Décomposons TE d'une nouvelle manière

$$T_{(s,m,g)}E = V \oplus W,$$

où $V = \{G(u,v) = (u, g(u), kg(v)); \forall u, v \in TS\}$ et $W = \{(u, -g(u) + \alpha n, \beta n)\}$, et munissons TE de la structure presque complexe J telle que

$$J\,|_V \colon G(u,v) \mapsto G(v,-u)$$

et

$$J\,|_W \colon (u, -g(u) + \alpha n, \beta n) \mapsto (J_0 u, -g(J_0 u) + \beta n, -\alpha n)$$

(en fait, la structure complexe sur W n'a aucune incidence sur la suite). Munissons V de la métrique hermitienne μ :

$$\mu(G(u_1, v_1), G(u_2, v_2)) = k\langle u_1 \mid u_2\rangle + k\langle v_1 \mid v_2\rangle,$$

où $\langle\ |\ \rangle$ désigne la métrique de S. On munit ensuite W de n'importe quelle métrique.

1.6. Courbes pseudo-holomorphes

Nous allons maintenant exhiber des courbes pseudo-holomorphes.

PROPOSITION 1.6.1. — *Si f est une immersion isométrique elliptique de S dans M, $j^1 f(S)$ est une courbe pseudo-holomorphe de O. De plus la structure presque complexe induite sur S est celle donnée par la deuxième forme fondamentale. L'élément d'aire ω induit est $H\omega_0$ où H désigne la courbure moyenne de l'immersion et ω_0 l'elément d'aire de la métrique initiale.*

C'est une conséquence de la proposition 1.4.1. □

Il existe d'autres courbes pseudo-holomorphes. Celles que nous allons introduire sont celles qui produisent les dégénérescences du théorème 1.1.2.

PROPOSITION 1.6.2. — *Soient $\gamma(t)$ une géodésique de S et $\Gamma(t)$ une géodésique de M, toutes deux paramétrées par l'arc. La surface $F = F(\gamma, \Gamma)$ de E constituée des isométries de $T_{\gamma(t)}S$ dans $T_{\Gamma(t)}M$ qui envoient $d\gamma/dt$ sur $d\Gamma/dt$, est une courbe pseudo-holomorphe.*

Nous l'appellerons *surface de pli*. A nouveau la preuve est immédiate. □

1.7. Idée de la démonstration du théorème 1.1.1

Nous avons vu que les immersions isométriques elliptiques se relèvent en courbes pseudo-holomorphes dans un espace de jets E se fibrant sur $S \times M$, la structure conforme induite étant celle de la deuxième forme fondamentale.

Une première étape de la démonstration consiste à démontrer que chaque fibre de cette fibration admet un voisinage calibré (A.1.1). Si (f_n) est une suite d'immersions isométriques de S dans M, convergeant de façon C^0, nous pouvons alors supposer en prenant un voisinage U d'un point x_0 de S — homéomorphe au disque — que $j^1 f_n(U)$ est inclus dans ce voisinage.

On considère maintenant g_n la représentation conforme de D dans $j^1 f_n(U)$ envoyant l'origine dans le relevé de x_0. D'après le lemme de Schwarz, nous pouvons extraire de (g_n) une sous-suite convergeant sur tout compact vers une application g_0. Il nous reste donc à comprendre ce qu'est l'image de g_0, et en particulier de montrer que c'est une surface immergée. Supposons donc que l'intégrale de la courbure moyenne soit uniformément bornée sur U, ce qui est l'hypothèse du théorème 1.1.1. Ceci entraîne que l'aire de $g_n(D)$ est bornée. Par ailleurs, le bord de $g_n(D)$ est inclus dans $\pi_S^{-1}(\partial U)$. Le lemme A.4.1 nous assure alors que le bord de $g_0(D)$ est également inclus dans $\pi_S^{-1}(\partial U)$. Comme $\pi_S g_0(0)$ est x_0, $g_0(D)$ n'est pas réduit à un point.

Ensuite, on considère $\mathbf{CP}^1(W)$, le fibré sur E dont la fibre en tout point est l'espace des droites complexes du plan complexe W. Notre application g_n se relève alors par l'application de Gauss en une application $\bar{g}_n$ à valeurs dans $\mathbf{CP}^1(W)$. De plus en chaque point de E, $\mathbf{CP}^1(W)$ est séparé en deux hémisphères par le cercle des droites complexes dont la projection sur TS est singulière. Or $\bar{g_n}$ ne rencontre jamais ce cercle. Le lemme A.1.2 nous assure alors que $(\bar{g_n})$ converge également. Le lemme A.3.1 nous assure ensuite que g_0 est bien une immersion.

La suite de la démonstration consiste alors à montrer que $g_0(D)$ est soit un graphe d'immersion isométrique, soit inclus dans une surface de pli. □

2. Courbure de Gauss prescrite

2.1. Le problème

Nous allons montrer que de nombreux problèmes à courbure de Gauss prescrite sur les surfaces, en particulier le problème de Minkowski, mais aussi le problème de Monge-Ampère et celui des plongements radiaux (cf [9], [3]) s'interprètent en terme de courbes pseudo-holomorphes. On s'interessera ici au problème suivant qui contient les cas cités :

PROBLÈME A. — *Soit M une variété riemannienne orientée de dimension 3, et g une fonction définie sur $U(M)$, le fibré unitaire de M, et strictement supérieure à la fonction courbure sectionnelle k. On cherche alors une surface orientée, dont la courbure en tout point x vaille $g(n)$, n étant la normale orientée en ce point.*

On remarquera que de telles surfaces sont nécessairement localement convexes, grâce à nos hypothèses sur k.

La traduction pseudo-holomorphe nous a été utile dans [7], où nous l'avons appliquée au problème de Minkowski dans les variétés hyperboliques.

Le problème de Minkowski dans l'espace euclidien s'énonce ainsi : Quelles fonctions sur la sphère sont obtenues en poussant la fonction courbure d'une surface immergée par l'application de Gauss ? Lorsque nous nous intéressons aux fonctions positives, c'est-à-dire aux surfaces convexes, ce problème une solution dont l'énoncé est simple et élégant ([8] et dans un autre cadre [10]) :

THÉORÈME (Nirenberg, Alexandrov, Pogorelov). — *Pour qu'il existe une surface convexe solution du problème de Minkowski pour une fonction k positive définie sur la sphère S^2, il faut et il suffit que*

$$\int_{S^2} \frac{u}{k} ds = 0,$$

où u est le vecteur position sur la sphère et ds l'élément d'aire. Une telle surface est alors unique à translation près.

Nous nous sommes intéressés à un problème analogue dans le cadre des variétés hyperboliques. La solution de ce problème nous a permis d'étudier les surfaces à courbure de Gauss constante de ces variétés. Enonçons ce problème :

Soit M une variété hyperbolique géométriquement finie et sans cusp, c'est-à-dire à cœur de Nielsen N compact. La variété $M \setminus N$ est alors la réunion finie de variétés (voir [12]) B_i homéomorphes à $S_i \times \mathbf{R}$, où les S_i sont les surfaces de Riemann compactes qui sont les composantes connexes du bord M_∞ de M à l'infini. Dans une telle variété ou plus généralement dans un *bout géométriquement fini* B (voir [7]) associé à une lamination géodésique mesurée, dont le bord à l'infini est B_∞, se pose naturellement un problème de Minkowski.

Soit en effet S, une surface convexe lisse incompressible dans B, on définit l'application de Gauss φ de S dans B_∞ en associant à tout point de S la géodésique normale en ce point (figure 15).

On associe donc à toute telle surface convexe S une fonction $F(S)$ sur B_∞ donnée par

$$F(S) = k \circ \varphi^{-1} \tag{2.1.1}$$

où k est la fonction courbure de S. Le problème que nous considérons est, étant donnée une fonction k sur B_∞, de résoudre (2.1.1). Notre résultat est le suivant

THÉORÈME 2.1.2. — *Soit k une fonction C^∞ définie de B_∞ dans $]-1,0[$. Il existe alors une unique surface convexe S incompressible dans B, telle que $k = F(S)$.*

Ce résultat est, pour le problème de Minkowski, l'analogue du résultat de M.S. Berger ([2]) pour le problème de Nirenberg :

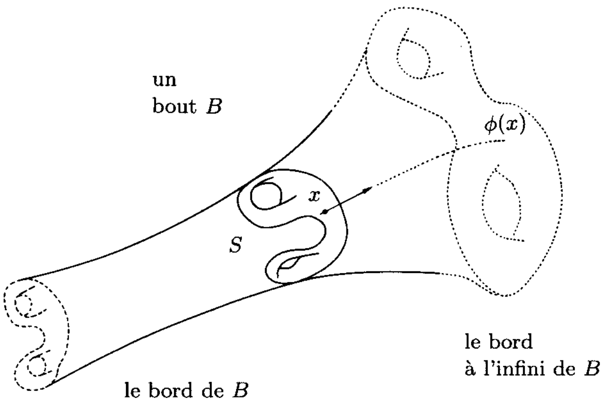

Figure 15

THÉORÈME (M.S. Berger). — *Toute fonction négative sur une surface de Riemann de genre au moins 2 est la courbure d'une unique métrique conforme à une métrique donnée.*

En appliquant notre résultat au cas où k est une fonction constante, on en déduit l'existence et l'unicité d'une surface S_k à courbure constante k. Un exemple bien connu de cette situation est le cas où M est le quotient de l'espace hyperbolique par un groupe fuchsien : les surfaces équidistantes de l'hyperplan de symétrie sont alors à courbure constante et de plus forment un feuilletage.

2.2. Traduction pseudo-holomorphe

Revenons au problème sous sa forme génerale, et montrons

LEMME 2.2.1. — *Il existe une structure presque complexe sur un sous-fibré W du fibré tangent à $U(M)$, telle que les champs de vecteurs normaux unitaires $N(S)$, aux surfaces S solutions du problème A soient des courbes pseudo-holomorphes tangentes à W.*

La structure conforme induite est alors celle donnée par la deuxième forme fondamentale. Enfin, il existe une métrique hermitienne sur W telle que l'aire d'une courbe pseudo-holomorphe est l'intégrale de la courbure moyenne de la surface sous-jacente.

Démonstration. — $U(M)$ est un fibré sur M muni d'une connexion qui se déduit de la connexion de Levi-Civita sur M. Grâce à cette connexion le fibré tangent à $U(M)$ en un point n se décompose en

$$T_nU(M) = P \oplus TM,$$

où P est le plan orthogonal à n. Remarquons que P étant orienté, il est muni naturellement d'une structure complexe J_0. Le sous fibré W est alors en tout point n de $U(M)$ donné à l'aide de cette décomposition par

$$W(n) = P \oplus P.$$

Il est facile de vérifier qu'une surface tangente à W dont la projection sur M est régulière est un champ de vecteurs normaux $N(S)$ à une surface S de M. Dans cette décomposition, l'espace tangent à une telle surface $N(S)$ est constitué des vecteurs de la forme $(u, A(u))$, où A est l'opérateur deuxième forme fondamentale. Soit maintenant c la fonction telle que

$$c^2 = g - k.$$

On munit alors W de la structure complexe J, telle que

$$J(u, v) = (c^{-1} J_0 v, c J_0 u).$$

On vérifie maintenant aisément que dire que $N(S)$ est une courbe pseudo-holomorphe entraîne

$$\text{dét}(A) = c^2,$$

c'est à dire que S est solution du problème A. Enfin, si g_0 est la métrique de M, si l'on munit W de la métrique donnée par

$$g((u_1, v_1), (u_2, v_2)) = c\, g_0(u_1, u_2) + c^{-1} g_0(v_1, v_2),$$

la métrique induite sur S, solution du problème A, est alors la métrique

$$g_1(u, v) = \text{dét}(A)^{-1/2}\, \text{tr}(A) g_0(A(u), v).$$

Ceci finit de démontrer notre lemme. □

A nouveau comme dans l'exemple précédent, des courbes parasites tubulaires apparaissent.

PROPOSITION 2.2.2. — *Soit γ une géodésique de M, la surface $S(\gamma)$ de $U(M)$ constituée des vecteurs normaux à γ est une courbe pseudo-holomorphe.*

Elles apparaissent comme dégénérescences dans l'exemple déjà cité en 1.1.2.

3. Autres exemples et constructions

3.1. Courbure moyenne prescrite

Soit M une variété riemannienne, $G = G_2(M)$ la grassmannienne des 2-plans orientés de M et $F \to G$ le fibré sur G dont la fibre en un point P est l'orthogonal P^o du plan P. Si S est une surface orientée immergée dans M on notera également $\Sigma(S)$ son relevé par l'application de Gauss dans G. On se pose alors le problème suivant

PROBLÈME B. — *Soit H une section du fibré F, existe-t-il une surface S orientée et immergée dans M dont le vecteur courbure moyenne en un point x est $H(T_xS)$?*

Les surfaces minimales sont des solutions du cas particulier de ce problème correspondant à la section nulle. En dimension 3, le fibré F a une section n canonique correspondant au vecteur normal et les surfaces à courbure moyenne constante λ sont les solutions du problème pour la section λn. Notre but est de démontrer le lemme suivant, analogue à celui sur la courbure de Gauss :

LEMME 3.1.1. — *Il existe une structure presque complexe sur un sous-fibré W du fibré tangent à G, telle que les relevés $\Sigma(S)$ par l'application de Gauss d'une surface S solution du problème B soient des courbes pseudo-holomorphes tangentes à W. La structure conforme induite est alors celle de la métrique induite S.*

Démonstration. — G est un fibré sur M muni d'une connexion qui se déduit de la connexion de Levi-Civita sur M. Grâce à cette connexion le fibré tangent à G en un point P se décompose en

$$T_PG = \mathrm{Hom}(P, P^\circ) \oplus TM,$$

où P° est l'orthogonal de P. Remarquons que, P étant orienté, il est muni naturellement d'une structure complexe J_0. Le sous-fibré W est alors en tout point P de G donné à l'aide de cette décomposition par

$$W(P) = \mathrm{Hom}(P, P^\circ) \oplus P.$$

Il est facile de vérifier qu'une surface tangente à W, dont la projection sur M est régulière, est un relevé $\Sigma(S)$ d'une surface S de M. Dans cette décomposition, l'espace tangent à une telle surface $\Sigma(S)$, est constitué des vecteurs de la forme $(u, A(u))$, où A est l'opérateur deuxième forme fondamentale.

Nous allons construire à l'aide de H (la section de F associée à notre problème) une structure presque complexe sur W. Pour cela soit u un vecteur de P, nous pouvons lui associer l'élément $B(u)$ de $\mathrm{Hom}(P, P^o)$ défini par

$$B(u) : v \mapsto g(J_0u, v)H(P).$$

Ceci nous permet de définir l'endomorphisme J de W dans lui-même donné par

$$J(A, u) = (A \circ J_0 + B(u), J_0u),$$

dont on vérifie aisément qu'il a les propriétés requises. □

Les courbes pseudo-holomorphes parasites sont maintenant incluses dans la fibre, et correspondent à l'apparition de singularités sur notre surface.

3.2. Applications harmoniques dont la source est une surface

Rappelons que si f est une application différentiable d'une variété riemannienne (S, g) dans une variété riemannienne (M, h), on peut définir l'énergie de f comme

$$E(f) = \frac{1}{2}\int_S \|Dg\|^2 v_g.$$

Les points critiques de l'énergie sont, par définition, les *applications harmoniques.* Ces points critiques sont ceux qui vérifient l'équation :

$$\mathrm{tr}(\nabla Df) = 0.$$

On considére ici ∇Df comme une 2-forme sur TS à valeurs dans TM, définie par

$$\nabla_X Df(Y) = \nabla^M_{Df(X)} Df(Y) - Df(\nabla^S_X Y),$$

où X et Y sont deux champs de vecteurs de S, et ∇^M (resp. ∇^S) désigne la connexion de Levi-Civita de M (resp. S). La trace est prise relativement à g.

Dans le cas où S est une surface, l'énergie est en fait un invariant conforme. Notons maintenant $E = J^1(S, M)$ l'espace des 1-jets d'applications de S dans M. On peut donc maintenant associer à toute application f de S dans M une surface $\Sigma(f)$ dans E. On montre alors aisément le résultat suivant :

LEMME 3.2.1. — *Il existe une structure presque complexe sur un sous fibré W du fibré tangent à E, tel que le relevé $\Sigma(f)$ d'une application harmonique f, soit une courbe pseudo-holomorphe tangente à W. La structure conforme induite est alors celle de la métrique de S.*

Démonstration. — E est un fibré sur $M \times S$ muni d'une connexion qui se déduit des connexions de Levi-Civita sur $M \times S$. Un point g de E est la donnée d'un point s de S, d'un point m de M et d'une application linéaire L de T_sS dans T_mM. Grâce à la connexion le fibré tangent à E en un point g se décompose en

$$T_gE = TS \oplus TM \oplus \mathrm{Hom}(TS, TM).$$

L'espace tangent à $\Sigma(f)$ est constitué des vecteurs de la forme $(u, Df(u), \nabla_u Df)$. On considère donc le sous-fibré W dont la fibre en $g = (s, m, L)$ est l'ensemble des vecteurs de la forme $(u, L(u), H)$. Si J_0 est la structure complexe de S, la structure complexe à construire sur W est celle donnée par

$$J(u, L(u), H) = (J_0u, L(J_0u), H \circ J_0),$$

dont on vérifie aisément qu'elle a toutes les propriétés requises. □

3.3. Equations non linéaires elliptiques du deuxième ordre

Soit donc S une surface et considérons $E = T^\star S \times \mathbf{R}$ l'espace des 1-jets de fonctions de S dans $\mathbf{R}$. Soit maintenant F une fonction de E dans $\mathbf{R}$. Nous allons montrer

LEMME 3.3.1. — *Il existe une structure complexe J_F sur le sous fibré d'holonomie W, telle que $\Sigma(f)$, le graphe du 1-jet d'une fonction f, soit pseudo-holomorphe si et seulement si, la fonction f est solution de l'équation $\Delta(f) = F(j^1(f))$.*

Démonstration. — L'espace tangent à E se décompose, à l'aide de la connexion, en

$$TE = TS \oplus T^\star S \oplus \mathbf{R},$$

Considérons alors le sous-fibré d'holonomie W consitué des vecteurs de la forme $(u, \omega, g(u))$ au point g de E où u est un vecteur tangent à S et ω un vecteur cotangent et munissons le de la structure complexe

$$J_F(u, \omega, g(u)) = (J_0 u, \omega \circ J_0 + F < J_0 u, . >, g(J_0 u)),$$

où J_0 désigne la structure complexe de S. A nouveau, cette structure presque complexe remplit bien son office. □

Appendice : Convergence d'applications pseudo-holomorphes

Dans cet appendice, nous allons, d'une part rappeler le lemme de Schwarz (A.1.1) qui exhibe un critère permettant de faire converger une suite d'applications pseudo-holomorphes, et d'autre part, démontrer deux lemmes (A.3.1 et A.4.1) qui nous permettent de décrire la limite.

Nous avons vu en 1.7 comment ces lemmes s'empilent pour démontrer le théorème 1.1.1.

A.1. Lemme de Schwarz

Nous allons extraire de la démonstration du théorème de compacité des courbes cusps de Gromov (voir [4] et les chapitres VII et VIII) le lemme fondamental qui nous sera utile par la suite. Nous considérons une variété E munie d'une structure presque complexe J et d'une métrique hermitienne μ. Le lemme est le suivant :

LEMME A.1.1 (Lemme de Schwarz). — *Soit g une application pseudo-holomorphe de D, le disque unité de $\mathbf{C}$, dans E. Si $g(D)$ est incluse dans un compact de E calibré, c'est-à-dire sur lequel il existe une 1-forme β telle que, pour tout x vecteur non nul*

$$(**) \qquad d\beta(x, Jx) > 0,$$

alors toutes les dérivées à tous les ordres de g à l'origine sont majorées a priori.

Ce lemme est le résumé de la première partie du théorème de compacité des courbes cusps de Gromov tel qu'il est exposé dans le chapitre VIII. Dans le cas, qui nous intéresse ici, des courbes pseudo-holomorphes tangentes à un sous-fibré, il suffit de supposer (∗∗) vérifiée pour les vecteurs tangents au sous-fibré considéré.

On peut trouver des compacts calibrés de la manière suivante : soit $F \longrightarrow E$ un fibré, muni d'une connexion, sur une variété presque complexe E et dont la fibre est kählérienne. L'espace total est alors également une variété presque complexe. Si maintenant K est un compact de la fibre, qui possède un voisinage, dans la fibre, dont le deuxième groupe de cohomologie réelle est nul, K est alors calibré. Cette remarque et le lemme de Schwarz permettent de montrer le lemme (voir le chapitre VII) :

LEMME A.1.2. — *Soit $\pi : F \longrightarrow E$ un fibré, muni d'une connexion, sur une variété presque complexe E et dont la fibre est kählérienne. On munit l'espace total de la structure presque complexe induite. Soit ensuite (f_n) une suite d'applications pseudo-holomorphes de D dans F telle que*

1. $(\pi \circ f_n)$ *converge,*
2. $f_n(D)$ *est inclus dans un compact K,*
3. *K possède un voisinage U, tel que $H^2(U \cap \pi^{-1}(x))$ soit nul, pour tout x de E.*

Alors les dérivées des f_n sont uniformément majorées.

En fait, dans le chapitre VII, on montre le lemme de Schwarz pour la première dérivée et l'on applique le lemme que nous venons d'énoncer (sur la première dérivée) à l'espace des jets. Ceci permet d'amorcer la récurrence démontrant le lemme de Schwarz à tous les ordres.

A.2. Hypothèses

A l'aide du théorème d'Ascoli et du lemme de Schwarz, nous pourrons extraire des suites convergentes d'applications pseudo-holomorphes pour la topologie $\mathcal{C}^\infty$ sur tout compact. Il nous reste à décrire un peu mieux la limite et en particulier de nous intéresser a l'apparition de singularités et au comportement du bord. Nous nous intéresserons donc dans la suite de cette section à la situation suivante :

1. une variété E munie d'une suite (J_n) de structures presque complexes convergeant de façon $\mathcal{C}^\infty$ vers J_0,
2. une suite (f_n) d'applications pseudo-holomorphes de D dans (E, J_n) convergeant de façon $\mathcal{C}^\infty$ sur tout compact vers f_0, application pseudo-holomorphe de D dans (E, J_0),
3. nous supposerons également que E est muni d'une suite (μ_n) de métriques hermitiennes convergeant vers μ_0.

A.3. Singularités

Considérons $G_2(E)$, la grassmannienne des 2-plans tangents à E. Si maintenant f est une immersion d'une surface S dans E, on peut lui associer une application $\bar{f}$ de S dans $G_2(E)$ qui, à un point, associe le plan tangent à l'image. Nous allons montrer le

LEMME A.3.1. — *Si les hypothèses suivantes sont vérifiées :*

1. *les f_n sont des immersions,*
2. *la suite $(\bar{f}_n)$ converge,*
3. *f_0 n'est pas constante,*

alors f_0 est une immersion.

Démonstration. — Remarquons, tout d'abord, que f_0 étant pseudo-holomorphe et non constante, ses singularités sont isolées. Nous pouvons donc supposer, pour simplifier, que f_0 est une immersion sur D moins l'origine. Nous voulons montrer que f_0 est une immersion sur D. Considérons les métriques $g_n = f_n^\star \mu_n$ induites par les f_n sur D. Nous savons que $g_n = \lambda_n g$ où g est la métrique canonique. De même, $g_0 = f_0^\star \mu_0$ est de la forme $\lambda_0 g$. Nous allons voir que λ_0 est partout non nulle.

Nous allons tout d'abord montrer que l'hypothèse 2 entraîne que, si k_n est la courbure de la métrique g_n, il existe une constante C telle que sur un voisinage de l'origine

$$|k_n| \leq \frac{C}{\lambda_n}.$$

Notre deuxième hypothèse peut se traduire de la manière suivante : soient $F_n = {f_n}^\star(TE) \longrightarrow D$ les fibrés induits par les f_n du fibré tangent à E, alors les fibrés $D_n = {f_n}^\star(TD)$ vus comme sous-fibrés de F_n, convergent. En particulier :

- d'une part la courbure des plans tangents aux images est uniformément bornée,
- d'autre part, les deuxièmes formes fondamentales S_n de D_n dans F_n sont uniformément bornées : si u est un vecteur de D_n et v un vecteur tangent à D alors
$$|S_n(f_{n\star}(v), u)|^2 \leq C\, g(v,v)\, \mu_n(u,u).$$

Ceci entraîne que si w et u sont des vecteurs tangents à l'image,

$$|S_n(w,u)|^2 \leq \frac{C}{\lambda_n} \mu_n(w,w)\mu_n(u,u).$$

En combinant ces deux remarques et l'équation de Gauss pour la surface $f_n(D)$, nous en déduisons l'inégalité recherchée. Maintenant nous savons que

$$k_n = \frac{\Delta(\log(\lambda_n))}{2\lambda_n},$$

et donc $\Delta(\log(\lambda_n)) \leq C$. Nous en déduisons que sur D moins l'origine

$$\Delta(\log(\lambda_0)) \leq C.$$

Il est alors facile de montrer à l'aide de la formule de Green que λ_0 est non nulle partout. □

A.4. Bord

On note dans ce paragraphe $f(\partial D)$, l'ensemble des points d'adhérence des suites $(f(x_n))$ lorsque (x_n) tend vers le bord du disque unité. Notre but est de montrer le

LEMME A.4.1. — *Si l'aire de $f_n(D)$ est uniformément bornée, alors $f_0(\partial D)$ est inclus dans la limite de Haussdorff des $f_n(\partial D)$.*

Montrons tout d'abord la proposition suivante :

PROPOSITION A.4.2. — *Fixons une métrique riemannienne auxilliaire d, on suppose que notre suite (f_n) d'applications pseudo-holomorphes vérifie de plus*

1. *$d(f_n(\partial D), f_n(0))$ est minorée par un nombre strictement positif indépendant de n,*
2. *l'aire de $f_n(D)$ est uniformément majorée.*

Alors f_0 n'est pas constante.

Démonstration. — Raisonnons par l'absurde : si (f_n) tend vers une fonction constante alors, pour tout disque $D(r)$ de rayon r et tout ε strictement positif, $f_n(D(r))$ est inclus dans $B(f_0(0), \varepsilon)$, pour tout n assez grand.

Considérons l'anneau $A(r) = D \setminus D(r)$ et notons A_0 et A_1 les deux composantes de son bord, $A_1 = \partial D$ et $A_0 = \partial D(r)$. D'après nos hypothèses, nous avons alors pour tout n assez grand

$$d_n(f_n(0), f_n(A_0)) \leq \varepsilon \tag{A.4.3}$$

et

$$d_n(f_n(0), f_n(A_1)) \geq \beta. \tag{A.4.4}$$

Rappelons maintenant la notion de module d'un anneau. Tout anneau métrique A est conforme à $S^1 \times [0, L(A)]$, le réel $L(A)$ est alors appelé *module* de l'anneau A. On peut également le calculer par la formule

$$\frac{1}{L(A)} = \inf_{u \in B} \int_A |du|^2 \tag{A.4.5}$$

où B désigne l'ensemble des fonctions u telles que $u|_{A_0} \leq 0$ et $u|_{A_1} \geq 1$, A_0 et A_1 désignant les deux composantes connexes du bord.

Revenons à notre anneau $A(r)$, si nous faisons tendre r vers 1 son module tend vers 0. Par ailleurs $A(r)$ est conforme à son image par f_n, et nous pouvons minorer son module à l'aide de (A.4.5) : nous injectons dans la formule la fonction $u(y) = d_n(y, f_n(0))$. Son gradient ayant une norme inférieure à 1, nous obtenons pour n grand grâce à l'hypothèse 2, (A.4.3) et (A.4.4),

$$L(f_n(A(r))) \geq C(\varepsilon, \beta, K).$$

Ceci nous fournit la contradiction recherchée. □

Nous pouvons maintenant démontrer notre lemme.

Démonstration. — Soit y un point de $f_0(\partial D)$. Par définition il existe une suite de points (y_n) tel que $y = \lim(f_0(y_n))$. Puisque f_n converge sur tout compact vers f_0, nous pouvons supposer que pour toute suite p_n telle que $p_n \geq n$,

$$y = \lim_{n\to\infty}(f_{p_n}(y_n)). \tag{A.4.6}$$

Munissons D de la métrique hyperbolique et supposons que les boules B_n de rayon 1 autour des y_n soient disjointes. Puisque l'aire de $f_n(D)$ est uniformément majorée, l'aire de $f_0(D)$ est bornée et donc

$$\lim_{n\to\infty}(\mathrm{Aire}(f_0(B_n))) = 0.$$

Il existe donc une suite p_n telle que $p_n \geq n$ et

$$\lim_{n\to\infty}(\mathrm{Aire}(f_{p_n}(B_n))) = 0. \tag{A.4.7}$$

Considérons maintenant la famille g_n de transformations conformes de D telle que $g_n(0) = y_n$ et la famille d'applications pseudo-holomorphes $\bar{f}_n = f_{p_n} \circ g_n$. D'après (A.4.7), g_n tend sur tout compact vers une application constante. Par ailleurs, $g_n(D) = f_{p_n}(D)$ a une aire uniformément majorée. Notre proposition précédente nous permet d'affirmer que

$$\lim_{n\to\infty}(d(g_n(\partial D), g_n(0))) = 0,$$

et donc que

$$\lim_{n\to\infty}(d(f_{p_n}(y_n), f_{p_n}(\partial D))) = 0.$$

Ceci et (A.4.6) nous permettent de conclure que y appartient à $\lim(f_n(\partial D))$. □

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Chapter X

Symplectic rigidity: Lagrangian submanifolds

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Introduction

This chapter is supposed to be a summary of what is known today about Lagrangian embeddings. We emphasise the difference between *flexibility* results, such as the h-principle of Gromov applied here to Lagrangian immersions (and also to the construction of examples of Lagrangian embeddings) and *rigidity* theorems, based on existence theorems for pseudo-holomorphic curves.

The most famous result in the family of rigidity theorems is the Arnold conjecture (corollary 2.2.5), according to which there exists no *exact* (see § 2.1) closed Lagrangian submanifold in $\mathbf{C}^n$ and its application to the existence of exotic structures on $\mathbf{R}^{2n}$. An equivalent statement is that the *symplectic area class* (see § 2.1), which belongs to the first de Rham cohomology group of a Lagrangian submanifold, *cannot* vanish. Very analogous are *Maslov class* rigidity results: if it is true that essentially any integral cohomology class can be the Maslov class of a Lagrangian immersion, the situation is quite different for embeddings.

We have tried to give complete proofs—at least of the results which are either easy to prove, or which rely on pseudo-holomorphic curves' methods, as this is the subject of the book. This chapter is based mainly on the results and techniques of Gromov but also on work of the authors, of course. We have also tried to present new results, in the text as well as in exercises, as for instance in § 4. The proofs of the basic results come more or less directly from [24], although very often via the unobtainable "corollaires symplectiques" [44]. We thank Jean-Claude Sikorav, who encouraged us to plunder his paper.

Let us now describe in detail the contents of this chapter.

We begin by recalling in § 1 the h-principle for Lagrangian immersions and by giving many examples of Lagrangian submanifolds and of ways to construct them, mainly as submanifolds of standard symplectic space $\mathbf{C}^n$.

In the next section, we state the main theorem, the rigidity result 2.2.4 and derive some symplectic corollaries, as for example the above mentioned Arnold conjecture and the solution of some Lagrangian intersection problems (in § 2.3).

In § 3, we concentrate on Lagrangian submanifolds of $\mathbf{C}^n$. We show how the previous rigidity results give obstructions to realising certain manifolds as Lagrangian submanifolds of $\mathbf{C}^n$. It is customary (see [26]) to call these obstructions "hard". For the sake of completeness, we recall also the simplest "soft" obstructions. In this way, no sphere (except the circle) can be a Lagrangian submanifold. Actually, the hard techniques are needed only in the case of the 3-sphere, where they are both essential and very poorly used: as it is simply connected, any Lagrangian embedding has to be exact. However we give other examples: a 3-manifold with non zero H^1, no soft obstruction, but which cannot be embedded as a Lagrangian submanifold due to the rigidity results (see theorem 3.3.3).

The next section is devoted to rigidity results in cotangent bundles and applications to mechanics: we look especially at Lagrangian tori in $T^\star T^n$, where the topology is large enough to contain both symplectic and topological invariants, and interesting relations between them. This example is also very interesting in mechanics, particularly if one is willing to look at perturbations of integrable systems.

In § 5, we concentrate on pseudo-holomorphic curves and, finally, we give a proof of the main theorem 2.2.4.

In a short appendix, for the sake of completeness, we construct exotic symplectic structures on $\mathbf{R}^{2n}$. Surprisingly enough, we know of no way to prove that these structures are different from the standard one other than via the rigidity theorems.

1. Lagrangian constructions

In this §, we give some systematic ways of constructing Lagrangian submanifolds, mainly in standard linear symplectic space. We begin by recalling Gromov's h-principle—a flexibility result—for Lagrangian immersions in § 1.1, then show how it may also be used to construct examples of Lagrangian submanifolds. In the following §§, we give as many examples of systematic constructions as we know. Perhaps nothing is really original here (we did not invent either symplectic reduction, Lagrangian cobordisms or wavefronts) but it turns out that folkloric and "well known" examples are not *that* well known, hence the list. We stress the fact that most of these constructions derive, more or less directly, from the symplectic reduction process of Marsden and Weinstein [34] and the Arnold papers [3] on Lagrangian cobordisms.

1.1. Lagrangian immersions in $\mathbf{C}^n$ and the h-principle

Recall from chapter I that $\lambda \subset \mathbf{C}^n$ is a Lagrangian subspace if and only if $\lambda \perp i\lambda$ and $\dim \lambda = n$. Hence an immersion $f : L^n \to \mathbf{C}^n$ is Lagrangian if and only if

$$\forall x \in L, \qquad T_x f\,(T_x L) \perp i T_x f\,(T_x L)\,.$$

In particular, T_xf defines a complex linear isomorphism

$$\begin{array}{ccccc} T_xL & \otimes_{\mathbf{R}} & \mathbf{C} & \longrightarrow & \mathbf{C}^n \\ v & \otimes & (a+ib) & \longmapsto & aT_xf(v)+ibT_xf(v) \end{array}$$

and, varying x, a complex vector bundle isomorphism

$$TL\otimes_{\mathbf{R}}\mathbf{C}\xrightarrow{\;Tf\;}L\times\mathbf{C}^n.$$

Such an isomorphism will be called a *U-parallelisation* of L. We have just proved that, in order for L to admit Lagrangian immersion in $\mathbf{C}^n$, it is necessary that it is U-parallelisable. The h-principle for Lagrangian immersions in $\mathbf{C}^n$, also called the Gromov-Lees theorem, asserts that the converse is true (see theorem 1.1.3 below for a precise statement).

Thus, for instance, any n-dimensional *stably* parallelisable manifold admits Lagrangian immersions into $\mathbf{C}^n$: complexifying a stable parallelisation, gives an isomorphism

$$\varphi:(TL\otimes\mathbf{C})\oplus\left(L\times\mathbf{C}^k\right)\longrightarrow L\times\mathbf{C}^{n+k}.$$

Recall the

PROPOSITION 1.1.1 (see e.g. [29]). — *Let E be a complex n-dimensional vector bundle on a manifold of (real) dimension n. Then E is trivial if and only if it is stably trivial. Moreover, any stable trivialisation is homotopic to the suspension of an instable one.*

This says that φ is homotopic to the sum of an isomorphism

$$\psi:TL\otimes\mathbf{C}\longrightarrow L\times\mathbf{C}^n$$

and the identity.

Using the normal vector field of a codimension 1 embedding, we see that all spheres, all orientable surfaces satisfy $TV\oplus$ trivial line bundle $=V\times\mathbf{R}^{n+1}$. Thus these manifolds are stably parallelisable. Consequently, they can have Lagrangian immersions, and the h-principle asserts thet they do have. The next exercise gives an explicit construction for the sphere.

1.1.2. Exercise. — Let $S^n=\left\{(x,y)\in\mathbf{R}^n\times\mathbf{R}\mid\|x\|^2+y^2=1\right\}$ and let $\mathcal{W}:S^n\to\mathbf{C}^n$ be given by $\mathcal{W}(x,y)=(1+iy)x$. Draw a picture of $\mathcal{W}$ for $n=1$, and show, in general, that $\mathcal{W}$ is a Lagrangian immersion with one double point[1].

[1]The existence of an immersion of S^n with one double point was very important in Whitney's study of immersions in twice the dimension. Hence the name $\mathcal{W}$.

Remark. — One can show that the U-parallelisation of S^n defined by the Lagrange immersion $\mathcal{W}$ is homotopic to (the destabilisation of) the complexification of a stable parallelisation of S^n (see chapter 0 of [5]).

Two Lagrangian immersions $f_0, f_1 : L \to \mathbf{C}^n$ are said to be *regularly homotopic* (implicitly "among Lagrangian immersions") if there exists a map

$$f : L \times [0,1] \longrightarrow \mathbf{C}^n$$

such that f_t is a Lagrangian immersion for any t. Now we can state the Gromov-Lees theorem in $\mathbf{C}^n$.

THEOREM 1.1.3 (Gromov [23], Lees [32]). — *The set of Lagrangian regular homotopy classes of Lagrangian immersions $L^n \to \mathbf{C}^n$ is in one-to-one correspondence with the set of homotopy classes of U-parallelisations of L.*

In other words, giving a Lagrangian immersion of L into $\mathbf{C}^n$ is equivalent to giving a U-parallelisation of L.

1.1.4. Exercise. — Show that if a Lagrangian immersion $f : L^n \to \mathbf{C}^n$ exists, then the set of regular homotopy classes of all Lagrangian immersions $L \to \mathbf{C}^n$ is in one-to-one correspondence with the set $[L, U(n)]$ of homotopy classes of maps $L \to U(n)$.

1.1.5. Exercise. — Let V be any closed surface.

1. Show that the set $[V, U(2)]$ is in one-to-one correspondence with $H^1(V; \mathbf{Z})$ (Hint: recall that V can be represented by a polygon, whose edges are identified in pairs, see [35] for instance).

2. Show that $TV \otimes \mathbf{C}$ is a trivialisable complex vector bundle if and only if the Euler characteristic $\chi(V)$ is even. If this is the case, show that the regular homotopy class of a Lagrangian immersion is well defined by its Maslov class (see the appendix to chapter I).

Let us mention that the "Gromov-Lees theorem" is a very special case of the general h-principle of Gromov. We recall here this principle, for the general case of Lagrangian immersions in symplectic manifolds.

Let $f : L^n \to W^{2n}$ be a Lagrangian immersion into a symplectic manifold. From f, we can extract two topological data: the homotopy class of the map $f : L \to W$ (which is such that the cohomology class of $f^\star\omega$ vanishes) and the Lagrangian subbundle TL of the symplectic bundle $f^\star TW \to L$.

THEOREM 1.1.6 ([23]). — *Let L be a closed manifold and (W, ω) be a symplectic manifold. Then there is a weak homotopy equivalence between the two spaces:*

- *The space of Lagrangian immersions $L \to W$.*
- *The space of all bundle maps $\varphi : TL \to TW$ such that the induced map on the bases $\psi : L \to W$ pulls back the cohomology class of ω to 0 and the induced map on the fibres sends any fibre of TL to a Lagrangian subspace.* □

Recall that a weak homotopy equivalence is a map that induces an isomorphism of homotopy groups. We shall use this statement only at the π_0 level.

When W is $\mathbf{C}^n$, $f : L \to \mathbf{C}^n$ is homotopic to a constant map, so that the first datum is redundant. The statement 1.1.3 is then 1.1.6 at the π_0 level. The condition $\psi^\star[\omega] = 0$ is also automatically satisfied when ω is exact, e.g. when it is the canonical structure on a cotangent bundle.

1.1.7. Exercise. — Let $W = T^\star X$ be a cotangent bundle. Prove that the regular homotopy classes of Lagrangian immersions $L \to W$ are in one-to-one correspondence with the disjoint union

$$\coprod_{f \in [L,X]} U(f)$$

where $U(f)$ is the set of homotopy classes of complex isomorphisms

$$\varphi : TL \otimes \mathbf{C} \longrightarrow f^\star TX \otimes \mathbf{C}.$$

What happens when $L = X = T^2$?

1.2. First examples of Lagrangian embeddings

The study of Lagrangian embeddings is far more difficult (the present chapter is devoted to this problem). Let us begin with a list of examples in $\mathbf{C}^n$.

Of course any embedding of S^1 in $\mathbf{C}$ is Lagrangian. Taking products gives Lagrangian embeddings of any torus T^n into $\mathbf{C}^n$ and in particular of T^2 into $\mathbf{C}^2$. It is easy to see (cf. 3.2.1 below) that no other orientable surface can have a Lagrangian embedding into $\mathbf{C}^2$. Beautiful examples of Lagrangian embeddings of all non orientable surfaces L with $\chi(L) < 0$ and divisible by 4 were constructed by Givental (see [22], [8] and below exercise 1.5.5). The condition that $\chi(V) \equiv 0 \bmod 4$ is necessary (see [7] and § 3.1) but nobody knows what happens for $\chi = 0$: the Klein bottle is open!

Another example, taken from [5], is the "classifying space", the Lagrangian grassmannian Λ_n (see chapter I).

1.2.1. Exercise. — Consider the complex vector space Sym $(n, \mathbf{C})$ of all complex symmetric $n \times n$ matrices, endowed with its natural hermitian (and symplectic) structure. Show that

$$\begin{array}{rcl} \widetilde{\Phi} : U(n) & \longrightarrow & \operatorname{Sym}(n, \mathbf{C}) \\ A & \longmapsto & A^t A \end{array}$$

induces a Lagrangian embedding $\Phi : \Lambda_n \to \operatorname{Sym}(n, \mathbf{C})$.

1.2.2. Exercise. — Deduce from 1.2.1 that $U(n)/SO(n)$ admits Lagrangian immersions into $\mathbf{C}^{n(n+1)/2}$ and that $SU(n)/SO(n)$ is U-parallelisable and thus admits Lagrangian immersions into $\mathbf{C}^{n(n+1)/2-1}$ (Hint: show that $U(n)/SO(n)$ is diffeomorphic to $S^1 \times SU(n)/SO(n)$).

We have already used the elementary fact that the product of two Lagrangian embeddings is a Lagrangian embedding. Almost as elementary—but more useful—is the following generalisation.

PROPOSITION 1.2.3. — *Assume V and W are compact manifolds, $f : V^n \to \mathbf{C}^n$ is a Lagrangian immersion and $g : W^m \to \mathbf{C}^m$ a Lagrangian embedding ($m \geq 1$). Then, in the Lagrangian regular homotopy class of the immersion $f \times g$, there is a Lagrangian embedding of $V^n \times W^m$ into $\mathbf{C}^{n+m}$.*

Proof. — Consider the *isotropic* immersion obtained as the composition

$$V^n \xrightarrow{\ f\ } \mathbf{C}^n \subset \mathbf{C}^{n+m}$$

and apply the general position lemma:

LEMMA 1.2.4. — *Let $f : V^n \to \mathbf{C}^{n+m}$ ($m \geq 1$) be an isotropic immersion. Then, among isotropic immersions, there exists an approximation of f which is an embedding.* □

We can approximate by an isotropic embedding $f_\varepsilon : V^n \to \mathbf{C}^{n+m}$. A tubular neighbourhood of $f_\varepsilon(V)$ is symplectomorphic to a neighbourhood of the zero section in $T^\star V \times \mathbf{C}^m \to V$ (see chapter I). We can assume that $g(W)$ is contained in a small enough neighbourhood of 0 in $\mathbf{C}^m$ and embed $V \times W$ as a Lagrangian submanifold of this neighbourhood via the zero section and g. □

An easy consequence of 1.2.3, 1.1.1 and of the Gromov-Lees theorem 1.1.3 is the following corollary:

COROLLARY 1.2.5. — *The three following assertions are equivalent:*
(i) V has a Lagrangian immersion in $\mathbf{C}^n$.
(ii) $V \times T^m$ ($m \geq 1$) has a Lagrangian immersion in $\mathbf{C}^{n+m}$.
(iii) $V \times T^m$ ($m \geq 1$) has a Lagrangian embedding in $\mathbf{C}^{n+m}$. □

Here is an application:

1.2.6. Exercise. — Use 1.2.5 and the existence of a Lagrangian embedding of Λ_n (see 1.2.1) to improve 1.2.2 and show that $U(n)/SO(n)$ actually has a Lagrangian *embedding* into $\mathbf{C}^{n(n+1)/2}$.

1.3. Symplectic reduction

It is a classical fact that one can use symplectic reduction to construct Lagrangian immersions in the "small" symplectic manifold starting from Lagrangian immersions in the "big" one (see [3] and [5] for instance). In this §, we use symplectic reduction to construct examples of Lagrangian embeddings. We use the constructions and the notations of chapter II, where the following diagram is considered:

$$(1.3.1)\qquad \begin{array}{ccccccc} M & \xrightarrow{f} & \mu^{-1}(\xi) & \xrightarrow{j} & W^{2N} & \xrightarrow{\mu} & \mathfrak{g}^\star \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} & & & & \\ L^n & \xrightarrow{i} & V^{2n} & & & & \end{array}$$

Here μ is the moment mapping for a G-action on W^{2N}, $\xi \in \mathfrak{g}^\star$ is a regular value such that the induced G_ξ-action on $\mu^{-1}(\xi)$ is free and $p : \mu^{-1}(\xi) \to V^{2n}$ is the corresponding principal G_ξ-bundle. The manifold V is endowed with the reduced symplectic structure σ induced from the symplectic structure ω of W.

Assume that i is a Lagrangian embedding and that $M \to L^n$ is the principal G_ξ bundle induced by i.

Then $j \circ f$ is an *isotropic embedding*: it is obviously the embedding of a dimension $n + \dim G_\xi$ submanifold, and

$$(j \circ f)^\star \omega = f^\star j^\star \omega = f^\star p^\star \sigma = p^\star i^\star \sigma = 0.$$

In particular, if $G_\xi = G$, for instance when G is abelian, or, more generally, if ξ is a fixed point of the coadjoint action, then $\dim M = N$ and $j \circ f$ is the inclusion of a *Lagrangian submanifold.*

At first sight, this does not seem very useful: you begin with a Lagrangian submanifold of something complicated like V and you get a Lagrangian submanifold of something simple. In fact, as the reader has probably already understood, it is very hard to construct Lagrangian submanifolds of the standard symplectic vector space $\mathbf{C}^N$. On the other hand, there are a lot of complex algebraic manifolds which are obtained as symplectic reductions... and they have obvious Lagrangian submanifolds: their real parts, from which one can construct lots of examples.

Circle bundles over real projective spaces. — This example appears in [50] but is a consequence of the remark above in the simplest case where the reduced manifold is the complex projective space: $G = S^1$, $W = \mathbf{C}^{n+1}$, $V = \mathbf{P}^n(\mathbf{C})$ and

$L = \mathbf{P}^n(\mathbf{R})$. We then get a Lagrangian submanifold M of $\mathbf{C}^{n+1}$ which is the circle bundle of the complexified canonical bundle over $\mathbf{P}^n(\mathbf{R})$.

1.3.2. Exercise.

1. Show that M is diffeomorphic to $S^1 \times S^n/(z,x) \sim (-z,-x)$, the Lagrangian embedding into $\mathbf{C}^{n+1}$ being given by $(z,x) \mapsto zx$.
2. Write M as a bundle onto S^1 with fibre S^n. Such a bundle can be understood as $([0,1] \times S^n)/(0,x) \sim (1,\varphi(x))$ where φ is a diffeomorphism of the fibre. Show that our M is given by $\varphi =$ antipodal map, and deduce that for n odd, M is diffeomorphic to $S^1 \times S^n$.
3. With the notations of the appendix to chapter I, show that $\|\mu_M\| = n+1$.

Torus bundles over real Hirzebruch surfaces. — The symplectic reduction process described above can be applied in the case of torus bundles over real toric manifolds. Once complex toric manifolds are described by symplectic reduction, as for instance in [9], one gets a lot of examples. For Hirzebruch surfaces, this was done in chapter II.

Here the diagram (1.3.1) is

$$\begin{array}{ccccccc} M^4 & \overset{f}{\hookrightarrow} & \mu^{-1}(\xi) & \overset{j}{\hookrightarrow} & \mathbf{C}^4 & \overset{\mu}{\longrightarrow} & \mathbf{R}^2 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} & & & & \\ L^2 & \overset{i}{\hookrightarrow} & W_k & & & & \end{array}$$

The real part of the Hirzebruch manifold W_k is (topologically) the torus $S^1 \times S^1$ (the real part of $\mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$) if k is even and the Klein bottle $\mathbf{P}^2(\mathbf{R})\#\mathbf{P}^2(\mathbf{R})$ ($\cong \widetilde{\mathbf{P}}^2(\mathbf{R})$, the real part of $\widetilde{\mathbf{P}}^2(\mathbf{C})$) if k is odd (recall from chapter II that $\widetilde{\mathbf{P}}^2(\mathbf{C})$ is $\mathbf{P}^2(\mathbf{C})$ blown up at one point). Thus we get Lagrangian embeddings of T^2-bundles over T^2 or over the Klein bottle into $\mathbf{C}^4$.

Principal bundles over real Grassmannians. — Let us look now at an example with a non-abelian group. Apply the construction above to the complex Grassmann manifolds obtained as symplectic reductions as in chapter II. The diagram (1.3.1) is now:

$$\begin{array}{ccccccc} M & \overset{f}{\hookrightarrow} & \mu^{-1}(\xi) & \overset{j}{\hookrightarrow} & M_{(n+k)\times k}(\mathbf{C}) & \overset{\mu}{\longrightarrow} & \mathfrak{u}(k)^\star \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} & & & & \\ G_k\left(\mathbf{R}^{n+k}\right) & \overset{i}{\hookrightarrow} & G_k\left(\mathbf{C}^{n+k}\right) & & & & \end{array}$$

and M is the $U(k)$-principal bundle over the real Grassmannian associated with the complexified canonical bundle.

1.4. Surgeries

It is very easy to remove the double points of a Lagrangian immersion... provided that you are allowed to change the source manifold! The surgery operation we use is a variant of a classical topological process which removes an $S^k \times D^l$ in a $(k+l)$-manifold and pastes in a $D^{k+1} \times S^{l-1}$ along the boundary $S^k \times S^{l-1}$ (this is an *index* k surgery). In the context of Lagrangian immersions, the construction is directly connected with Lagrangian cobordisms and thus comes from [3], see also [31] and [42]. Consider, following Arnold, the function

$$S_y(q,a) = \frac{a^3}{3} - aQ(q) + ay$$

where $q \in \mathbf{R}^n$, Q is a nondegenerate quadratic form, $a \in \mathbf{R}$ and $y \in \{-1,1\}$ is a parameter.

Let L_y be the submanifold (a quadric) of $\mathbf{R}^n \times \mathbf{R}$ defined by

$$L_y = \left\{ (q,a) \in \mathbf{R}^n \times \mathbf{R} \mid \frac{\partial S_y}{\partial a} = a^2 - Q(q) + y = 0 \right\}$$

and let $f_y : L_y \to \mathbf{C}^n$ be defined by

$$f_y(q,a) = q + i\frac{\partial S_y}{\partial q} = q - ia\frac{\partial Q}{\partial q}.$$

This is a Lagrangian immersion by construction. As the index of the quadratic form varies, we get examples of surgeries of all indices.

1.4.1. Exercise. — Consider the function $S(q,a,y) = \dfrac{a^3}{3} - aQ(q) + ay$ and the $n+1$-manifold with boundary (*elementary cobordism*)

$$L = \left\{ (q,a,y) \in \mathbf{R}^n \times \mathbf{R} \times [-1,1] \mid \frac{\partial S}{\partial a} = 0 \right\}.$$

Check that $f : (q,a,y) \mapsto (q,y) + i\left(\dfrac{\partial S}{\partial q}, \dfrac{\partial S}{\partial y}\right)$ defines a Lagrangian immersion of L into $\mathbf{C}^{n+1}$ and that the restriction of f to $y = \pm 1$ projects to f_y (thus f is a *Lagrangian cobordism* between f_{-1} and f_1).

Assume now that Q is positive definite. Then f_{-1} is a Lagrangian immersion of the disjoint union of two n-balls, with one double point and f_1 is a Lagrangian embedding of a cylinder $S^{n-1} \times \mathbf{R}$ (see figure 16). It is not hard to embed these local models in order to replace a Lagrangian immersion of a manifold V by another Lagrangian immersion of a manifold V' having one double point less and thus to get an embedding after a finite number of steps. The manifold V' is obtained from V by adding a handle of index 1.

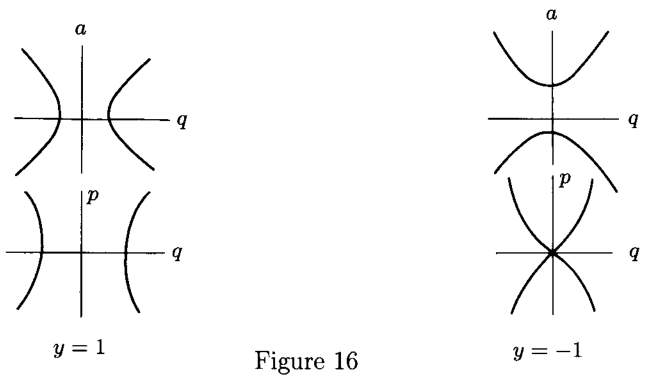

Figure 16

In the next application, we construct, following [30], a Lagrangian immersion of the Klein bottle in $\mathbf{C}^2$ which will be shown later to be, in some sense, as close as possible to an embedding (see 2.2.9).

1.4.2. Exercise.

1. Show that there exist two copies S_1 and S_2 of the Whitney sphere (see exercise 1.1.2) which intersect at two points p_1 and p_2, which can be chosen arbitrarly close to each other. Check that these two points have different signs, and that after an index 1 surgery on each of these two points, one gets a Lagrangian immersion j of the Klein bottle into $\mathbf{C}^2$ with two double points.

2. Assume p_1 and p_2 were chosen very close to each other. Consider a small loop $\gamma = \gamma_1 \cup \gamma_2$ where γ_1 joins p_1 to p_2 in S_1 and γ_2 joins p_2 to p_1 in S_2. The loop γ has two corners at p_1 and p_2. The index 1 surgery on p_1 and p_2 transforms γ into a smooth loop γ'. Show that one can assume that γ and γ' lie in $\mathbf{C} \times 0 \subset \mathbf{C}^2$. Compute the Maslov index of γ' by looking at the normal field of lines $X_{\gamma'}$ along γ' defined by $X_{\gamma'}(p) =$ orthogonal complement of $T_p(\gamma')$ in $T_p(j(K^2))$. The Maslov index is the sum of the rotation indices along γ' of $T(\gamma')$ and $X_{\gamma'}$. Deduce that $\mu(j) = a^\star$, where $a^\star$ generates $H^1(K^2, \mathbf{Z})$.

1.5. Wavefronts and Lagrangian surfaces

In this §, we give some examples based on Givental's beautiful paper [22]. The idea is to construct Lagrangian immersions or submanifolds in $\mathbf{R}^4$ by drawing them in some $\mathbf{R}^3$. Here it must be understood that $4 = 2n$ and $3 = n + 1$. Consider canonical coordinates $(q, p) \in \mathbf{R}^{2n}$, add a coordinate z and endow $\mathbf{R}^{2n} \times \mathbf{R}$ with the (contact) 1-form $\alpha = dz - pdq$.

An immersion $f : L \to \mathbf{R}^{2n}$ which is Lagrangian is *exact* (see below § 2.1) if (and only if) there exists a lift $\tilde{f} : L \to \mathbf{R}^{2n} \times \mathbf{R}$ such that $\tilde{f}^\star(\alpha) = 0$ (a *Legendre immersion*): choose a primitive h of $f^\star(pdq)$ and set $\tilde{f}(x) = (f(x), h(x))$.

Note that, as $dz = pdq$ on L, the projection $\pi \circ f : L \to \mathbf{R}^n \times \mathbf{R}$, which forgets the p coordinates, suffices to determine $\widetilde{f}$ (and f). This is the *wavefront* of f. For instance the "eye" in figure 17 is the wavefront of the Whitney immersion in $\mathbf{R}^2$ while the "flying saucer", obtained by rotating the eye about the z-axis is that of the Whitney immersion in $\mathbf{R}^4$. Double points of the Lagrangian immersion correspond to points in the front which project to the same $q \in \mathbf{R}^n$ and have parallel tangent spaces.

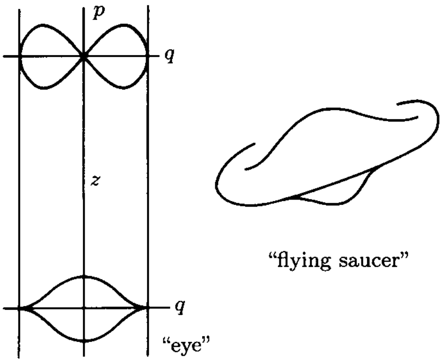

Figure 17

1.5.1. Exercise. — Show that, for a hypersurface in $\mathbf{R}^n \times \mathbf{R}$ to be a wavefront, it is necessary that its tangent space never contains the "vertical" direction (z axis). Interpret the singularities in figure 17 in terms of the Lagrangian immersion. Deduce that the cohomology class dual to the singularities of the wavefront is the Maslov class of the Lagrangian immersion.

If f is a Lagrangian immersion which is not exact, the construction gives a Legendre immersion and a wavefront of the smallest cover of L on which $f^\star(pdq)$ becomes exact. For instance, the zigzag in figure 18 is the wavefront of a Lagrangian immersion of $\mathbf{R}$ into $\mathbf{R}^2$... which is just the standard embedding of S^1. It is still possible to "draw" the Lagrangian submanifold by drawing a significative part (image of a fundamental domain for the deck transformations) of the wavefront (as the bold part in the zigzag).

Rotating the zigzag about the z-axis (figure 18) gives the wavefront of a lagrangian immersion of $\mathbf{R} \times S^1$ in $\mathbf{R}^4$... which defines an embedded Lagrangian torus, so that the "flying saucer with a hole" in figure 18 represents a genuine Lagrangian subtorus of $\mathbf{R}^4$. The following exercises rely on this technique and show how to construct the Givental surfaces according to [8].

1.5.2. Exercise. — Show that figure 19 represents a Lagrangian immersion of an orientable genus g surface with $g - 1$ double points, all with the same sign.

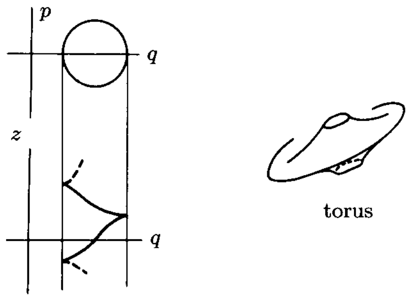

Figure 18

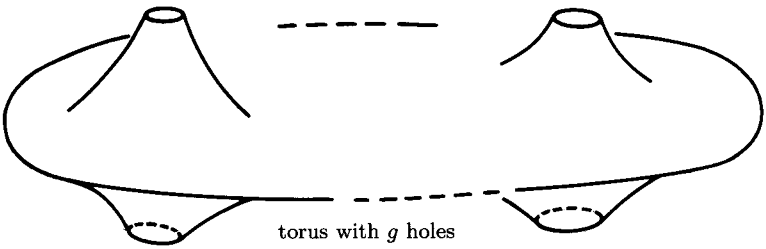

Figure 19

Now consider the Lagrangian submanifold L_y used in § 1.4.

1.5.3. Exercise (see [3]*).* — Give formulas for the wavefronts of the immersions $f_y : L_y \to \mathbf{R}^{2n}$. For $n = 1$ and Q definite positive, draw the wavefront of the elementary cobordism.

To solve the next exercises, one must be rather careful with signs. We concentrate on *surfaces*, and use the orientation of $\mathbf{C}^2$ defined by the symplectic form.

1.5.4. Exercise. — Show that the double point in the Whitney 2-sphere is positive, and that the double points in figure 19 are all negative.

At this point, it should be obvious that the embedded torus in figure 17 is what one obtains by suppressing the double point of the Whitney immersion.

1.5.5. Exercise. — Suppress the $g-1$ double points of the Lagrangian immersion in figure 19 and show that the result is a Lagrangian embedded non orientable surface V_g. Compute the Euler charcteristic $\chi(V_g)$. Show that one obtains Lagrangian embeddings of all non orientable surfaces with Euler characteristic divisible by 4, except the Klein bottle.

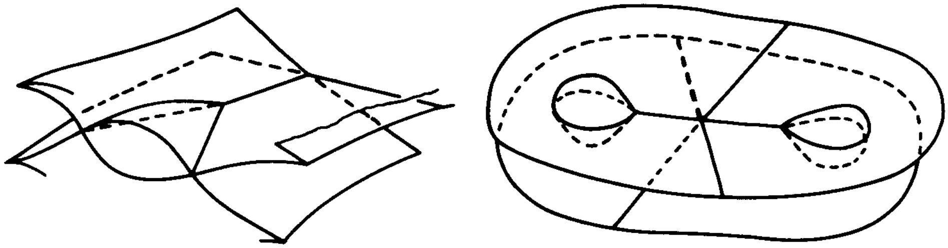

Figure 20

We shall see in 3.2 that the condition that χ is divisible by 4 is necessary. For the Klein bottle, we already did the best we could (up to now) in 1.4.2. Of course, it is easy, from the construction given there and 1.5.3, to get the wavefront of the immersion constructed. A nice definition of Lagrangian non oriented handles, coming from [22], allows to give an alternative construction.

1.5.6. Exercise.

1. Show that the map $\mathbf{R}^2 \to \mathbf{R}^4$ given by

$$(u,v) \longmapsto \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 2uv \\ u^2 - v^2 \\ u \\ v \end{pmatrix}$$

is an exact Lagrangian embedding. Draw its wavefront.

2. Show that the left part of figure 20 represents an embedded non oriented handle. Use it to construct alternative pictures of nonoriented Lagrangian surfaces.

3. Show that the right part of figure 20 is a Lagrangian Klein bottle with two double points. Use exercise 1.5.1 to show that its Maslov class is a generator of $H^1(K;\mathbf{Z})$, thus, according to exercise 1.1.5 the Klein bottle is regularly homotopic to that of exercise 1.4.2.

Remark. — Each half of the Klein bottle in figure 20 is a Möbius strip, obtained by adding a handle to a disc (half of the Whitney sphere). The wavefront of an alternative (exact) Lagrangian Möbius strip is drawn in [3] and is probably the ancestor of all these constructions.

2. Symplectic area and Maslov classes—rigidity in split manifolds

2.1. Symplectic area class, exactness and monotonicity

Let (V, ω) be a symplectic manifold. It is *exact* if the 2-form ω is exact. This is the case for cotangent bundles, for instance, where the symplectic form is the differential of the Liouville form. Suppose that $f : L \to V$ is a Lagrangian immersion in an exact symplectic manifold. Let λ be a primitive of the symplectic form. Of course the 1-form $f^\star\lambda$ is closed on L. Indeed,

$$d(f^\star\lambda) = f^\star d\lambda = f^\star\omega = 0 .$$

The class $[f^\star\lambda] \in H^1(L; \mathbf{R})$ is called the *symplectic area class* of the immersion f.

2.1.1. Exercise. — Show that the symplectic area class does not depend on the choice of λ.

2.1.2. Exercise (the Lagrangian suspension). — Let L and X be two n-dimensional manifolds and let

$$\begin{array}{ccc} L \times [0,1] & \xrightarrow{f} & T^\star X \\ (x,t) & \longmapsto & f_t(x) \end{array}$$

be a regular homotopy of Lagrangian immersions. Show that there exists a Lagrangian immersion of the form

$$\begin{array}{ccc} L \times [0,1] & \xrightarrow{F} & T^\star X \times \mathbf{C} \\ (x,t) & \longmapsto & (f_t(x), t + ig_t(x)) \end{array}$$

if and only if the symplectic area class $[f_t^\star\lambda_X]$ does not depend on t (Hint: set $h_t(x) = i_{\partial/\partial t}\,(f^\star\lambda_X)_{(x,t)} - g_t(x)$ and show that F is Lagrangian if and only if $\frac{\partial}{\partial t}\,(f_t^\star\lambda) = dh_t$).

Remark. — This well known result will be very useful in § 2.3. It was essential in the theory of Lagrangian cobordisms and thus comes from[2] [3].

A Lagrangian immersion into an exact symplectic manifold will be called *exact* if its symplectic area class vanishes.

2.1.3. Exercise. — Let α be a 1-form on a closed manifold X. We consider α as a section of the cotangent bundle $T^\star X$, and thus its graph as a submanifold in $T^\star X$. When is it a Lagrangian submanifold?, an exact Lagrangian submanifold?

[2] Actually it is only implicit in [3]; it was written explicitly, in [5] for instance where one can find the solution of the exercise.

2.1.4. Exercise. — Show that the Whitney immersion $W : S^1 \to \mathbf{C}$ is an exact Lagrangian immersion (see figure 17). Use the Jordan theorem to show that there exists no *exact* Lagrangian embedding of S^1 into $\mathbf{C}$ (see 2.2.5 for the Gromov theorem which generalises that remark to higher dimensions).

2.1.5. Exercise. — Let X, Y and Z be three n-dimensional manifolds. Suppose $f : X \to T^\star Y$ and $g : Y \to T^\star Z$ are two *exact* Lagrangian immersions. Show that their composition (in the sense of the appendix to chapter I) is exact.

In an exact symplectic manifold (V, ω), the integration map

$$\int_{\cdot} \omega : \pi_2 V \longrightarrow \mathbf{R}$$

is zero. If (V, ω) satisfies this weaker condition, it will be called *weakly exact.*

There is a notion of *weakly exact* Lagrangian submanifold L in any symplectic manifold: one requires that the integration morphism

$$\int_{\cdot} \omega : \pi_2(V, L) \longrightarrow \mathbf{R}$$

is zero. The following exercises are straightforward but unavoidable.

2.1.6. Exercise. — Check that an exact Lagrangian submanifold in an exact symplectic manifold is weakly exact.

2.1.7. Exercise. — If a connected symplectic manifold contains a weakly exact Lagrangian submanifold, then it is weakly exact.

The next notion relates the two 1-cohomology classes we have. A Lagrangian immersion $f : L \to \mathbf{C}^n$ is called *monotone* if its Maslov class is proportional to the symplectic area class: $\mu = c \cdot \lambda$ for some real number $c > 0$.

2.1.8. Exercise. — Show that the standard Lagrangian torus $L = \{p_j^2 + q_j^2 = 1, j = 1, \ldots, n\} \subset \mathbf{C}^n$ is monotone.

2.1.9. Exercise (see [39]). — Show that the Lagrangian immersion $M \to \mathbf{C}^{n+1}$ of exercise 1.3.2 is monotone (Hint: evaluate the Maslov class).

2.2. Rigidity in split manifolds

In this §, we state two main general results of rigidity for both the area and the Maslov classes. Both theorems were originally proved by infinite dimensional techniques: the first one using the pseudo-holomorphic curves approach (this proof is given in § 5) and the second one using calculus of variations for the action functional.

DEFINITION 2.2.1. — *Let (V, ω) be a symplectic manifold without boundary. (V, ω) is geometrically bounded if there exist on V an almost complex structure J and a complete Riemannian metric μ such that the following properties are satisfied:*

1. *J is uniformly tamed by ω, that is: there exist strictly positive constants α and β such that:*
$$\omega(X, JX) \geq \alpha \|X\|_\mu^2$$
and
$$|\omega(X, Y)| \leq \beta \|X\|_\mu \|Y\|_\mu$$
for all $X, Y \in TV$. This obviously implies that ω tames J (that is: $\omega(X, JX) > 0$ for all non zero tangent vector X).

2. *There exist an upper bound for the sectional curvature of (V, μ) and a strictly positive lower bound for the injectivity radius of (V, μ).*

Remark. — If V is compact, condition 1 is equivalent to the fact that ω tames J. Condition 2 is also obviously true in this case, thus any compact symplectic manifold is geometrically bounded.

2.2.2. Other examples.

1. The cotangent bundle $(T^\star M, \omega_M)$ of a closed manifold M, endowed with the standard symplectic structure, is geometrically bounded; in fact, one may choose a ω_M-tame almost complex structure J, homogeneous with respect to uniform dilatations in the fibres, and a metric μ on $T^\star M$ induced by any Riemannian metric on M. This is also true if M is an open subset of a closed manifold, or if M is of finite topological type.

2. Any product of a geometrically bounded manifold (V, ω) with the standard $(\mathbf{C}^n, \omega_0, J_0, \mu_0)$ is geometrically bounded.

DEFINITION 2.2.3. — *Let L be a Lagrangian submanifold of a symplectic manifold (V, ω). The pair (V, L) is geometrically bounded if:*

1. *there exist structures J and μ on V such that (V, ω, J, μ) is geometrically bounded;*

2. *L is properly embedded in V (that is: L has no boundary and is closed as a subset of V);*

3. *the second fundamental form of $L \subset (V, \mu)$ is bounded above, and there exist constants $\delta > 0$ and $K > 0$ such that any pair of points $\ell_1, \ell_2 \in L$ with distance (in V) $d_\mu(\ell_1, \ell_2)$ smaller or equal to δ can be joined by a path in L of μ-length smaller or equal to $K\delta$.*

We shall sometimes abbreviate "geometrically bounded" to "g. bounded". Of course, any compact Lagrangian submanifold without boundary in a g. bounded manifold gives rise to a g. bounded pair. We can now state the main theorem of this chapter. We recall that a Lagrangian distribution on a symplectic manifold (V, ω) is simply a smooth field that associates to each point $x \in V$ a Lagrangian vector subspace of $T_x V$.

THEOREM 2.2.4. — *Let (V, ω) be a product $(V' \times \mathbf{C}, \omega' \oplus \omega_0)$ where (V', ω') is a geometrically bounded weakly exact symplectic manifold. Let L be a geometrically bounded Lagrangian submanifold of V. If the projection of L on the $\mathbf{C}$-factor is bounded, then there exists a loop in L which bounds a holomorphic disc (thus L is not weakly exact).*

Moreover, if $\mathcal{L}'$ is any Lagrangian distribution on V', and if L is as above and monotone, then there exists a loop γ which bounds a holomorphic disc and which is such that

$$1 \leq \|\mu_L(\gamma)\| \leq \dim L + 1$$

(with respect to the Lagrangian distribution $\mathcal{L} = \mathcal{L}' \times \mathbf{R}$ of V). Thus $1 \leq \|\mu_L\| \leq \dim L + 1$.

The proof of this theorem using pseudoholomorphic techniques is postponed to § 5. This theorem shows that the requirement of being embedded impose severe constraints on both the symplectic area and the Maslov classes of a Lagrangian submanifold. This is in sharp contrast with an immersed Lagrangian submanifold, whose behaviour is entirely characterised by the h-principle and thus much more flexible. The formulation of theorem 2.2.4 is quite general and has many useful corollaries. The following was conjectured by Arnold in the 60's and proved by Gromov in 1985 and in [24]:

COROLLARY 2.2.5. — *There exists no closed exact Lagrangian submanifold in $(\mathbf{C}^n, \omega_0)$.* □

Remark. — As a consequence, there is no closed simply connected Lagrangian submanifold in $\mathbf{C}^n$. For instance, for $n \geq 2$, the n-sphere has *no* Lagrangian embedding into $\mathbf{C}^n$. If $n \neq 3$, this is more or less[3] easy to prove using standard algebraic topology methods, but for $n = 3$, corollary 2.2.5 is needed (see § 3.2 for these questions).

Consider now the constraints imposed on the Maslov class of a Lagrangian embedding in $\mathbf{C}^n$. The Maslov class μ_f of any Lagrangian immersion $f : L \to \mathbf{C}^n$ is clearly an invariant of Lagrangian regular homotopy. It turns out, as theorem 2.2.4 shows, that the invariant $\|\mu_f\|$ can distinguish those Lagrangian regular homotopy classes that may contain embeddings. For $n = 1$, we have already seen in the

[3] Depending on the actual value of n.

appendix to chapter I that any even integer is the Maslov class of a Lagrangian immersion $f : S^1 \to \mathbf{C}$ but that Lagrangian *embeddings* have $\|\mu_f\| = 2$. For $n \geq 2$, the following two results are known, the first one being an obvious corollary of theorem 2.2.4:

COROLLARY 2.2.6 ([39]). — *Let L be a closed monotone Lagrangian submanifold of $\mathbf{C}^n$. Then $1 \leq \|\mu_L\| \leq n+1$.* □

The second result was proved by Viterbo using "soft" techniques... in this case infinite dimensional Hamiltonian systems (once again, soft may be very hard!)

THEOREM 2.2.7 (Maslov class rigidity [49]). — *Let L be any closed manifold admitting a Riemannian metric of non negative sectional curvature (a torus for instance). Then for every Lagrangian embedding $f : L \to \mathbf{C}^n$, $1 \leq \|\mu_f\| \leq n+1$.*

Remarks. — There is no known example of a closed Lagrangian submanifold $L \subset \mathbf{C}^n$ that does not satisfy the inequality $1 \leq \|\mu_L\| \leq n+1$.

Corollary 2.2.6 is almost optimal in the following sense: for every pair of integers $2 \leq k \leq n$, there exists a monotone Lagrangian embedding f of a closed manifold into $\mathbf{C}^n$ with $\|\mu_f\| = k$ (see [39]).

Surprisingly enough, there exist no rigidity theorems for higher[4] Lagrangian characteristic classes. Using the h-principle, one can say a lot about the realisability of such or such a class by Lagrangian immersions (see [6]), but nothing is known (at least to us) in the case of embeddings.

2.2.8. Example. — Note that theorem 2.2.4 implies that a simply connected closed manifold L admits no Lagrangian embedding in $(V' \times \mathbf{C}, \omega' \oplus \omega_0)$ if V' is weakly exact. For such an embedding, there would exist a disc D with boundary in L, of strictly positive area. Let D' be a disc in L bounded by ∂D. It would have a non zero symplectic area because V is weakly exact, which is absurd. Does there exist a non weakly exact V' such that $V' \times \mathbf{C}$ contains a closed simply connected Lagrangian submanifold? Here is an example: let ω be an area form on S^2 such that the pull-back of ω by the Hopf fibration $f : S^3 (\subset \mathbf{C}^2) \to S^2$ is equal to the restriction of the standard form ω_0 on $\mathbf{C}^2$. Then the graph of $f : S^3 \to S^2$ is a Lagrangian submanifold of $(\mathbf{C}^2 \times S^2, \omega_0 \oplus -\omega)$ since $(gr(f))^\star(\omega_0 \oplus -\omega) = \omega_0 - \omega_0 = 0$.

Now let us generalise the preceeding example: let m, k be two non zero integers, X^{2m+2k} and Z^{2k} be any geometrically bounded symplectic manifolds, with X weakly exact and split, $Y^{m+2k} \subset X$ any geometrically bounded simply connected submanifold whose projection on the $\mathbf{C}$-factor is bounded.

If Z is weakly exact, there is no symplectic map $f : Y \to Z$ (that is such that $f^\star(\omega_0) = \omega_{|_Y}$). In particular, one may replace the condition "weakly exact" on Z by

[4]Actually, pseudo-holomorphic curves methods are typically 1 and/or 2-dimensional, and they are the only tool we have up to now for proving rigidity results.

the stronger condition "$\pi_2(Z) = 0$" and obtains: there is no symplectic map from Y to Z if $\pi_1(Y) = 0$ and $\pi_2(Z) = 0$!

Here is a consequence: let Y be a closed coisotropic submanifold of a geometrically bounded split weakly exact manifold X ($\mathbf{R}^{2n}$ for instance). Suppose that the characteristic foliation of Y is globally integrable (that is: Y admits a reduction). Then $\pi_1(Y) = 0$ implies $\pi_1(K) \neq 0$, where K is a leaf of the foliation.

As a final application, the next exercise will show that any (*if* any!) Lagrangian embedding of the Klein bottle must be regularly homotopic to the Lagrangian immersions of K^2 constructed above (1.4.2 and 1.5.6).

2.2.9. Exercise (see [30]).

1. Suppose that $i : K^2 \to \mathbf{C}^2$ is a Lagrangian embedding, and let $\mu(i) = xa^\star$ (where $a^\star$ generates $H^1(K^2;\mathbf{Z})$), $x \in \mathbf{Z}$. Use the Weinstein tubular neighbourhood theorem and the summation formula (appendix to chapter I) to show that i induces a Lagrangian embedding of T^2 in $\mathbf{C}^2$ of Maslov class $2xa_1^\star$. Deduce, using Viterbo's theorem 2.2.7, that $x = \pm 1$, that is : $\mu(i) = \pm a^\star$.

2. Consider the Lagrangian immersion j constructed in 1.4.2. Let $f : T^2 \to K^2$ be the orientation twofold covering of K^2, and let $\{a_1, b_1\}$ be a basis of $H_1(T^2;\mathbf{Z})$ with $f_\star(a_1) = 2a$ and $f_\star(b_1) = b$ where b is torsion and a is a torsion-free primitive class. Let $\{a_1^\star, b_1^\star\}$ be the dual basis and $\theta \in H^1_{dR}(T^2)$ the 1-form corresponding to $b_1^\star$ by the de Rham isomorphism. Note that the graph $gr(\theta)$ of $\theta : T^2 \to T^\star T^2$ is Lagrangian submanifold and that the composition
$$h : T^2 \xrightarrow{gr(\theta)} T^\star T^2 \xrightarrow{T^\star f} T^\star K^2$$
is a Lagrangian embedding which is a lifting of f. Show that $f^\star(H^1(K^2;\mathbf{Z})) \subset 2H^1(T^2;\mathbf{Z})$ and that $\mu(h) = 0$.

3. Using the result of exercise 1.1.5, prove that the Lagrangian Klein bottles j constructed in 1.4.2 and 1.5.6 are regularly homotopic to i.

2.3. Lagrangian intersections

In the sixties, Arnold stated a set of conjectures in global symplectic geometry, based on Poincaré's last geometric theorem (the Poincaré-Birkhoff theorem), a theorem about fixed points of certain diffeomorphisms of the annulus which preserve the area: that is, about fixed points of some symplectic diffeomorphisms. Using the classical "diagonal trick" explained in chapter I, one easily converts a question about fixed points of symplectic diffeomorphisms into a problem about the intersection of two Lagrangian submanifolds (a very good reference on these conjectures and their mutual relations is the survey by Chaperon [19]). The general conjectures are now theorems of Gromov (here theorem 2.3.6 and corollaries 2.3.9 and 2.3.7). Following Gromov, we give proofs of these results, based on theorem 2.2.4.

The main result in this § asserts that you cannot disconnect certain Lagrangian submanifolds from themselves by deformation. We first need to explain which deformations are allowed.

Exact Lagrangian isotopies. — Let $L \subset (V, \omega)$ be a Lagrangian submanifold, and let $f_t : L \to V$ be an isotopy (among Lagrangian embeddings): f_t is a Lagrangian embedding for all t and $f_0 = \mathrm{Id}$. We shall also consider all the f_t's together as a map

$$f : L \times [0,1] \longrightarrow V.$$

We say that f is an *exact Lagrangian isotopy* if the 1-form

$$i_{\partial/\partial t} f^\star \omega$$

on $L \times [0,1]$ is exact.

2.3.1. Exercise. — Show that f is an exact Lagrangian isotopy if and only if, for any loop α in L and any t, the area of $f(\alpha \times [0,t])$ is zero.

2.3.2. Exercise. — In the neighbourhood of a Lagrangian submanifold, we know that the symplectic form is exact, being isomorphic to that of a cotangent bundle. Show that f is an exact Lagrangian isotopy if and only if the symplectic area class of f_t does not depend on t.

2.3.3. Exercise. — Assume L is compact. Show that a Lagrangian isotopy is exact if and only if, for any given t, the $f_s(L)$ (s close to t) are graphs of *exact* 1-forms dh_s in any symplectic tubular neighbourhood of $f_t(L)$ (isomorphic to $T^\star L$).

Hamiltonian isotopies. — Recall that a diffeomorphism φ of (V, ω) is Hamiltonian if it is of the form $\varphi = \varphi_1$, φ_t being a Hamiltonian isotopy, i.e. the flow of a (time-dependent) Hamiltonian vector field.

If (V, ω) is any symplectic manifold, we endow $V \times V$ with the symplectic form $\omega \otimes -\omega$ so that the diagonal Δ_V is a Lagrangian submanifold.

2.3.4. Exercise. — Show that an isotopy φ_t of V is a Hamiltonian isotopy if and only if $f_t : x \mapsto (x, \varphi_t(x))$ is an exact Lagrangian isotopy of V into $V \times V$.

2.3.5. Exercise (ambient isotopies, see e.g. [19]*).* — Let L be a compact submanifold of the symplectic manifold (V, ω). Let f_t be an isotopy of L in V. Show that the following two properties are equivalent:

- f_t is an exact Lagrangian isotopy of L,
- L is a Lagrangian submanifold (i.e. f_0 is a Lagrangian embedding) and there exists a compactly supported Hamiltonian isotopy φ_t of (V, ω) such that $f_t = \varphi_t \circ f_0$ for any $t \in [0,1]$.

Now the main results of this § are the following:

THEOREM 2.3.6 ([24]). — *Let (V, ω) be a g. bounded weakly exact symplectic manifold. Let L be a closed weakly exact Lagrangian submanifold and let $f_t : L \to (V, \omega)$ be a Hamiltonian isotopy. Then $f_1(L)$ meets L.*

COROLLARY 2.3.7. — *Let (V, ω) be a weakly exact compact symplectic manifold. Then any Hamiltonian diffeomorphism of V has a fixed point.*

Following Gromov, we prove theorem 2.3.6 by using the isotopy f to construct a Lagrangian immersion of $L \times S^1$ into $(V \times \mathbf{C}, \omega \oplus \omega_0)$, and relate the double points of the later with the points of $L \cap f_1(L)$; we then apply theorem 2.2.4 to conclude. The construction is based on an elegant trick known as the "figure eight trick". Notice first that, according to the Lagrangian suspension process described in exercise 2.1.2, we have a Lagrangian immersion

$$\begin{array}{ccc} L \times [0,1] & \xrightarrow{F} & V \times \mathbf{C} \\ (x,t) & \longmapsto & (f_t(x), t + ig(x,t)) \,. \end{array}$$

Now, we use the Whitney immersion (the "figure eight"). Consider a symplectic immersion $\varphi : S^1 \times [a,b] \to \mathbf{C}$ giving a tubular neighbourhood (figure 21) of the Whitney sphere: we use an interval $[a,b] = g(L \times [0,1])$ where we can assume that $a < 0 < b$ and consider $S^1 \times [a,b]$ as a part of the (trivial) cotangent bundle $T^\star S^1$.

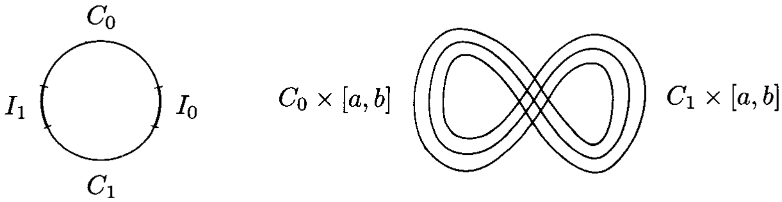

Figure 21

We decompose S^1 as a union of four closed intervals $I_0 \cup C_0 \cup I_1 \cup C_1$ as in figure 21: $\varphi\,|_{(C_0 \cup C_1)\times[a,b]}$ is an embedding. Now, we describe the desired map

$$L \times S^1 \xrightarrow{G} V \times \mathbf{C}$$

on each of the four intervals. To keep notation simple, we rescale and assume that $I_j = C_k = [0,1]$. Then

- On $L \times I_0$, $G(x,t) = (f_0(x), \varphi\,(t, g_0(x)))$: L does not move.
- On $L \times C_0$, $G(x,t) = (f_t(x), \varphi(t, g(x,t)))$. Here, φ has no double point, so we allow L to move.

- On $L \times I_1$, the situation is the same as it was on $L \times I_0$, and $G(x,t) = (f_1(x), \varphi(t, g(x,1)))$.
- On $L \times C_1$, we just have to come back: $G(x,t) = (f_{1-t}(x), \varphi(1-t, g(x, 1-t)))$.

The smoothing of these maps is left as an exercise to the reader:

2.3.8. *Exercise.* — Construct a smooth map $\rho : S^1 \to [0,1]$ such that

$$G(x,z) = \left(f(x, \rho(z)), \varphi\left(z, h_{\rho(z)}(x)\right)\right)$$

is a smooth map $L \times S^1 \to \mathbf{C}$ with the above properties (the formula comes from [24]).

Now, by its very definition, G is a Lagrangian immersion and its double points in $V \times \mathbf{C}$ are the points in

$$L \cap f_1(L) \times 0 \subset V \times 0 \subset V \times \mathbf{C}.$$

If $L \cap f_1(L) = \emptyset$ we thus have a Lagrangian embedding, which is weakly exact because L was (and because the Whitney map is exact). This concludes the proof. □

Remark. — The same proof gives a more general result: we may assume that L is non compact provided the isotopy has compact support; of course, the conclusion is that L and $f_1(L)$ meet inside the support.

We apply the previous result to the Lagrangian intersection problem in a cotangent bundle: any *exact* Lagrangian submanifold must meet the zero section:

COROLLARY 2.3.9. — *Let L be a Lagrangian submanifold in the cotangent bundle $T^\star X$ of a closed manifold. If L is exact, it intersects the zero section.*

Proof. — Consider the isotopy F_t given by scalar multiplication in the fibres $(q,p) \mapsto (q, e^t p)$. If L misses the zero section, $F_t(L) \cap L = \emptyset$ for t large enough. Now $F_t^\star \omega_X = e^t \omega_X$ thus F_t induces an isotopy of L, which is Lagrangian exact if L is exact. Now the symplectic form ω_X itself is exact thus L is weakly exact (see 2.1.6), and we get a contradiction to theorem 2.2.4. □

Proof of corollary 2.3.7. — We use the "diagonal trick" (see chapter I) which converts results on Lagrangian submanifolds into results on fixed points of symplectic diffeomorphisms. Let φ_t be the Hamiltonian isotopy, $\varphi = \varphi_1$ the diffeomorphism. Consider

$$F_t = \mathrm{Id} \times \varphi_t : V \longrightarrow V \times V,$$

and endow $V \times V$ with the symplectic form $\omega \oplus (-\omega)$ so that the diagonal, which is also $F_0(V)$, is a Lagrangian submanifold. The symplectic manifold (V, ω) is weakly exact if and only if its Lagrangian submanifold Δ_V is. We may apply 2.3.6 thus getting points in $\Delta_V \cap F_1(V)$. Of course, they correspond to fixed points of φ. □

Remark. — One can relax the compactness condition, replacing it by compactness of the support of the isotopy.

3. Soft and hard Lagrangian obstructions to Lagrangian embeddings in $\mathbf{C}^n$

3.1. Lagrangian and totally real embeddings

We begin with a straightforward observation.

PROPOSITION 3.1.1. — *In order for there to exist a Lagrangian embedding of V^n into $\mathbf{C}^n$, it is necessary that there exists, in the same regular homotopy class of (ordinary) immersions, both a Lagrangian immersion and an (ordinary) embedding.*

The example of S^3 (see below and § 2.2) shows that the converse is false. However, it becomes true if one relaxes somewhat the constraint of being Lagrangian. The "soft" version of a Lagrangian embedding is a *totally real*[5] embedding: instead of requiring that

$$T_x f\,(T_x L) \perp i T_x f\,(T_x L)$$

you just insist that those two subspaces be *transversal*. This is equivalent to requiring that $T_x f\,(T_x L)$ contain no non trivial *complex* subspace, or equivalently no complex line, hence the name. This really is a softer notion: there exists an analogue of the Whitney lemma (elimination of double points) which gives the converse to 3.1.1.

THEOREM 3.1.2 ([23]). — *Any Lagrangian or totally real immersion which is regularly homotopic to an embedding is regularly homotopic to a totally real embedding, among totally real immersions.* □

Note that there is no homotopic distinction between an (exact) Lagrangian immersion and a totally real immersion (this is an obvious consequence of the Gromov-Lees theorem 1.1.3).

[5]Totally real embeddings are interesting in their own right, especially in complex analysis (see e.g. [46]).

3.2. Soft obstructions

Now, if $f : V^n \to \mathbf{C}^n$ is any immersion, there is a simple and classical (due to Whitney) obstruction to the existence of an embedding in the same regular homotopy class: you just count appropriately (if V is oriented and n is even, any transversal double point carries a natural sign) the number of double points (assumed to be transversal). You get a number

$$d(f) \in \begin{cases} \mathbf{Z} & \text{if } n \text{ is even and } V \text{ orientable} \\ \mathbf{Z}/2 & \text{otherwise.} \end{cases}$$

It is classical (and very easy to prove in the orientable case) that

PROPOSITION 3.2.1. — *Let $n = 2k$ be even, and let $f : V^n \to \mathbf{C}^n$ be a Lagrangian immersion with normal crossings. Then*

$$d(f) = (-1)^{k+1}\frac{\chi(V)}{2}$$

(equality mod 2 *if V is non orientable).*

As usual, χ is the Euler characteristic[6]. The general result is due to Whitney, at least in the oriented case, and relates $d(f)$ to the Euler characteristic of the *normal* bundle of f. As f is Lagrangian, this is isomorphic to the tangent bundle (the sign comes from the consideration of orientations). Hence 3.2.1 (for the non-orientable case, see [7] for a complete proof). □

3.2.2. Exercise. — Carefully taking orientations into account, check that the formula in 3.2.1 is compatible with the suppression of double points allowed by the surgery operation of § 1.4.

3.2.3. Exercise. — Let V be an oriented surface of genus g. What is the minimal number of double points of a Lagrangian immersion of V? Compare with 1.5.2.

The situation in odd dimensions ($n = 2k + 1$) is more complicated. Very often, the Kervaire semi-characteristic

$$\widehat{\chi}_{\mathbf{Z}/2}(V) = \sum_{i=0}^{k} \dim H^i(V; \mathbf{Z}/2) \mod 2$$

plays the same role as χ, but this is not so easy to prove: "soft" obstructions may come from... hard algebraic topology. We shall not give details here—after all, this

[6]Note that $\chi(V)$ is automatically even when a Lagrangian immersion exists: it is plus or minus the Euler characteristic of the normal bundle, and normal bundles of manifolds are known to have rather few Stiefel-Whitney classes, due to Wu formulas.

is a book on pseudo-holomorphic curves' methods. The reader is invited to consult [7] for precise statements and proofs. The easy-to-state consequences are grouped in the following theorem.

THEOREM 3.2.4. — 1. *A closed connected n-manifold V admits totally real embeddings if and only if it is U-parallelisable and*

(a) *if n is even, $\chi(V) = 0$ (equality* $\bmod 4$ *if V is nonorientable)*

(b) *if $n \equiv 1 \bmod 4$ and V is orientable, $\widehat{\chi}_{\mathbf{Z}/2}(V) = 0$.*

2. *If V is a non parallelisable stably parallelisable manifold, it has no totally real embedding.* □

The applications of this theorem to the *non* existence of Lagrangian embeddings are obvious.

As a conclusion to this §, let us discuss in some detail the case of the sphere S^n. If n is even, 3.2.1 shows that there is no totally real, hence no Lagrangian embedding. If $n > 1$ is odd, it is a classical and basic result in differential topology (Smale-Hirsch theory of immersions) that there are exactly two regular homotopy classes of immersions $S^n \to \mathbf{R}^{2n}$, that of the standard embedding $S^n \subset \mathbf{R}^{n+1} \subset \mathbf{R}^{2n}$ and that of $\mathcal{W}$. Assume now that there exists a Lagrangian embedding. Then the standard embedding must be regularly homotopic to a Lagrangian immersion, in particular its normal bundle (which obviously is trivial) must be isomorphic to TS^n... Due to the non parallelisability of almost all the spheres, we conclude that $n = 1$ (we know S^1 has Lagrangian embeddings into $\mathbf{C}$), 3 or 7.

If $n = 3$ or 7, one looks carefully at the forgetful map from the space of Lagrangian immersions to the set of ordinary immersions, which turns out, homotopically, to be a map $\pi_n(U(n)) \to \pi_n(O/O(n))$, that is $\mathbf{Z} \to \mathbf{Z}/2$, induced by the natural inclusions and projection[7] $U(n) \subset O(2n) \subset O = O(\infty) \to O/O(n)$. This map is onto for $n = 3$, but not for $n = 7$. In other words, any immersion is homotopic to a Lagrangian immersion for $n = 3$, but not for $n = 7$. Due to the existence of the Lagrangian Whitney immersion $\mathcal{W}$ (see 1.1.2), S^7 has *no* Lagrangian (or totally real) embedding in $\mathbf{C}^7$.

We can conclude that S^3 has a totally real embedding[8]. But we need all the Gromov machinery (see § 2.2) to conclude that S^3 has no Lagrangian embedding in standard[9] $\mathbf{C}^3$!

Remark. — The same arguments show that all orientable closed 3-dimensional manifolds have totally real embeddings into $\mathbf{C}^3$.

[7]According to the Smale-Hirsch theory, the h-principle for immersions, $\pi_n(O/O(n))$ is in one-to-one correspondence with the set of regular homotopy classes of immersions $S^n \to \mathbf{R}^{2n}$.

[8]An explicit one, i.e. formulas, is given in [1].

[9]But there are symplectic structures on $\mathbf{C}^3$ in which you can find a Lagrangian S^3, see [37].

3.3. Hard obstructions to Lagrangian embeddings

We just saw that there is *no* soft obstruction to the existence of Lagrangian embeddings of closed orientable 3-manifolds into $\mathbf{C}^3$. Indeed, each such manifold admits a totally real embedding into $\mathbf{C}^3$ while the algebraic topology cannot distinguish between the totally real and Lagrangian cases. However the rigidity results of § 2 allow us to produce further obstructions which work in dimension 3 as well.

Lagrangian obstructions via the symplectic area class. — According to 2.2.5 every closed Lagrangian submanifold of $\mathbf{C}^n$ has a non-vanishing symplectic area class, and hence a non-trivial first cohomology group. Therefore we have the following propositon:

PROPOSITION 3.3.1. — *Let L^n be a closed manifold with $H^1(L, \mathbf{Z}) = 0$. Then L does not admit a Lagrangian embedding into $\mathbf{C}^n$.* □

For instance, as already discussed, the 3-dimensional sphere S^3 does not admit a Lagrangian embedding into $\mathbf{C}^3$. This is also the case for the homogeneous space $SU(n)/SO(n)$ (see 1.2.2).

Lagrangian obstructions via the Maslov class. — In the present section we describe a sequence of *flat* manifolds K^n $(n \geq 3)$ which do not admit a Lagrangian embedding into $\mathbf{C}^n$ (see [41]). These manifolds can be considered as "multidimensional Klein bottles". Our approach is based on the Maslov class rigidity for flat manifolds (see 2.2.7).

Let T^n be the torus with coordinates $x_1, \ldots, x_n$ (mod 1). Consider the map

$$\alpha : (x_1, \ldots, x_n) \mapsto (x_1 + \frac{1}{2n-2}, x_3, \ldots, x_n, -x_2).$$

3.3.2. Exercise. — Show that α generates a group, say G, of transformations of the torus which is isomorphic to $\mathbf{Z}/(2n-2)$ and acts freely. Denote by K^n the quotient T^n/G and by $\theta : T^n \to K^n$ the natural projection. Show that $H^1(K^n; \mathbf{Z}) \cong \mathbf{Z}$. Show that K^3 is orientable.

Our main result in this § is the following theorem:

THEOREM 3.3.3. — *The manifold K^n does not admit a Lagrangian embedding into $\mathbf{C}^n$.*

Before proving theorem 3.3.3, let us notice that at least the case of K^3 cannot be treated with the methods described above. Indeed, K^3 is orientable and hence there are no obstructions of an algebraic topological nature. Moreover, non-vanishing of the symplectic area class cannot be applied because $H^1(K^3) \neq 0$ as shown in 3.3.2.

Suppose that a Lagrangian embedding, say f, does exist. Let $\tilde{\theta} : T^\star T^n \to T^\star K^n$ be the covering induced by θ, and let $\pi : T^\star K^n \to K^n$ be the natural projection.

3.3.4. Exercise. — Show that there exists $p_0 \in \mathbf{R}^n$ such that the restriction of $\tilde{\theta}$ to a torus $L = \{p = p_0\} \subset T^\star T^n$ is *an embedding* .

Denote this restriction by g. Note that g is a Lagrangian embedding since L is a Lagrangian submanifold. Let

$$A : H^1(K^n; \mathbf{Z}) \to H^1(L; \mathbf{Z})$$

be the homomorphism induced by πg.

3.3.5. Exercise. — Show that A *expands* in the sense that: $\|Aa\| = (2n - 2)\,\|a\|$ for all $a \in H^1(K^n, \mathbf{Z})$.

3.3.6. Exercise. — Show that the Maslov class of g vanishes (Hint: use the fact that L is *transversal* to the fibres of the bundle $T^\star T^n$).

We are now in a position to finish the proof. Let $h : L \to \mathbf{C}^n$ be a composition (in the sense of the appendix to chapter I) of f and g. Then the summation formula implies that $\mu_h = A\mu_f$. Since K^n and L are *flat*, it follows from 2.2.7 that $\|\mu_h\| \leq n + 1$, and $\|\mu_f\| \geq 1$. On the other hand 3.3.5 implies that $\|\mu_h\| = (2n - 2)\,\|\mu_f\|$. Therefore we obtain that $n + 1 \geq 2n - 2$ which is impossible for $n > 3$. In order to treat the case $n = 3$ recall that the manifold K^3 is orientable and therefore in this case $\|\mu_f\| \geq 2$ (we saw in the appendix to chapter I that $\|\mu_f\|$ is even). Previous arguments give us that

$$n + 1 \geq (2n - 2) \cdot 2,$$

and we get a contradiction with $n = 3$. □

4. Rigidity in cotangent bundles and applications to mechanics

In the present section we establish constraints on various invariants of Lagrangian submanifolds of cotangent bundles. A specific feature of this problem is that non-linear methods, which allow us to attack the rigidity phenomenon in linear spaces and split manifolds, do not generally work for cotangent bundles. Fortunately, in many important cases one can get around this difficulty with the help of quite elementary (but clever!) constructions which reduce rigidity questions in cotangent bundles to rigidity questions in split manifolds.

We start our discussion with embedded Lagrangian tori in $T^\star T^n$ (see § 4.1). There are at least two reasons for this. Firstly, in this case one can clearly see non-trivial interrelations between symplectic and topologic invariants of Lagrangian embeddings. Secondly, Lagrangian tori arise in various qualitative problems of mechanics (see § 4.3 below). In order to treat the problem we need certain techniques which are described below in a more general context. In § 4.2 we give an account of Maslov class rigidity in cotangent bundles. We use a simple algebraic formalism in which a crucial role is played by the summation formula for the composition of Lagrangian embeddings.

We then give applications to mechanics: Hamiltonian mechanics provides us with a wide variety of different types of dynamic behaviour. However one can distinguish a class of systems with the simplest dynamics, the so called integrable systems, whose phase spaces (up to measure zero) are foliated by invariant Lagrangian tori carrying quasi periodic motion. An important problem is to understand what happens to these tori when perturbed. In §§ 4.3 and 4.4 we discuss several recent results in this direction in the context of classical mechanical systems on $T^\star T^n$.

4.1. Lagrangian tori in $T^\star T^n$

In the following theorem we sum up our knowledge on embedded Lagrangian tori in $T^\star T^n$.

THEOREM 4.1.1. — *Let $L \subset T^\star T^n$ be an embedded Lagrangian torus, and let $A : H^1(T^n; \mathbf{R}) \to H^1(L; \mathbf{R})$ be the homomorphism induced by the restriction of the natural projection to L. Then*

1. *L is either homologous to the zero section with suitable orientation or homologous to zero;*
2. *in the first case the Maslov class of L vanishes;*
3. *in the second case the symplectic area class λ of L does not vanish and the Maslov class μ satisfies $2 \leq \|\mu\| \leq n+1$. Moreover, neither λ nor μ lies in the image of A.*

Comments and proof. — The first statement is due to Arnold (see [4]) who realised that it is closely related to the Lagrangian intersections problem (see the discussion in § 2.3). We present his argument below. Certain generalisations to exact Lagrangian submanifolds of cotangent bundles can be found in [31]. Statement 2 was proved in [49] (see also [39], [31] and [41] for alternative approaches and generalisations). Below we use a method invented by Lalonde and Sikorav in [31]. To our knowledge, the last statement never appeared in the literature before (except the Maslov class computations in the case $n = 2$, see [39]).

Proof of statement 1. — We use an indirect argument. Assume that L is homologous to a non-trivial multiple of the zero section.

4.1.2. Exercise. — Show that this assumption implies that there exists a non-trivial covering $T^\star T^n \to T^\star T^n$ such that all lifts of L are disjoint Lagrangian tori homologous to the zero section.

Let L' and L'' be two such lifts. Then they differ by a deck transformation, say Q, which in canonical coordinates (p, q) can be written $Q(p, q) = (p, q + k \bmod 1)$, for some constant vector k.

4.1.3. Exercise. — Show that Q is a time-one map of a Hamiltonian flow on $T^\star T^n$ (Hint: choose the Hamiltonian function to be linear in p).

We now have two disjoint Lagrangian tori L' and L'' in $T^\star T^n$ homologous to the zero section, and a time-one map Q of a Hamiltonian flow such that $Q(L') \cap L'' = \emptyset$. But this is impossible due to the theorem on Lagrangian intersections (see § 2.3 above). This contradiction proves the desired statement. □

Proof of statement 2. — Note that if L is homologous to the zero section then A is an isomorphism. Hence the statement will follow from the first assertion in 4.2.9 below. □

Proof of statement 3. — Note that if L is homologous to zero then A has a non-trivial kernel. All assertions about the Maslov class now follow from 4.2.9 below.

It remains to show that λ does not lie in the image of A (of course this will imply that $\lambda \neq 0$). Obviously it is sufficient to find a loop, say γ, on L which bounds a disc with positive symplectic area.

4.1.4. Exercise. — Use our assumption on L in order to show that there exists a covering $\tau : V = T^\star(T^{n-1} \times \mathbf{R}^1) \to T^\star T^n$ such that every lift of L is compact.

Denote such a lift by L'. Note that V is symplectomorphic to $T^\star T^n \times \mathbf{C}$ and hence is a split manifold. Therefore 2.2.4 gives us a loop, say γ' on L' which bounds a disc of positive symplectic area. Obviously the loop $\tau(\gamma')$ has the desired properties. This completes the proof. □

4.2. The Maslov class rigidity in cotangent bundles

As we have already mentioned, a basic tool for understanding Maslov class rigidity in cotangent bundles is the summation formula for the composition of Lagrangian embeddings (see the appendix to chapter I). We will use (essentially) ideas developed in [31]: an iteration method based on the summation formula for Maslov classes. Let us fix some notation. All manifolds considered below are connected and without boundary. We write π_X for the natural projection of the cotangent bundle $T^\star X$. Given two manifolds X and Y and a homomorphism $A : H^1(X; \mathbf{Z}) \to H^1(Y; \mathbf{Z})$, we denote by $F(Y, X; A)$ the set of all Lagrangian embeddings $f : Y \to T^\star X$ with $(\pi_X f)^\star = A$. The main objects of interest are the sets $I(Y, X; A)$ consisting of all elements of $H^1(Y, \mathbf{Z})$ which can be represented by the Maslov class of an embedding from $F(Y,X;A)$. An important algebraic invariant of such a set is the value

$$\|I\| = \sup_{v \in I} \|v\| \in [0; +\infty].$$

We set $\|\emptyset\| = -1$.

Reparametrisations. — Let us describe the behaviour of the sets $I(Y,X;A)$ under diffeomorphisms of manifolds Y and X. We denote by D_X the group of all automorphisms of $H^1(X,\mathbf{Z})$ generated by diffeomorphisms of X.

PROPOSITION 4.2.1. — *For all $C \in D_Y$, $C\,(I(Y,X;A)) = I(Y,X;C(A))$.*

Proof. — Note that every diffeomorphism φ of Y defines a bijective map

$$F(Y,X;A) \to F(Y,X;\varphi^\star A)$$

which takes a Lagrangian embedding f into its composition with φ. Moreover, $\mu_{f\varphi} = \varphi^\star \mu_f$. The proposition follows. □

PROPOSITION 4.2.2. — *For all $C \in D_X$, $I(Y,X;AC) = I(Y,X;A)$.*

Proof. — Note that every diffeomorphism ψ of X admits a unique lift to a (fibrewise linear) symplectic diffeomorphism, say ψ' of $T^\star X$. Therefore ψ defines a bijective map

$$F(Y,X;A) \to F(Y,X;A\psi^\star)$$

which takes each Lagrangian embedding f to its composition with ψ'. The desired assertion follows from the fact that $\mu_{\psi' f} = \mu_f$. □

4.2.3. Exercise. — Show that $I(Y,Y;\mathrm{Id})$ and $I(Y,X;0)$ are D_Y-invariant sets.

The cocycle property. — Recall that the *sum* of two subsets of a lattice is a set consisting of all pairwise sums of their elements. By definition, $I + \emptyset = \emptyset$.

Our main technical tool is given by the following "cocycle property" which is just a reformulation of the summation formula of chapter I.

PROPOSITION 4.2.4. — *If Z is compact, then $I(Z,Y;B) + BI(Y,X;A) \subset I(Z,X;BA)$.*

Proof. — If h is a composition of Lagrangian embeddings $f \in F(Z,Y;B)$ and $g \in F(Y,X;A)$ then $h \in F(Z,X;BA)$ and moreover $\mu_h = \mu_f + B\mu_g$. □

Geometric properties. — Recall that each $I(Y, X; A)$ is a subset of an integral lattice $H^1(Y; \mathbf{Z})$.

PROPOSITION 4.2.5. — *The set $I(Y, X; A)$ is central symmetric.*

Proof. — Let σ be the standard anti-symplectic involution of $T^\star X$, that is $\sigma(\xi) = -\xi$ for every co-vector ξ. One can easily check that for each $f \in F(Y, X; A)$ the composition σf belongs to $F(Y, X; A)$ and $\mu_{\sigma f} = -\mu_f$. The proposition follows. □

4.2.6. Exercise. — Prove that if X is compact then $I(X, X; \mathrm{Id})$ is star-shaped, that is $\mathbf{Z}v \subset I(X, X; \mathrm{Id})$ provided $v \in I(X, X; \mathrm{Id})$. In particular, either $I(X, X; \mathrm{Id}) = \{0\}$ or $\|I(X, X; \mathrm{Id})\| = +\infty$ (Hint: use the cocycle property 4.2.4).

Rigidity in cotangent bundles via rigidity in $\mathbf{C}^n$. — From now on we assume that X is a compact n-dimensional manifold satisfying the following conditions:

1. the group D_X acts transitively on the set of primitive elements of $H^1(X; \mathbf{Z})$;
2. X admits a Lagrangian embedding into $\mathbf{C}^n = T^\star \mathbf{R}^n$;
3. moreover, $I(X, \mathbf{R}^n; 0)$ does not contain 0 and $\|I(X, \mathbf{R}^n; 0)\| < +\infty$.

A basic example of such a manifold is the torus T^n (see theorem 4.1.1).

Our first observation is that in this situation the structure of the set $I(X, \mathbf{R}^n; 0)$ is quite simple.

4.2.7. Exercise. — There exists a finite set M of positive integers such that $I(X, \mathbf{R}^n; 0)$ consists of all $v \in H^1(X; \mathbf{Z})$ with $\|v\| \in M$ (Hint: use that $I(X, \mathbf{R}^n; 0)$ is D_X-invariant (see 4.2.3) and the fact that the primitive elements are exactly the elements of norm 1).

Now we are in position to formulate the main result.

THEOREM 4.2.8. — *For every homomorphism A of $H^1(X; \mathbf{Z})$ the following holds:*

1. *$I(X, X; A) \subset I(X, \mathbf{R}^n; 0)$ provided A has a non-trivial kernel;*
2. *If A is a monomorphism, $I(X, X; A)$ is equal to $\{0\}$ or empty;*
3. *$I(X, X; A) \cap \mathbf{Q} \cdot \mathrm{Image}(A)$ is $\{0\}$ or empty.*

Proof. — The cocycle property 4.2.4 implies that

$$I(X,X;A) + AI(X,\mathbf{R}^n;0) \subset I(X,\mathbf{R}^n;0).$$

If A has a non-trivial kernel then 4.2.7 implies that the set $AI(X,\mathbf{R}^n;0)$ contains 0 because $\operatorname{Ker} A$ contains at least one primitive element and therefore some multiple of that element must belong to $I(X,\mathbf{R}^n;0)$, and assertion 1 follows. Note that if A is a monomorphism then $\mathbf{Q} \cdot \operatorname{Image}(A)$ covers the whole of $H^1(X;\mathbf{Z})$. Therefore assertion 2 is an immediate consequence of assertion 3.

We prove assertion 3 by using an indirect argument. Assume that there exist primitive elements e_1, e_2 of $H^1(X;\mathbf{Z})$ and positive integers p, q such that $Ae_1 = pe_2$ and $qe_2 \in I(X,X;A)$. Set $m = \max M$, and note that $me_1 \in I(X,\mathbf{R}^n;0)$ due to 4.2.7. We obtain

$$qe_2 \in I(X,X;A),$$

$$A(me_1) = mpe_2 \in AI(X,\mathbf{R}^n;0),$$

and hence the cocycle property gives us that $(q+mp)e_2 \in I(X,\mathbf{R}^n;0)$. This is a contradiction to the assumption $\|I(X,\mathbf{R}^n;0)\| < m+1$. □

Let us apply the previous theorem to the case $X = T^n$.

COROLLARY 4.2.9.

1. *If A is an isomorphism then $I(T^n,T^n;A) = \{0\}$.*
2. *If A has a non-trivial kernel then $2 \le \|\mu\| \le n+1$ for all $\mu \in I(T^n,T^n;A)$.*
3. *Moreover, in the last case, $I(T^n,T^n;A) \cap \mathbf{Q} \cdot \operatorname{Image}(A) = \emptyset$.*

Proof. — Statement 1 follows from 4.2.8 (second condition) and the fact that $I(T^n,T^n;A)$ is not empty (it contains a suitable parametrisation of the zero section). Statement 2 follows from 4.2.8 (first condition) and the Maslov class rigidity theorem 2.2.7. Statement 3 follows from 4.2.8 (third condition) and the fact that $I(T^n,T^n;A)$ does not contain 0 due to statement 2. □

Exactness and monotonicity. — Propositions 4.2.1, 4.2.2, 4.2.3, 4.2.4, 4.2.5, 4.2.6 and their proofs remain true with no change if one considers *exact* Lagrangian embeddings instead of general ones (that is the sets $I_{ex}(Y,X;A)$ consisting of all elements from $H^1(X,\mathbf{Z})$ representing the Maslov classes of exact Lagrangian embeddings from $F(Y,X;A)$). The reason is that that *the class of exact Lagrangian embeddings is closed under the composition operation* as we already mentioned in § 2.1. The situation with regard to 4.2.7, 4.2.8 and 4.2.9 is quite different since there are no exact Lagrangian embeddings of a compact manifold into $\mathbf{C}^n$! Nevertheless one can overcome this difficulty using *monotone* Lagrangian embeddings into $\mathbf{C}^n$ (see [41]). As an illustration we present the following exercise:

4.2.10. Exercise. — Let $X = T^n \times S^2$. Show that $I_{ex}(X, X; \mathrm{Id}) = \{0\}$. Due to the "exact" version of 4.2.6 it is sufficient to show that $\|I_{ex}(X, X; \mathrm{Id})\| < +\infty$. Suppose that this is not true, then there is an exact Lagrangian embedding, say f from $F(X, X; \mathrm{Id})$ with $\|\mu_f\| > n$. Recall now that X admits a monotone Lagrangian embedding, say g into $\mathbf{C}^{n+2}$ (see 2.1.8). Moreover, using the action of D_X one can choose g in such a way that the vectors μ_g and μ_f are proportional with a positive coefficient. Consider now the composition h of f and g. One can check that h is monotone and that $\|\mu_h\| > n + 1$, in contradiction with 2.2.6.

4.2.11. Exercise. — Let X be as in the previous exercise. Show that $I(X, X; \mathrm{Id}) = \{0\}$ (Hint: show that each embedding from $F(X, X; \mathrm{Id})$ is Lagrangian isotopic to an exact one, and apply 4.2.10).

4.3. Applications to mechanics: from order to chaos

In the following we use canonical coordinates (p, q) on $T^\star T^n$. We say that a Hamiltonian function H on $T^\star T^n$ satisfies the *Legendre condition* if it is strictly convex with respect to momenta: $H_{pp} > 0$. Note that the Legendre condition appears in most physically important examples. Consider for instance the motion of a free particle. The corresponding Hamiltonian function is just the kinetic energy $H(p, q) = (A(q)p, p)/2$, where each matrix $A(q)$ is symmetric and positive definite.

4.3.1. Exercise. — Set $(a^{ij}) = A^{-1}$. Show that trajectories of the Hamiltonian flow generated by H project to geodesics of the Riemannian metric $a^{ij}dq_i dq_j$ on T^n.

It turns out that the Legendre condition has a nice geometric meaning which is crucial for our considerations. Roughly speaking, if a Hamiltonian function satisfies this condition, then the corresponding Hamiltonian flow *twists* every Lagrangian subspace into the positive direction with respect to the vertical Lagrangian distribution $\mathcal{F} = \{dq = 0\}$. Let us express this *twisting property* in a more precise way. In what follows we identify every tangent space $T_x T^\star T^n$ with the standard symplectic vector space $\mathbf{R}^{2n}$, and we write $\Lambda = \Lambda_n$ for the Lagrangian Grassmann manifold. We denote by α the subspace $\{dq = 0\}$, and by Λ^α the set off all Lagrangian subspaces from Λ which are not transversal to α.

PROPOSITION 4.3.2 (Twisting property). — *Let g^t be the Hamiltonian flow generated by a Hamiltonian function H which satisfies the Legendre condition. Then for all $x \in T^\star T^n$, $\lambda \in \Lambda^\alpha$ the vector*

$$\frac{d}{dt}\left(g^t_\star(x)\lambda\right)\Big|_{t=0} \in T_\lambda \Lambda$$

is α-positive.

Proof. — Given the definition of an α-positive vector (see the appendix to chapter I), we have to check that for every non-zero vector $\xi \in \lambda \cap \alpha$ the following inequality holds:

$$\omega\left(\xi, \frac{d}{dt}\left(g_\star^t(x)\xi\right)\Big|_{t=0}\right) > 0.$$

In (p,q)-coordinates such a vector ξ can be written as $(a,0)$. Linearising the Hamiltonian equations we obtain

$$\frac{d}{dt}\left(g_\star^t\xi\right)\Big|_{t=0} = (-H_{qp}a, H_{pp}a),$$

and the desired inequality can be reformulated as $(H_{pp}a, a) > 0$. But this follows from the Legendre condition[10]. □

The simplest integrable system on $T^\star T^n$ is given by a Hamiltonian function $H(p,q) = (p,p)/2$ associated with the Euclidean metric on the torus (cf. 4.3.1). The corresponding Hamiltonian flow can be written as

$$(p,q) \mapsto (p, q + pt),$$

and hence each torus $\{p = \text{ const}\}$ is invariant under the dynamics.

A striking fact which follows from the celebrated Kolmogorov-Arnold-Moser theory (see e.g. [2], appendix 8) is that most of these tori survive under a *small* perturbation, namely a set of large measure in the phase space is still foliated by invariant tori of the perturbed system. Let us emphasise two properties of those invariant tori which are exhibited by the KAM-theory:

- each such torus is homologous to the zero section;
- in some angular coordinates φ mod 1 on such a torus the dynamics is given by a linear shift $\varphi \mapsto \varphi + \sigma t$, where the components $(\sigma_1, \ldots, \sigma_n)$ of σ are independent over $\mathbf{Z}$.

A smooth embedded n-dimensional torus which is invariant under a Hamiltonian flow on $T^\star T^n$ and satisfies these two conditions is an *essential invariant torus.* One might expect that the geometry of invariant tori becomes very complicated when the perturbation becomes very large. An obstruction to such a behaviour is given by the following theorem, the proof of which will be given in § 4.4.

THEOREM 4.3.3. — *If a Hamiltonian function satisfies the Legendre condition, every essential invariant torus of its Hamiltonian flow is the graph of a smooth section of the cotangent bundle.*

---

[10]This twisting property makes sense in any symplectic manifold endowed with a Lagrangian distribution. Functions with this property are investigated in [16], [17].

Let us make some comments. The first result of this kind was obtained by Birkhoff for invariant curves of area preserving maps of the annulus (see [13], and also [27], [21]) and is usually known as *Birkhoff's second theorem.* Our formulation for the multidimensional case comes from [40] (see also [15], [12] for the case $n = 2$, and [28] for some other interesting extensions of the Birkhoff's theory). We refer the reader to [16] for a detailed discussion and far reaching generalisations.

As we shall show in § 4.4, theorem 4.3.3 gives us a tool to understand the *invariant tori breaking mechanism.* In order to formulate the result, we specify our deformation of the Euclidean metric on the torus. For simplicity, we restrict our attention to the case $n = 2$ (the multidimensional case can be treated in exactly the same manner). Let γ be an embedded closed curve on T^2 which bounds a disc, say B, and fix a point K inside the disc. Let G_ε, ($\varepsilon \geq \varepsilon_0$) be a Riemannian metric on the torus such that

$$\text{distance}(K, \gamma) > \varepsilon \cdot \text{length}(\gamma).$$

In other words, increasing the parameter ε corresponds to a bump growing up inside the disc B and whose top point is K. The proof of the next theorem will be given in § 4.4.

THEOREM 4.3.4 ([14], see also [10] and [40]). — *There exists a critical value of the parameter, say ε_{cr}, such that for all $\varepsilon > \varepsilon_{cr}$ the Hamiltonian flow associated with the metric G_ε has no essential invariant tori.*

Remark. — Thus tori disappear. The question of the dynamical properties of the system is far from being understood. Nevertheless, there are some very interesting results in dimension 2: more complicated invariant sets appear, the Aubry-Mather Cantor sets (see [14], [10] and [20] in which Donnay constructed a metric on T^2 whose geodesic flow is ergodic—a classical property of *negatively curved manifolds*).

4.4. Symplectic geometry and variational properties of invariant tori

The following simple observation is crucial for proving the results of the previous section.

PROPOSITION 4.4.1 ([28]). — *Every essential invariant torus of a Hamiltonian flow on $T^\star T^n$ is Lagrangian.*

Proof. — Let φ mod 1 be angular coordinates on an essential invariant torus L such that in these coordinates the restriction of the Hamiltonian flow to L is given by an irrational shift. Denote by Ω the restriction of the symplectic form to L. Since Ω is preserved by the dynamics, and since every trajectory is dense on L we conclude that Ω is a *translation invariant* form with respect to φ-coordinates. Moreover, Ω is exact since the symplectic form is. Hence Ω vanishes, and L is a Lagrangian torus. □

Now we are in position to prove 4.3.3. The proof given below incorporates a nice argument which we learned from G. Paternain (see [38]). We use the notations of § 4.3.

Proof of 4.3.3. — Let L be an essential invariant torus of a Hamiltonian flow g^t, which is generated by a Hamiltonian function H satisfying the Legendre condition. Note first of all that it is sufficient to check that L is *transversal* to the Lagrangian distribution $\mathcal{F} = \{dq = 0\}$. Assume for contradiction that there exists a point $x \in L$ such that the tangent subspace $\lambda = T_x L$ belongs to Λ_n^α.

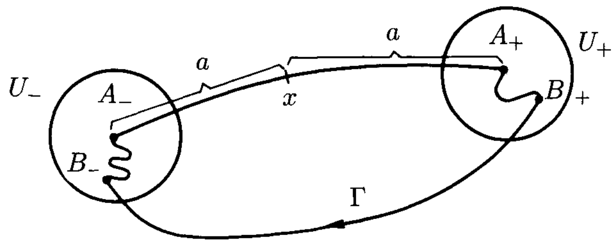

Figure 22

Our plan is to construct a loop, say γ, on L with positive value of the Maslov class. By 4.3.2, the vector

$$\frac{d}{dt}\left(g^t_\star(x)\lambda\right)\Big|_{t=0}$$

is α-positive and hence transversal to Λ_n^α. Therefore there exists some positive a such that the for every $t \in [-a, a]$ the subspace

$$g^t_\star(x)\lambda = T_{g^t(x)}L$$

is transversal to α (see figure 22). Set $A_+ = g^a(x), A_- = g^{-a}(x)$. Let U_+, U_- be small neighbourhoods in L of the points A_+ and A_- respectively such that for each point, say y, of these neighbourhoods the tangent subspace $T_y L$ is still transversal to α. Note that every trajectory on L is dense, and hence there exists a segment of trajectory, say Γ, whose end points B_+ and B_- belong to U_+ and U_- respectively. Now join B_- with A_- by a smooth path inside U_-, A_- with A_+ by a piece of the trajectory of the point x for $t \in [-a, a]$, and A_+ with B_+ by a smooth path inside U_+. Adding the segment Γ we obtain a loop γ on L. Note that the corresponding loop $z \mapsto T_z L$ $(z \in \gamma)$ in Λ_n has a non-empty intersection with Λ_n^α. Moreover, our construction together with 4.3.2 imply that its tangent vector in each intersection point is α-positive. Therefore the value of the Maslov class of L on γ is positive as noticed in the appendix to chapter I. On the other hand L is an embedded Lagrangian torus homologous to the zero section and hence its Maslov class vanishes by theorem 4.1.1. This contradiction proves the theorem. □

Before we give the proof of 4.3.4, let us recall that a fundamental property of a geodesic is that locally it minimises the length. A geodesic is called (globally) *minimal* if any segment minimises length in the homotôpy class of paths with the same endpoints (see chapter III).

4.4.2. Exercise. — Show that for ε sufficiently large there is no complete globally minimal geodesic of the metric G_ε which passes through the top of the bump K (Hint: if ε is sufficiently large then a globally minimal geodesic will prefer to go round the bump!).

We now give the proof of 4.3.4 in two exercises. A basic observation is the following:

4.4.3. Exercise. — Let L be a smooth Lagrangian section of $T^\star T^n$ which is invariant under a Hamiltonian flow associated to a Riemannian metric on T^n and which is contained in a regular level $\{h = \text{ cont}\}$. Show that each trajectory which lies on L projects to a globally minimal geodesic (Hint: use a classical Weierstrass theorem of the variational calculus (see [33] for details)).

4.4.4. Exercise (proof of 4.3.4*).* — Show that 4.3.4 is a consequence of the two previous exercises together with 4.3.3 and 4.4.1.

4.4.5. Exercise. — Find an estimate for the critical value ε_{cr} of the parameter.

5. Pseudo-holomorphic curves: proof of the main rigidity theorem

5.1. The Riemann-Hilbert problem

In order to prove theorem 2.2.4, we will first establish a theorem on the existence and unicity of solutions of the equation $\overline{\partial}_J f = g$ on *geometrically bounded* almost complex manifolds containing a given totally real submanifold. This is a generalisation of the so-called Riemann-Hilbert problem.

The simplest form of the Riemann-Hilbert problem consists, given a map $g : D^2 \to \mathbf{C}^n$ on the closed unit disc $D^2 \subset \mathbf{C}$, in finding a solution $f : D^2 \to \mathbf{C}^n$ of the equation $\overline{\partial} f = g$ which sends the boundary ∂D^2 into the totally real subspace $\mathbf{R}^n = \langle x_1, \dots, x_n \rangle \subset \mathbf{C}^n = \langle x_1, y_1, \dots, x_n, y_n \rangle$. This problem always admits a solution in appropriate Hölder spaces (and therefore in $\mathcal{C}^\infty$ spaces by elliptic regularity). More precisely, for any fixed real non integral number $r > 0$, the map

$$\overline{\partial} : \mathcal{C}^{r+1}(D^2, \partial D^2, 1; \mathbf{C}^n, \mathbf{R}^n, 0) \longrightarrow \mathcal{C}^r(D^2, \mathbf{C}^n)$$

is onto, where the domain is the space of maps of Hölder class $r+1$ from D^2 to $\mathbf{C}^n$ which send ∂D^2 in $\mathbf{R}^n$ and 1 on 0, and where the target space is the space of maps from D^2 to $\mathbf{C}^n$ of Hölder class r (see the classical reference [47], pp. 56 and 355-356).

This problem can be generalised in the following way (see [24], [43], we follow here [44]). Let (V, J) be an almost complex manifold. Denote by i the complex structure on $\mathbf{C}$ (multiplication by i). For a $\mathcal{C}^1$ map $f : D^2 \to V$, define

$$\overline{\partial}_J f = \frac{1}{2}\left(\frac{\partial f}{\partial x} + J\frac{\partial f}{\partial y}\right) = \frac{1}{2}(df + Jdfi)\left(\frac{\partial}{\partial x}\right)$$

which is a section of the vector bundle $f^\star(TV)$ on D^2. It will be more convenient to consider this section as a section of the bundle $pr_2^\star TV \to D^2 \times V$ defined over the graph of f (here $pr_2 : D^2 \times V \to V$ is the projection on the second factor). The generalisation of the Riemann-Hilbert problem is then:

GENERAL PROBLEM. — *Given a totally real submanifold L of an almost complex manifold (V, J) and a global section g of $pr_2^\star TV$, find $f : (D^2, \partial D^2) \to (V, L)$ homotopic to a point in $\pi_2(V, L)$ and such that $\overline{\partial}_J f = g|_{\text{graph}(f)}$ (which we will simply denote by $\overline{\partial}_J f = g$).*

Note that this problem does not always admit a solution; for instance, if L is a closed, totally real submanifold embedded in $\mathbf{C}^n$ endowed with the standard complex structure, and if $g : D^2 \times \mathbf{C}^n \to \mathbf{C}^n$ is equal to a constant g_0, there is no solution when $\|g_0\|$ is sufficiently large (see the harmonicity argument in the proof of theorem 2.2.4).

The following theorem (which provides a solution to the Riemann-Hilbert problem under certain hypotheses) is proved by Sikorav in [44].

THEOREM 5.1.1. — *Let V be endowed with any complete Riemannian metric μ. The general problem above has a solution for all $\mathcal{C}^\alpha$ bounded (with respect to μ) sections g of Hölder class $\mathcal{C}^\alpha$ (for $\alpha > 0$) if the following conditions hold:*

1. *if $g \equiv 0$, the only solutions of $\overline{\partial}_J f = g$ which belong to $\mathcal{C}^{\alpha+1}(D^2, \partial D^2; V, L)$ are the constants;*
2. *for any $B > 0$, the family $\{f \in \mathcal{C}^{\alpha+1}(D^2, \partial D^2; V, L) \mid \left\|\overline{\partial}_J f\right\|_{\mathcal{C}^\alpha} \leq B\}$ is equicontinuous.*

Moreover, one may impose $f(1) = \ell_0$ for any given point $\ell_0 \in L$.

It may be embarassing to try to apply this theorem directly, because the second condition is not easily verifiable in practice. Sikorav shows that this condition holds true under some hypotheses on the geometry of V and L. His proof does not make use of the compactness theorem for J-holomorphic discs and, for this reason, the hypotheses on V and L needed to establish the surjectivity of $\overline{\partial}_J f = g$ are stronger than those needed by Gromov in [24] theorem 2.3.B. We will now prove the stronger version of the theorem due to Gromov, using a compactness theorem.

THEOREM 5.1.2. — *Let (V, ω) be a symplectic manifold and $L \subset V$ be a Lagrangian submanifold such that (V, L) is g. bounded (with respect to structures J and μ on V). If there is no non-constant J-holomorphic 2-sphere in V and no non-constant J-holomorphic disc with boundary in L, then for any C^∞ bounded section g of $pr_2^\star TV \to D_2 \times V$ and any point $\ell_0 \in L$, there exists a solution to the equation $\overline{\partial}_J f = g$ which sends ∂D^2 in L and the point 1 to the point ℓ_0, and such that f is homotopic to a point in $\pi_2(V, L, \ell_0)$.*

Remarks.

1. Since ω tames J, any non-constant J-holomorphic disc $f : (D^2, \partial D^2) \to (V, L)$ satisfies $\int_{D^2} f^\star\omega > 0$. Therefore, the hypotheses of the theorem concerning J-holomorphic discs and spheres are satisfied if L is a *weakly exact* Lagrangian submanifold (recall that this means that the integration map $\int \omega$ from $\pi_2(V, L)$ to $\mathbf{R}$ is zero). Thus, if (V, L) is g. bounded and L is weakly exact, there always exists a solution to the Riemann-Hilbert problem. This is the case, for example, when $(V, \omega, J, \mu) = (\mathbf{C}^n, \omega_0, J_0, \mu_0)$ and L is the image of $\mathbf{R}^n \subset \mathbf{C}^n$ by any bounded symplectic diffeomorphism.
2. The conclusion of theorem 5.1.2 is a Fredholm alternative (see the proof below): suppose (V, L) is g. bounded and V contains no non-trivial J-holomorphic sphere, then either the map $\overline{\partial}_J$ is "surjective" or there exists a non-trivial J-holomorphic disc in $\pi_2(V, L)$. Of course, this alternative may be useful in both directions: when we know that there is no non-trivial J-holomorphic disc in $\pi_2(V, L)$, we conclude that $\overline{\partial}_J$ is surjective as in the example above; when we know that $\overline{\partial}_J$ cannot be surjective, we conclude that there exists a non-trivial J-holomorphic disc in $\pi_2(V, L)$ and hence that L is not (weakly) exact (this applies to any closed (compact and without boundary) Lagrangian submanifold of $\mathbf{C}^n$, see below).

Proof of theorem 5.1.2. — We first bring the problem into a Fredholm framework with a given operator $\overline{\partial}_J$: we do not yet have a well defined operator since the section $\overline{\partial}_J f$ of $f^\star TV$ is defined only on a subset of $D^2 \times V$ which depends on f. For $\ell_0 \in L$ and a non integral $r > 0$ fixed, let us define:

- The set F^{r+1} of Hölder class C^{r+1} maps $f : (D^2, \partial D^2, 1) \to (V, L, \ell_0)$ which are homotopic to $f_0 \equiv \ell_0$ in $\pi_2(V, L, \ell_0)$,
- the set G^r of C^r sections of $pr_2^\star TV \to D^2 \times V$ which are C^r bounded,
- the set $H^{r+1} = \left\{(f, g) \in F^{r+1} \times G^r \text{ such that } \overline{\partial}_J f = g\right\}$,
- the projection map $\Delta^r : H^{r+1} \to G^r$ defined by $\Delta^r(f, g) = g$.

The following four properties hold for any totally real submanifold L of an almost complex manifold (V, J) (see chapter V).

1. G^r is a complex Banach vector space, F^{r+1} is an almost complex Banach manifold with tangent space

   $$T_f F^{r+1} = \{\mathcal{C}^{r+1} \text{ sections of } (f^\star(TV), (f\,|_{\partial D^2})^\star TL) \text{ which vanish at } 1\},$$

   and the charts of F^{r+1} are given by the exponential map of a Riemannian metric on V for which L is totally geodesic (that is such that any geodesic tangent to L at any point of L lies entirely in L). Finally, H^{r+1} is a $\mathcal{C}^\infty$ submanifold of $F^{r+1} \times G^r$. Indeed, let $f_0 \in F^{r+1}$ and $\mathcal{U}(f_0)$ be a $\mathcal{C}^0$-small neighbourhood of f_0. For any $f \in \mathcal{U}(f_0)$, denote by $\Lambda^r(f)$ the space of $\mathcal{C}^r$-sections of $pr_2^\star(TV)$ over the graph of f. Let V be endowed with any $\mathcal{C}^\infty$ connection compatible with J; then the parallel transport induced by this connection gives a complex isomorphism:

   $$PT(f, f_0) : \Lambda^r(f) \to \Lambda^r(f_0)$$

   for any $f \in \mathcal{U}(f_0)$. On $\mathcal{U}(f_0) \times G^r$, define $\Phi : \mathcal{U}(f_0) \times G^r \to \Lambda^r(f_0)$ by

   $$\Phi(f, g) = PT(f, f_0)\left(\overline{\partial}_J f - g|_{\mathrm{graph}(f)}\right).$$

   It is clear that $\Phi^{-1}(0) = H^{r+1} \cap (\mathcal{U}(f_0) \times G^r)$, that Φ is a smooth map, and that 0 is a regular value of Φ because $\Phi(f, .) : G^r \to \Lambda^r(f_0)$ is affine and onto. Finally, the kernel of the associated linear map, $K(f) = \{g \mid g|_{\mathrm{graph}(f)} = 0\}$ admits a closed supplement in G^r. Since these are the sufficient conditions for the inverse image of a point by a smooth map between Banach manifolds to be a smooth submanifold, we conclude that H^{r+1} is a $\mathcal{C}^\infty$ submanifold.

2. Δ^r is a Fredholm map of index 0. In fact, Δ^r is locally equivalent to a quasi-linear differential operator of order 1 whose linearisation is of the form

   $$(\mathrm{Id} + K) \circ \overline{\partial} : \mathcal{C}^{r+1}(D^2, \partial D^2, 1; \mathbf{C}^n, \mathbf{R}^n, 0) \longrightarrow \mathcal{C}^r(D^2; \mathbf{C}^n)$$

   where K is a compact operator. Since $\overline{\partial}$ is an isomorphism, $(\mathrm{Id} + K) \circ \overline{\partial}$ is Fredholm of index 0 (see chapter V).

3. Δ^r is regular at $(\ell_0, 0)$(see chapter V).

4. Elliptic regularity: if f is of class $\mathcal{C}^1$ and if $\overline{\partial}_J f = g$ with g of class $\mathcal{C}^r$, then f is of class $\mathcal{C}^{r+1}$ (see chapter V or [36]).

5.2. Proof of theorem 5.1.2

Let us now prove theorem 5.1.2: given elliptic regularity, it is sufficient to prove that Δ^r is surjective for any non integral $r > 0$. The non-existence of non-constant

J-holomorphic disc in $\pi_2(V, L)$ means that $\Delta^{-1}(0) = (\ell_0, 0)$, and together with the third property above, this implies that 0 is a regular value of Δ^r. In order to prove the theorem, it is therefore enough to show that Δ^r is proper. By Smale's generalisation (see [45]) of Sard's theorem to Fredholm maps between Banach manifolds, one can deduce (see figure 23):

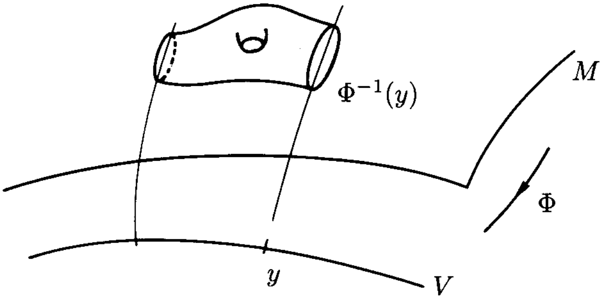

Figure 23

THEOREM 5.2.1. — *Let $\Phi : M \to V$ be a C^∞ proper Fredholm map between Banach manifolds of nonnegative index. Then, for any regular value $y \in V$, $\Phi^{-1}(y)$ is a smooth submanifold (either empty, or of dimension* ind Φ*), and the non oriented cobordism class of $\Phi^{-1}(y)$ does not depend on the choice of the regular value y of V. Denoting by $\gamma(\Phi)$ this invariant in the ring of non oriented cobordism, we have: Φ is onto when $\gamma(\Phi) \neq 0$.* □

Let us apply this theorem to Δ^r, assuming that it is proper. Since 0 is a regular value whose inverse image is a single point, $\gamma(\Delta^r) \neq 0$, and hence Δ^r is onto. Let us describe this more explicitely: let $g \in G^r$ be a regular value of Δ^r and let $\gamma : [0,1] \to G^r$ be a path from 0 to g, everywhere transversal to Δ^r. Since Δ^r is a proper Fredholm map of index 0, $(\Delta^r)^{-1}(\gamma)$ is a non empty compact submanifold of real dimension ind $\Delta^r + 1 = 1$, whose boundary is included in $(\Delta^r)^{-1}(0) \cup (\Delta^r)^{-1}(g)$. In fact, $\partial(\Delta^r)^{-1}(g)$ equals $(\Delta^r)^{-1}(0) \cup (\Delta^r)^{-1}(g)$ because if there were a point p belonging say to $(\Delta^r)^{-1}(g)$ but not to $\partial(\Delta^r)^{-1}(\gamma)$, the differential of Δ^r would send the vector tangent to $(\Delta^r)^{-1}(\gamma)$ at p on the zero vector at $\gamma(1)$, hence the dimension of ker Δ^r at p would be strictly positive. Since ind $\Delta^r = 0$, this would contradict the regularity of g. Thus $(\Delta^r)^{-1}$ is a non oriented cobordism between $(\Delta^r)^{-1}(0)$ and $(\Delta^r)^{-1}(g)$, therefore their mod 2 cardinalities are equal. We conclude that $(\Delta^r)^{-1}(g)$is not empty. This shows that all regular values of Δ^r are in the image of Δ^r, and since this is true by definition for non regular values as well, Δ^r is onto. □

It remains to prove:

PROPOSITION 5.2.2. — *Δ^r is proper.*

Proof. — Suppose that Δ^r is not proper: then there exists a sequence (f_n, g_n) $(n \in \mathbf{N})$ which contains no convergent subsequence in H^{r+1} but whose projection $(g_n)_{n\in\mathbf{N}}$ has a convergent subsequence. Denoting by the same symbols $(g_n)_{n\in\mathbf{N}}$ this subsequence, we thus have a sequence $(f_n) \in (F^{r+1}, \|\ \|_{C^{r+1}})$ with no convergent subsequence, and a sequence (g_n) which C^r-converge to a value $g \in G^r$ such that $\overline{\partial}_J f_n = g_n$.

We now show that each section g of $pr_2^\star TV$ induces an almost complex structure J_g on $D^2 \times V$ with the properties described in the following lemma, due to Gromov (we do not specify the differentiability class, since the lemma clearly holds true for any class of maps).

LEMMA 5.2.3. — *There exists a unique structure J_g on $D^2 \times V$ such that the germs of J_g-holomorphic sections* $\mathrm{Id} \times f$ *of $pr_1 : D^2 \times V \to D^2$ correspond exactly to the solutions of $\overline{\partial}_J f = g$ on V and such that the restriction of J_g to any fibre $s \times V$ of $D^2 \times V \to D^2$ coincides with the original structure J on $V = s \times V$.*

Proof. — In the exact sequence of bundles over $D^2 \times V$:

$$0 \longrightarrow \mathrm{Hom}_{\mathbf{C}}(TD^2, TV) \xrightarrow{\text{incl}} \mathrm{Hom}_{\mathbf{R}}(TD^2, TV) \xrightarrow{\pi}$$
$$\mathrm{Hom}_{\mathbf{R}}(TD^2, TV)/\mathrm{Hom}_{\mathbf{C}}(TD^2, TV) \longrightarrow 0$$

the projection π has a right inverse whose image is the set $\mathrm{Hom}_{\text{anti-}\mathbf{C}}(TD^2, TV)$ of real homomorphisms C such that $C \circ i = -J \circ C$. Hence,

$$\mathrm{Hom}_{\mathbf{R}} = \mathrm{Hom}_{\mathbf{C}} \oplus \mathrm{Hom}_{\text{anti-}\mathbf{C}},$$

since

$$A = \frac{1}{2}(A - J \circ A \circ i) \oplus \frac{1}{2}(A + J \circ A \circ i)$$

(of course $\overline{\partial}_J$ is simply the projection on the second factor of this decomposition, that is the projection on the anti-complex part). Since D^2 is of complex dimension 1, an anti-complex homomorphism is determined by its value on $\partial/\partial x$ and this defines an identification of $\mathrm{Hom}_{\text{anti-}\mathbf{C}}(TD^2, TV)$ with $pr_2^\star(TV)$.

Let g be a section of $pr_2^\star(TV)$, which we consider as a section of the bundle $\mathrm{Hom}_{\text{anti-}\mathbf{C}}(TD^2, TV)$. We define the endomorphism J_g of $T(D^2 \times V)$ at $(s, v) \in D^2 \times V$ by

$$J_g | T_v V = J \text{ and } J_g | T_s D^2 = i + 2 g_{s,v} \circ i.$$

The fact that g is anti-complex obviously implies that $J_g^2 = -\mathrm{Id}$. Finally, the germ of a section $\mathrm{Id} \times f$ of $pr_1 : D^2 \times V \to D^2$ defined near $s \in D^2$ is J_g-holomorphic at s if and only if $J_g(d(\mathrm{Id} \times f)(s)) = d(\mathrm{Id} \times f)(s) \circ i$. But

$$\begin{aligned} J_g(d(\mathrm{Id} \times f)(s)(X)) &= J_g(X \oplus df(s)(X)) \\ &= iX \oplus (2g(iX) + Jdf(X)) \end{aligned}$$

is equal to $d(\mathrm{Id} \times f)(s)(iX) = iX \oplus df(iX)$ for all $X \in T_s D^2$ exactly when $g(Z) = \frac{1}{2}(df + Jdfi)(Z)$ for all $Z \in T_s D^2$, that is exactly when $g = \overline{\partial}_J f$. □

It is clear that the properties stated in the lemma determine an almost complex structure on $D^2 \times V$ in a unique way. Hence, the two sequences (f_n) and (g_n) induce a sequence of J_{g_n}-holomorphic sections $(\mathrm{Id} \times f_n)$ of $D^2 \times V \to D^2$. The sequence (J_{g_n}) C^r converges to an almost complex structure J_g but the sequence $(\mathrm{Id} \times f_n)$ of maps from D^2 to $D^2 \times V$ C^{r+1} diverges. We will deduce from this the existence of a non constant *rational* J-holomorphic curve in V (that is: a non constant J-holomorphic map S^2, in other words a J-holomorphic sphere) or the existence of a non-constant J-holomorphic disc in V whose boundary lies in L. In both cases, this will contradict the hypotheses of the main theorem. We need the following theorem of Gromov on the compactness of holomorphic discs (see chapter VIII for these compactness properties):

THEOREM 5.2.4. — *Let (M, ω_M) be a symplectic manifold, g. bounded for J_M and μ_M. Assume J_M is of class C^r. Let $W \subset M$ be a Lagrangian submanifold such that the pair (M, W) is g. bounded. Let*

$$\varphi_n : \left(D^2, \partial D^2, 1\right) \longrightarrow (M, W, w_0)$$

be a sequence of (i, J_M)-holomorphic discs of class C^{r+1} whose areas are bounded above. Then there exists a subsequence which either C^{r+1} converges to a (i, J_M)-holomorphic disc or weakly converges to a J_M-holomorphic cusp-disc whose image in M is the union of k_1 non-constant J_M-rational curves and k_2 non-constant J_M-holomorphic discs with boundary in W, where $k_1 \geq 0, k_2 \geq 1$, and $k_1 + k_2 \geq 2$.
□

This theorem is still true for a sequence $(J_M)_n \to J_M$ of almost complex structures C^r-converging to an almost complex structure J_M and a sequence φ_n of holomorphic discs (for i and $(J_M)_n$), provided that all the $((J_M)_n, \omega_M, \mu_M)$ for n sufficiently large and (J_M, ω_M, μ_M) are geometrically bounded with respect to the same constant α_M. Then the limit disc or cusp-disc is J_M-holomorphic.

To apply this theorem to the sequence $(\mathrm{Id} \times f_n)$ of J_{g_n}-holomorphic discs in $D^2 \times V$, we verify the hypotheses of the theorem. Here $M = D^2 \times V$, $W = \partial D^2 \times L$, and $\mu_M = \mu_0 \oplus \mu_1$ where μ_0 is the standard metric on D^2 and $\mu_1 = \mu$ is the metric given on V. Since $(D^2, \partial D^2)$ is obviously geometrically bounded with respect to (i, μ_0, ω_0), the conditions in the definition of "geometrically bounded" which depend only on the metric structure are satisfied for the pair $(D^2 \times V, \partial D^2 \times L)$. Therefore, it is enough to find a symplectic form ω_M on $M = D^2 \times V$ and a positive constant α_M such that

$$\omega_M(X, J_{g_n}X) \geq \alpha_M \left\|X\right\|^2_{\mu_M}$$

for all tangent vectors of $D^2 \times V$ and all sufficiently large n.

We set $\omega_M = (C\omega_0) \oplus \omega$, where ω_0 is the standard form on D^2, ω is the given form on V, and C is a constant depending on $\|g_n\|_{C^0}$ which we will now compute[11].

[11]The reader may consider the following estimations as a (rather technical) exercise and compute the constant C by her or himself.

Let G be an upper bound for all $\|g_n\|_{C^0}$. Then:

$$\begin{aligned}
\omega_M(X, J_{g_n}X) &= C\omega_0 \oplus \omega(X_0 \oplus X_1, iX_0 \oplus (2g_n(iX_0) + JX_1)) \\
&= C\|X_0\|^2 + \omega(X_1, JX_1) + 2\omega(X_1, g_n(iX_0)) \\
&\geq C\|X_0\|^2 + \alpha\|X_1\|^2 + 2\omega(X_1, g_n(iX_0)) \\
&\geq C\|X_0\|^2 + \alpha\|X_1\|^2 - 2\beta G\|X_0\|\,\|X_1\|
\end{aligned}$$

because $|\omega(X_1, g_n(iX_o))| \leq \beta\|X_1\|\,\|g_n(iX_0)\| \leq \beta\|X_1\|\,\|g_n\|_{C^0}\|X_0\|$. Thus, obviously,

$$\omega_M(X, J_{g_n}X) \geq \frac{\alpha}{2}\|X\|^2_{\mu_M}$$

if C is sufficiently large. Actually, if

$$\frac{\|X_1\|}{\|X_0\|} \leq \frac{4\beta G}{\alpha},$$

then:

$$\begin{aligned}
\omega_M(X, J_{g_n}X) &\geq \left(C - 2\beta G\frac{\|X_1\|}{\|X_0\|}\right)\|X_0\|^2 + \alpha\|X_1\|^2 \\
&\geq \left(C - \frac{8\beta^2G^2}{\alpha}\right)\|X_0\|^2 + \alpha\|X_1\|^2 \\
&\geq \frac{\alpha}{2}\|X\|^2_{\mu_M}
\end{aligned}$$

with

$$C = \frac{8\beta^2G^2}{\alpha} + \frac{\alpha}{2}.$$

When

$$\frac{\|X_1\|}{\|X_0\|} \geq \frac{4\beta G}{\alpha},$$

one gets

$$\begin{aligned}
\omega_M(X, J_{g_n}X) &\geq \left(\alpha - 2\beta G\frac{\|X_0\|}{\|X_1\|}\right)\|X_1\|^2 + C\|X_0\|^2 \\
&\geq \frac{\alpha}{2}\|X_1\|^2 + C\|X_0\|^2 \geq \frac{\alpha}{2}\|X\|^2_{\mu_M}.
\end{aligned}$$

Therefore, $M = D^2 \times V$ is geometrically bounded with $\mu_M = \mu_0 \oplus \mu_1$, $\omega_M = C\omega_0 \oplus \omega$ where

$$C = \frac{8\beta^2G^2}{\alpha} + \frac{\alpha}{2},$$

and with any of the given almost complex structures J_{g_n} or J_g.

Since every $f_n : D^2 \to V$ is contractible in $\pi_2(V, L)$, all discs $\mathrm{Id} \times f_n$ belong to the same homotopy class in $\pi_2(D^2 \times V, \partial D^2 \times L)$. Their ω_M-area is $\int_{D_2} C\omega_o + \int_{D^2} f_n^\star \omega = C\pi$ for all n, and their μ_M-area is uniformly bounded above. Writing $\rho = \mathrm{Id} \times f_n$,

$$\begin{aligned}
\mu_M\text{-area}(\rho) &= \int_{D^2} \left\| \frac{\partial \rho}{\partial X} \wedge \frac{\partial \rho}{\partial Y} \right\|_{\mu_M} \\
&\leq \frac{1}{2} \int_{D^2} \left(\left\| \frac{\partial \rho}{\partial X} \right\|_{\mu_M}^2 + \left\| \frac{\partial \rho}{\partial Y} \right\|_{\mu_M}^2 \right) \\
&\leq \frac{1}{\alpha} \int_{D^2} \left(\omega_M \left(\frac{\partial \rho}{\partial X}, J_{g_n} \frac{\partial \rho}{\partial X} \right) + \omega_M \left(\frac{\partial \rho}{\partial Y}, J_{g_n} \frac{\partial \rho}{\partial Y} \right) \right) \\
&\leq \frac{2}{\alpha} \int_{D^2} \rho^\star \omega_M
\end{aligned}$$

because ρ is J_{g_n}-holomorphic. Hence:

$$\mu_M\text{-area}(\mathrm{Id} \times f_n) \leq \frac{2C\pi}{\alpha}.$$

The compactness theorem thus applies to the C^{r+1} divergent sequence $\mathrm{Id} \times f_n$ and therefore there exists a J_g-cusp-disc whose components contain at least one non-constant J_g-holomorphic sphere S or two non constant J_g-holomorphic discs D with boundary in $\partial D^2 \times L$. But since this cusp-disc is a weak limit of sections of $D^2 \times V \to D^2$, S cannot be transverse to the fibres of the projection $D^2 \times V \to D^2$ and the same is true for D; that is to say, S (or D) lies entirely in a fibre of the projection. There exists therefore either a non-constant J-rational curve in V or a non-constant J-holomorphic disc in V whose boundary lies in L. This concludes the proof of theorem 5.1.2. □

5.3. Proof of theorem 2.2.4

Let (V, ω) be a product $(V' \times \mathbf{C}, \omega' \oplus \omega_0)$ where (V', ω') is a weakly exact geometrically bounded symplectic manifold. Let L be a g. bounded Lagrangian submanifold of V. If L is bounded with respect to the second factor (that is if the image of L by $pr_2 : V' \times \mathbf{C} \to \mathbf{C}$ is bounded), we show that there exists a loop in L that bounds a holomorphic disc (with respect to an ω-tamed almost complex structure).

First, $(V' \times \mathbf{C}, \omega' \oplus \omega_0)$ is also geometrically bounded. It is easy to see that, given any totally real submanifold L in $V' \times \mathbf{C}$ which is bounded with respect to $\mathbf{C}$, the equation $\overline{\partial}_J f = g$ has no solution when the factor g_2 of $g = g_1 \oplus g_2$ is equal to a constant c of sufficiently large norm. In fact, any solution f to the equation $\overline{\partial}_J f = g_1 \oplus c$ satisfies $\overline{\partial}(pr_2 o f) = g_2 = c$ and therefore $f_2 = pr_2 \circ f : D^2 \to \mathbf{C}$ is harmonic. The Poisson formula then gives

$$\overline{\partial}_J f_2(0) = -\frac{1}{2\pi} \int_0^{2\pi} e^{i\theta} f_2(e^{i\theta}) d\theta;$$

so $|c| = \left|\overline{\partial}_J f_2(0)\right| \leq \sup_{0 \leq \theta \leq 2\pi} \left|f_2(e^{i\theta})\right| \leq \mathrm{diam} f_2(\partial D^2) \leq \mathrm{diam}(pr_2(L))$.

Thus, there is no solution to $\overline{\partial}_J f = g_1 \oplus c$ when $|c| > \operatorname{diam}(pr_2(L))$. On the other hand, if the totally real submanifold L is Lagrangian and g. bounded, the Fredholm alternative of 5.1.2 shows that if $\overline{\partial}_J f = g$ has no solution for some g, there must exist either a holomorphic sphere or a holomorphic disc with boundary on L. Since V is weakly exact, only the second case can occur.

To prove the second part of theorem 2.2.4, let us quickly review the proof above from a slightly more general point of view. For each class $\alpha \in \pi_2(V, L)$, define $F_\alpha^{r+1}, H_\alpha^{r+1}$ and Δ_α^r as follows:

- F_α^{r+1} is the set of C^{r+1} Hölder maps $f : (D^2, \partial D^2) \to (V, L)$ such that $[f] = \alpha$,
- $H_\alpha^{r+1} = \{(f, g) \in F_\alpha^{r+1} \times G^r$ such that $\overline{\partial}_J f = g\}$,
- $\Delta_\alpha^r(f, g) = g$.

Note that since V is weakly exact, the symplectic area of $\alpha \in \pi_2(V, L)$ depends only on its image by $\pi_2(V, L) \xrightarrow{\partial} \pi_1(L) \xrightarrow{H} H_1(L; \mathbf{Z})$ where ∂ is the boundary homomorphism and H the Hurewicz map. Observe also that the Maslov index of L along $H\partial(\alpha)$ relative to the Lagrangian distribution $\mathcal{L}$ in V is independent of the choice of $\mathcal{L}$ because $H\partial(\alpha)$ is contractible in V (so we need only know that such a distribution exists). We will refer to this Maslov index as $\mu_L(H\partial\alpha)$ or $\mu_L(\alpha)$ indifferently.

Here, F_α^{r+1}, H_α^{r+1} are again Banach manifolds and $\Delta_\alpha^r : H_\alpha^{r+1} \to G^r$ is a Fredholm map of index $\operatorname{ind}(\Delta_\alpha^r) = \mu_L(\alpha) + n$.

The Poisson formula argument showed that there exists some value $g \in G^r$ such that there is no solution to the equation $\overline{\partial}_J f = g$ with $f \in F_0^{r+1}$. Let γ be a path in G^r from $\gamma(0) = 0$ to $\gamma(1) = g$ transverse to the Fredholm projections $\Delta_\alpha^r : H_\alpha^{r+1} \to G^r$ for all α. The proof above shows that the restriction of Δ^r to $(\Delta^r)^{-1}(\gamma)$ is not proper. Therefore, there exists a sequence $g^t \in \operatorname{Im}\gamma$ converging to $g_\infty \in \operatorname{Im}\gamma$ and a sequence f_t with $\overline{\partial}_J f_t = g_t$ such that the sequence $\{f_t\}$ has no convergent subsequence. By the compactness theorem and lemma 5.2.3, this implies (extracting a subsequence of f_t if needed) that the sequence f_t weakly converges to a union of k_1 non-constant rational curves in V with one disc f_∞ with boundary in L such that $\overline{\partial}_J f_\infty = g_\infty$, and $k_2 - 1$ non-constant holomorphic discs with boundary in L. Since V is weakly exact, we have therefore one disc f_∞ and $k = k_2 - 1 > 0$ holomorphic discs $f_1, \ldots, f_k$ with boundary in L. Denote by $\alpha_\infty, \alpha_1, \ldots, \alpha_k \in \pi_2(V, L)$ the homotopy classes of $f_\infty, f_1, \ldots, f_k$.

5.3.1. Exercise. — Using the index formula, show that the fact that γ is transversal to $\Delta_{\alpha_\infty}^r$ and the fact that $\left(\Delta_{\alpha_\infty}^r\right)^{-1}(g_\infty)$ is not empty together imply that $\mu_L(\alpha_\infty) \geq -n - 1$.

Since the discs $f_\infty, f_1, \ldots, f_k$ occur as a weak limit of a contractible disc, we have

$$\alpha_\infty + \alpha_1 + \cdots + \alpha_k = 0,$$

so that $\mu_L(\alpha_\infty)+\sum_{i=1}^k \mu_L(\alpha_i) = 0$. But each disc f_i is holomorphic and non-constant, so their symplectic area $\lambda(\alpha_i)$ is strictly positive, and by monotonicity $\mu_L(\alpha_i) > 0$ for all $1 \le i \le k$. Therefore

$$\sum_{i=1}^{k} \mu_L(\alpha_i) \le n+1$$

implies that $1 \le \mu_L(\alpha_i) \le n+1$ for at least one i. This concludes the proof. □

Appendix: Exotic structures on $\mathbf{R}^{2n}$

A.1. Exotic structures

A symplectic form ω on $\mathbf{R}^{2n}$ is *exotic* if there exist *no* symplectic embedding $(\mathbf{R}^{2n}, \omega) \hookrightarrow (\mathbf{R}^{2n}, \omega_0)$ in the standard structure. The fact that the proof of the Arnold conjecture (corollary 2.2.5) implies the existence of exotic structures on $\mathbf{R}^{2n}$ for $n \ge 2$ is now considered to be folklore. Nevertheless, it appeared in the written culture only in 1984 in a paper by Viterbo [48]. It is very surprising that we do not know any other method of constructing exotic structures. There are some variants (see e.g. [11]), but basically, you construct a symplectic structure which admits a closed exact Lagrangian submanifold. We already mentioned such a construction, due to M.-P. Muller in § 3.2. Here we content ourselves[12] with a simple proof of:

THEOREM A.1.1. — *For any $n \ge 2$, there exist on $\mathbf{R}^{2n}$ exotic symplectic structures.*

A.2. Proof of existence

It is enough to construct an immersion $F : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ that sends a given closed Lagrangian submanifold L of $(\mathbf{R}^{2n}, \omega_0)$ onto an immersed exact Lagrangian submanifold W of $(\mathbf{R}^{2n}, \omega_0)$. Indeed, $F^\star\omega_0$ is then such that L is a closed exact Lagrangian submanifold of $(\mathbf{R}^{2n}, F^\star\omega_0)$. Hence $(\mathbf{R}^{2n}, F^\star\omega_0)$ is exotic because a symplectic embedding

$$\varphi : (\mathbf{R}^{2n}, F^\star\omega_0) \to (\mathbf{R}^{2n}, \omega_0)$$

would send a primitive λ of $F^\star\omega_0$ (a form whose exterior derivative is $F^\star\omega_0$) to a form $\varphi^\star(\lambda)$ on Im φ whose derivative is ω_0. We would then have $d(\lambda_0 - \varphi^\star(\lambda)) = 0$ on Im φ and $\lambda_0 = \varphi^\star(\lambda)+$ exact form on Im φ, so that$\lambda_0|_{\varphi(L)}$ would be exact which contradicts the inexistence of a closed exact Lagrangian submanifold of $(\mathbf{R}^{2n}, \omega_0)$.

To construct the immersion $F : (\mathbf{R}^{2n}, L) \to (\mathbf{R}^{2n}, W)$, we take the simplest example of a closed Lagrangian submanifold, that is $L = T^n \subset \mathbf{R}^{2n}$. Let $f_t : S^1 \to$

[12]The construction in [37] is actually harder... but gives something more precise, as the title of the paper shows.

$\mathbf{R}^2$ be a regular homotopy from the standard inclusion to an exact (Lagrangian) immersion $f_1 : S^1 \to \mathbf{R}^2$ (that is f_1 has for instance two double points of opposite signs and the total area enclosed by Im f_1 is algebraically zero). The product of n copies of this regular homotopy gives a regular homotopy $F_t : T^n \to \mathbf{R}^{2n}$. Now, given any closed smooth submanifold X of a closed manifold Y and any manifold Z of dimension larger than dim Y, Smale's h-principle for immersions (see e.g. [25]) implies that the restriction map

$$r : \mathrm{Imm}(Y, Z) \to \mathrm{Imm}(X, Z)$$

between spaces of smooth immersions is a Serre fibration: that is, given any homotopy $H_t : A \to \mathrm{Imm}(X, Z)$ defined on a CW-complex A and any r-lift $\widetilde{H_0} : A \to \mathrm{Imm}(Y, Z)$ of H_0, there exists an r-lift of the whole homotopy, $\widetilde{H_t} : A \to \mathrm{Imm}(Y, Z)$, which coincides with the given lift at time $t = 0$. Let us apply this with $X = T^n$, $Y = S^{2n-1}$, $Z = \mathbf{R}^{2n}$ (note that $n < 2n-1$ because $n \geq 2$), $A = \{pt\}$, $H_t = F_t : T^n \to \mathbf{R}^{2n}$, and $\widetilde{H_0}$ the standard inclusion. Hence F_t can be extended to a regular homotopy $\widetilde{F_t} : S^{2n-1} \to \mathbf{R}^{2n}$ and so $\widetilde{F_1}$ extends F_1 (this is all we need). Let D^{2n-1} be a disc smoothly embedded in S^{2n-1}, such that T^n lies in D^{2n-1} (such a disc exists because $n < 2n-1$), and let U be a tubular neighbourhood of D^{2n-1} in $\mathbf{R}^{2n}$. Then the restriction $\widetilde{F_1} : D^{2n-1} \to \mathbf{R}^{2n}$ of $\widetilde{F_1} : S^{2n-1} \to \mathbf{R}^{2n}$ can be extended to an immersion $G : U \to \mathbf{R}^{2n}$. The composition of G with an immersion $\varphi : \mathbf{R}^{2n} \to U$ which is the identity on a smaller disc $D_1^{2n-1} \subset D^{2n-1}$ containing T^n, gives an immersion $G \circ \varphi : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ sending T^n onto an immersed exact Lagrangian submanifold of $(\mathbf{R}^{2n}, \omega_0)$.

Remark. — Note that by theorem 2.2.4, the exotic structure ω on $\mathbf{C}^n$ constructed in the above proof is not symplectomorphic to $(\mathbf{C}^{n-1} \times \mathbf{C}, \omega' \oplus \omega_0)$ for any geometrically bounded factor $(\mathbf{C}^{n-1}, \omega')$.

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