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to the memory of Jean Martinet

Preface

This book comes from a course I gave in Strasbourg in 1988–89. In the audience were in particular Julien Duval, Santiago Lopez de Medrano and Marcus Slupinski who helped me by their questions to understand many of the points I was supposed to be explaining. There were genuine students as well who, refusing to understand what I was badly explaining, suggested many improvements. I am thinking in particular of J. Fougeront, P. Gaucher, Li Jie and J.-M. Rinkel.

I learned a lot in discussions with Michel Brion, mainly about "Duistermaat-Heckman with singularities" and Jean-Yves Mérindol, who taught me in particular the Inoue surfaces. The reader will probably think that I said almost nothing about Lie groups and algebras, she or he must know that without Nicole Bopp and Hubert Rubenthaler I would have said even less. I thank them all.

The original version of this book was published in French by the *Institut de Recherche Mathématique Avancée*. The English version the reader has in hand is not very different: I corrected some of the misprints in the mathematics, I updated the bibliographical references, I made the presence of the exercises more evident by numbering them (thanks to the automatic system of cross references in LaTeX), I tried to suppress those of the bad jokes which were untranslatable, and I replaced a short discussion about "Duistermaat-Heckman with singularities" with a whole section in chapter V. I thank all the people who pointed out mistakes in the French version.

I thank also the Editors, who kindly accepted the book in this series and gave me good advice on the language problem, then I must thank again Marcus Slupinski who had the very difficult job of making my English readable. If there are still some gallicisms, I am the only responsible *(sic)*... and after all there are anglicisms in almost any French mathematical text!

In addition to the fact that I had to understand all the proofs and to write them down, I also typed both the French and English versions myself. I obviously cannot thank me, but I want to thank J.-M. Bony and C. Sabbah who made my work easier by their help in the use of LaTeX.

M. A.
December 7, 1990

Contents

Introduction

At least since the Felix Klein[1] Erlangen program, everybody knows that it is useful to make groups act on sets, thus providing information both on the group acting and on the set acted on.

In this book, we shall be interested in "differentiable" (*i.e.* locally linear) objects: the set will be a smooth manifold and the group a Lie group, acting by diffeomorphisms.

The group action allows one then to decompose the manifold into pieces (strata) corresponding to the various orbit types. When the manifold is not too complicated, it is possible to recover "everything" from this stratification: it is the case for example when the group is the circle S^1 and the manifold a surface ... but there are rather few examples of such a simple situation.

The two principal motivations to investigate group actions on *symplectic* manifolds might be:

1. It is a natural (?) framework for Hamilton mechanics: ever since Lagrange, we are able to consider the phase space of any mechanical system as a symplectic manifold, the group then represents the more or less hidden symmetries of the system[2].

2. From the point of view of the group itself, what is called the "orbits method" (invented by B. Kostant, etc...) and which is a tool to construct representations of the group, uses symplectic geometry in an essential way. The fundamental objects in the theory are the coadjoint orbits, which are naturally symplectic manifolds.

There is another good reason to investigate symplectic actions: it is *easier!* Actually the so-called *hamiltonian* actions are, by the very definition, those for which there is one (or many) *functions* on the manifold, the critical points of which correspond to the fixed points of the group action. It is thus possible to use the well-understood methods (invented by Morse, Thom, and many others) called *Morse theory* to study the group action.

For example, the central theorem in the first part of this book is the following statement (that any neophyte is allowed to find abstruse) due to Frankel [36] and Atiyah [18]:

[1]Always begin by citing a great mathematician.

[2]A good reference for this point of view is the book by J.-M. Souriau [12].

Theorem III-3.2.1 *Let X be an almost periodic vector field on a connected symplectic manifold (W, ω). Suppose H is a function $W \to \mathbf{R}$ such that $i_X\omega = dH$, then all the levels $H^{-1}(t)$ are (empty or) connected.*

For those who felt the statement actually was abstruse, here are two applications which should be clear.

In the former we enumerate the solutions of a particular system of algebraic equations. Consider a finite subset $S \subset \mathbf{Z}^n$ of multi-exponents, and the system of n equations and n unknowns:

$$\sum_{\alpha \in S} c_\alpha^j z^\alpha = 0 \qquad (1)$$

where $1 \leq j \leq n$, the parameters c_α^j being complex and the unknown z an element of the complex torus $(\mathbf{C}^\star)^n$.

For example,

- If $S = \{e_1, \ldots, e_n\}$ is the set of the vectors in the canonical basis of $\mathbf{Z}^n$, then (1) is a homogeneous linear system and has in general no solution (in $(\mathbf{C}^\star)^n$).
- If we add 0 (then $S = \{e_1, \ldots, e_n, 0\}$), it is a linear system and has in general one solution.

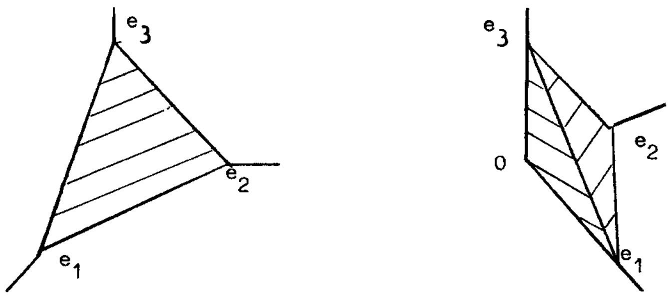

Figure 1

The following theorem, due to A. G. Kushnirenko [51], generalises the two previous examples:

Theorem III-4.4.1 *The number of solutions of (1) for general enough coefficients c_α^j is*

$$N(S) = n! Vol(\hat{S})$$

where $\hat{S}$ is the convex hull of S in $\mathbf{R}^n$.

The latter application[3] of III-3.2.1 looks rather different:

[3]I learned this theorem in V. A. Ginsburg's paper [37] where it is attributed to Hausdorff.

Theorem III-4.1.1 *Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{R}^n$ and let $\mathcal{H}_\lambda$ be the set of all order n complex hermitian matrices whose spectrum is λ. Then the image of*

$$\begin{array}{rccc} f_A : & \mathcal{H}_\lambda & \longrightarrow & \mathbf{C} \\ & X & \longmapsto & \operatorname{tr}(AX) \end{array}$$

(where A is any order n matrix) is a convex subset of $\mathbf{C}$.

For instance,

- If A itself is hermitian, $\operatorname{tr}(AX)$ is real and the image is an interval in $\mathbf{R} \subset \mathbf{C}$.
- If $n = 2$, $\lambda = (1, 0)$ and $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, the image is a disc in $\mathbf{C}$.

Although I did my best to present two rather different applications of III-3.2.1, I cannot hide that they have two important common points:

- in their hypotheses: the manifolds $(\mathbf{C}^\star)^n$ and $\mathcal{H}_\lambda$ are both endowed with an action of the torus T^n.
- in their conclusions: both contain the word "convex": actually, theorem III-3.2.1 is the essential step in the proof of the so-called *Atiyah, Guillemin and Sternberg convexity theorem.*

Our abstruse statement, in addition to giving unified proofs of previously different theorems, allows us to classify certain group actions on symplectic manifolds. A rather important part of this book will be devoted to show that we know essentially all the compact symplectic 4-manifolds on which a Lie group is acting, preserving the symplectic form.

As its title says, the object of this book is the study of some of the *topological* methods we use in symplectic geometry, but it is not a course in symplectic geometry, the market in this domain being full (*see* the references at the end).

As the reader may have understood, the theorems in chapter III are Morse theoretical results. Morse theory will also be used in an essential and rather nice way (following an idea of Dusa McDuff) in chapter IV together with the classical techniques of 3 and 4 dimensional topology (Seifert manifolds, investigated in chapter I, plumbing).

Up to this point, algebraic topology is used only occasionally. In chapter V, it will take up more room: it happens that the language of equivariant cohomology fits very well with the study of hamiltonian actions. Actually the existence of a moment map for the G-action on the manifold W is equivalent to the existence of an extension of the symplectic form to the Borel construction W_G. This remark allowed N. Berline and M. Vergne [22] to give a lucid proof of a rather spectacular theorem of Duistermaat and Heckman [34].

I shall not resist presenting this perfect example of a theorem which becomes practically tautological once the right language to state it in is found.

Chapter VI is a kind of appendix: I shall present a rather topological construction of the complex toric varieties which we shall see is what we were flirting with in the previous chapters. I understood this construction while reading the book [46] by F. Kirwan and the thesis of T. Delzant [31] and comparing with what I was able to understand in the reference paper by Danilov [30].

Before this, chapters I and II will have been devoted to laying the foundations and to introducing the necessary tools: generalities and examples about smooth actions with mainly the slice theorem and the classification of Seifert manifolds in the former, and (even!) some symplectic geometry in the latter.

A few remarks on prerequisites and on the material organisation of the book:

1. Prerequisites:

   - Smooth manifolds, definition, first examples, tangent spaces, bundles and maps, vector fields. I recommend Spivak's book [1] and, as a good general reference for all these notes, paragraph 5 in the (less heavy) book by Kirillov [10].
   - Classification of surfaces: there are a lot of good books, among which [5] and [4]; to stay in the spirit of the present book, I recommend that of Gramain [3], as it might also be used as an introduction to Morse theoretical methods.
   - Very little about Lie groups and algebras, essentially the paragraph 6 in [10].
   - As to algebraic topology: the reader is supposed to learn a few elementary notions as she or he goes along, being assumed to know nothing at the beginning (for instance in the proof of 2.2.4 in the first chapter I did not even use the words "exact homotopy sequence of the fibration" when I used that exact sequence) and a little more at the end. There will always be a reference or a hint of proof for what I shall use, but not always a complete proof.

2. Material organisation:

   - Exercises: some of the most interesting examples are presented in exercises, as are some easy, very classical or straightforward results. These results are then used in the text and it therefore goes without saying that the reader is supposed to solve them (or to make sure that she or he can solve them!).
   - Cross-references: references of type [?] refer (!) to the bibliography. Those of type (??) to an equation with a number in the same chapter. Figures, paragraphs and statements are numbered in each chapter with arabic digits and (sometimes) latin capitals. On the contrary, chapters are numbered with roman digits, any reference of the same type as above but beginning with a roman digit thus refering to the chapter in question.

Examples.

- In chapter I, reference 2.2.4 denotes the proposition with this number in this chapter.
- In the very same chapter I, reference IV-A would refer to appendix A of chapter IV, which appendix, mentioned in chapter IV, will only be called A.

Chapter I

Smooth Lie group actions on manifolds

1 Generalities

There is no question of our rewriting here in great detail things that can be found in many excellent books (I am thinking in particular of [7] and [6]), but we shall make a list of the basic results we want to use in the text, with bibliographical hints.

1.1 Notation

Let G denote a Lie group the unit element of which is 1 and Lie algebra $\mathfrak{g}$, W a smooth (*i.e.* C^∞) manifold on which G acts by diffeomorphisms. The action is written

$$\begin{array}{ccc} G \times W & \longrightarrow & W \\ (g, x) & \longmapsto & g \cdot x \end{array}$$

(it is a left action).

We shall sometimes say that W is a G-manifold. Similarly, we shall call G-vector bundle over a G-manifold W any vector bundle $E \longrightarrow W$ endowed with a G-action which is linear in the fibres and such that the projection is an equivariant map... where: if G acts in W and V, and if φ is a smooth map

$$\varphi : W \longrightarrow V$$

we say that φ is an *equivariant* map if

$$\forall x \in W, \forall g \in G, \varphi(g \cdot x) = g \cdot \varphi(x).$$

If x is a point in W, its orbit will be denoted $G \cdot x$, and its stabilizer G_x. Stabilizers of points in the same orbit are conjugate (and all possible conjugates appear). With each orbit is associated a conjugation class of subgroups of G, called the *type* of the orbit. The conjugation class of H is denoted by (H). Thus the type of $G \cdot x$ is (G_x).

1.2 Orbits and fundamental vector fields

For each $x \in W$,

$$\begin{array}{ccc} G & \longrightarrow & W \\ g & \stackrel{f_x}{\longmapsto} & g \cdot x \end{array}$$

is a smooth map, the *orbit map*. Its differential at 1 is a map:

$$\mathsf{T}_1 f_x : \mathsf{T}_1 G = \mathfrak{g} \to \mathsf{T}_x W.$$

With each $X \in \mathfrak{g}$, is associated a vector $\mathsf{T}_1 f_x(X) = \underline{X}_x \in \mathsf{T}_x W$ tangent to W at x. When x varies in W, we get a vector field, the *fundamental field* associated with X, denoted by ${}^W\underline{X}$, $\underline{X}$, or X when there is no risk of confusion. By definition, the flow of $\underline{X}$ is $\exp(tX) \cdot x$. Notice that the Lie bracket is related to the bracket of vector fields by the simple formula

(1) $$\underline{[X,Y]} = [\underline{X}, \underline{Y}].$$

The image of f_x is the orbit of x, the stabilizer of x is a closed subgroup of G, the quotient is thus a smooth manifold and one shows

Theorem 1.2.1 *The orbit map*

$$f_x : G/G_x \longrightarrow W$$

is an injective immersion.

To prove this, evaluate the kernel of

$$\mathsf{T}_g f_x : \mathsf{T}_g G \longrightarrow \mathsf{T}_{g \cdot x} W.$$

By invariance, it is sufficient to study the case where $g = 1$. Thus look at the kernel of $\mathsf{T}_1 f_x$, that is $\{X \in \mathfrak{g} \mid \underline{X}_x = 0\}$. It is easily checked that this set is the Lie algebra $\mathfrak{g}_x$ of G_x. □

Thus orbits are images of manifolds by injective immersions. It does not follow that they are submanifolds.

Exercise 1.2.2 Fix a real number α and let $\mathbf{R}$ act on the torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ by

$$t \cdot (x, y) = (x + t, y + \alpha t).$$

Find the orbits and show that these are submanifolds of T^2 if (and only if) α is rational (hint: if α is irrational, orbits are dense).

Recall that a *proper* injective immersion is an embedding, in particular the above problems do not occur when the group is compact.

Corollary 1.2.3 *If G is a compact Lie group, its orbits are submanifolds of W.* □

1.3 Examples

Example 0. $GL(n, \mathbf{R})$ acts on $\mathbf{R}^n$.

In general the groups under consideration in the present text will be compact (from time to time, we shall nevertheless need to use $\mathbf{C}^*$). Very often, the group will even be commutative.

Example 1. The circle S^1 acts on $\mathbf{C}^n$ by complex multiplication

$$t \cdot (z_1, \ldots, z_n) = (tz_1, \ldots, tz_n).$$

The point 0 is fixed; other orbits are circles. The spheres

$$S^{2n-1} = \left\{(z_1, \ldots, z_n) \mid \sum |z_i|^2 = 1\right\}$$

are stabilised by this action. This apparently trivial example is the fundamental one in the book!

Example 2. S^1 also acts on $S^3 \subset \mathbf{C}^2$ (for instance) in a more sophisticated way

$$t \cdot (z_1, z_2) = (t^{m_1} z_1, t^{m_2} z_2)$$

for all $m_1, m_2 \in \mathbf{Z}$.

Exercise 1.3.1 Find the orbits (they depend on m_1 and m_2).

Exercise 1.3.2 Imagine "analogous"(?) actions of the torus T^p in $\mathbf{C}^n$.

Example 3. The unitary group $U(n)$ acts on hermitian $n \times n$-matrices by conjugation

$$A \cdot M = AMA^{-1}.$$

Exercise 1.3.3 The orbits of this action are the manifolds $\mathcal{H}_\lambda$ mentioned in the introduction. Given a vector $\lambda \in \mathbf{R}^n$ what is the stabilizer of $\mathcal{H}_\lambda$? (hint: if $\lambda_1, \ldots, \lambda_n$ are all distinct, the stabilizer of the diagonal matrix is the torus T^n of all *diagonal* unitary matrices.)

1.4 More definitions

Definition 1.4.1 *The G-action is said to be effective if each element $g \neq 1$ moves at least one x in W, that is to say:*

$$\bigcap_{x \in W} G_x = \{1\}$$

Any action may be replaced by an effective one:

Proposition 1.4.2 *$\bigcap G_x$ is a closed normal subgroup in G, and the G-action in W induces an effective action of $G/\bigcap G_x$ in W.*

The proof is left as an exercise. □

Example 1. Let S^1 act in $\mathbf{C}$ by $t \cdot w = t^2 w$. This is not an effective action, the subgroup $\{\pm 1\}$ fixing all points. The quotient $S^1/\{\pm 1\}$ acts effectively.

Example 2. The S^1-action on S^3 by

$$t \cdot (z_1, z_2) = (t^{m_1} z_1, t^{m_2} z_2)$$

is effective if and only if m_1 and m_2 are relatively prime (exercise).

Example 3. If G is *simple*, the action is either effective or trivial. This is the case for instance for $G = SO(3)$.

Here are some more definitions:

Definition 1.4.3 *The action is transitive if it has only one orbit, free if all orbits have $\{1\}$ as stabilizer, semifree if they have G or $\{1\}$ as stabilizer (in other words if it is free outside fixed points).*

Exercise 1.4.4 The S^1-action on $\mathbf{C}^2$ is semifree. When is the S^1-action on S^3 by $(t^{m_1} z_1, t^{m_2} z_2)$ free?

1.5 Equivariant maps and orbit spaces

Let us begin with some properties of equivariant maps:

Exercises 1.5.1

1. If φ is a diffeomorphism and an equivariant map, then φ^{-1} is a diffeomorphism and is equivariant.
2. $G_x \subset G_{\varphi(x)}$.
3. If H et K are closed subgroups of G, in order that there exists an equivariant map
$$\varphi : G/H \longrightarrow G/K$$
it is necessary and sufficient that H is conjugate to a subgroup of K.

The last notion in this paragraph is that of orbit space: it is the quotient space W/G endowed with the quotient topology. In general, it is not even Hausdorff (*cf* the example of the irrational flow on the torus T^2). Anyway:

Proposition 1.5.2 *If G is compact, W/G is a Hausdorff space and the projection $W \longrightarrow W/G$ is proper and closed.*

The proof is left as an exercise. □

2 Equivariant tubular neighborhoods and orbit types decomposition

2.1 The slice theorem (equivariant tubular neighborhood)

Although it is not the most general possible hypothesis, assume that G is compact. In this case, its orbits are submanifolds of W. We now describe W in the neighborhood of an orbit.

Let $x \in W$. Denote V_x the vector space $\mathsf{T}_x W/\mathsf{T}_x(G\cdot x)$. If $g \in G_x$, the tangent map:

$$\mathsf{T}_x g : \mathsf{T}_x W \longrightarrow \mathsf{T}_{g\cdot x} W = \mathsf{T}_x W$$

is an isomorphism, sending the tangent space of the orbit into itself (by the identity map), and in particular it induces a linear isomorphism from V_x to itself. Thus, with each point x in W, is associated a *linear representation* of its stabilizer, *i.e.* a group homomorphism:

$$G_x \longrightarrow GL(V_x).$$

Theorem 2.1.1 (the "slice theorem"[50]) *There exists an equivariant diffeomorphism from an equivariant open neighborhood of the zero section in $G \times_{G_x} V_x$ to an open neighborhood of $G\cdot x$ in W, which sends the zero section G/G_x on the orbit $G\cdot x$ by the natural map f_x.*

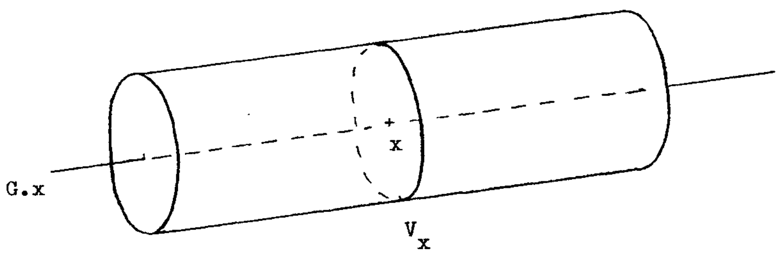

Figure 1

Explanation of notations. $G \times V_x$ is acted on by G_x (by multiplication on G and by the linear representation just described on V_x). More generally, if H is a closed subgroup of G, and if V is a vector space endowed with an H-linear action, there is a *free* action of H on $G \times V$ by:

$$h \cdot (g, v) = (gh^{-1}, h \cdot v).$$

The quotient is denoted $G \times_H V$, and $[g, v]$ represents the equivalence class of (g, v). This is a vector bundle on G/H with fiber V.

$$\begin{array}{ccc} G \times V & \longrightarrow & G \\ \downarrow & & \downarrow \\ G \times_H V & \longrightarrow & G/H \end{array}$$

and it is endowed with a G-action by

$$g' \cdot [g, v] = [g'g, v].$$

We may consider G/H as a submanifold in $G \times_H V$, the zero section of the vector bundle, $\{[g, 0] \mid g \in G\}$.

Theorem 2.1.1 says that there exists an extension $\bar{f}_x$ of the orbit map :

$$\begin{array}{ccc} G/G_x & \subset & G \times_{G_x} V_x \\ \big\downarrow f_x & & \big\downarrow \bar{f}_x \\ G \cdot x & \subset & W \end{array}$$

at least to a neighborhood of the zero section.

Sketch of a proof of theorem 2.1.1. In the non equivariant case, to prove the existence of a tubular neighborhood of a compact submanifold $Y \subset W$, one uses the *exponential map* of a *riemannian metric*[1] on W which induces a diffeomorphism from a suitable neighborhood of the zero section in the normal bundle of Y onto a neighborhood of Y in W (*see* [1] for instance).

To adapt this proof in the equivariant case, it suffices to choose the riemannian metric in such a way that it is G-invariant (*i.e.* G acts on W by isometries), thus making it possible to identify V_x to the orthogonal complement of $\mathrm{T}_x(G \cdot x)$.

To be convinced of the existence of an invariant metric:

1. G has a Haar measure (*i.e.* one invariant by left translations). As it is a Lie group, the proof of this fact is very easy: choose an n-linear alternating form[2] $\omega \in \Lambda^n(\mathfrak{g})^*$ on $\mathfrak{g} = \mathrm{T}_1 G$. Define an invariant differential form on G by $\omega_g = g \cdot \omega$. In particular it is a measure, it is moreover possible to normalise it, thanks to the compacity of G, multiplying ω by an *ad hoc* scalar such that $\int_G \omega_g = 1$.

[1]The reader is invited to have a look at her or his favourite differential geometry textbook to find a proof of the existence of such a metric; it goes without saying that here all manifolds are implicitely assumed to be paracompact (and Hausdorff)!

[2]n is the dimension of G.

2. Given a G-vector bundle $E \longrightarrow W$, we have a G-action on the sections of E by:
$$(g \cdot s)(x) = gs(g^{-1}x).$$
Given a section s, associate to it an equivariant section $\bar{s}$ by:
$$\bar{s}(x) = \int_G gs(g^{-1}x)\omega_g.$$
If E is the bundle of symmetric bilinear forms on the tangent bundle TW and if s is a riemannian metric, it is easily checked that the invariant section $\bar{s}$ is still a riemannian metric.

2.2 Applications

Proposition 2.2.1 *The union of all orbits of a given type is a submanifold of W.*

Proof. Fix a type, *i.e.* a conjugation class (H) of subgroups of G. Let $x \in W_{(H)} = \{x \in W \mid G_x \in (H)\}$. Let us prove that $W_{(H)}$ is a submanifold (local property!) in the neighborhood of the orbit of x. For this, look at the orbits of type (H) in $G \times_H V$.

If $g' \in G_{[g,v]}$, we have:
$$g' \cdot [g, v] = [g'g, v] \Leftrightarrow \exists h \in H \left\{ \begin{array}{rcl} g'g & = & gh^{-1} \\ v & = & h \cdot v \end{array} \right.$$
thus $G_{[g,v]} = gH_vg^{-1}$ is in the conjugation class of H_v and all conjugates do appear when g varies (and v is fixed).

The orbit of $[g, v]$ is of type (H), if and only if $H_v = H$, that is if and only if v is a fixed point of the H-action in V.

Let $F = \{v \in V \mid h \cdot v = v, \forall h \in H\}$ be the set of these fixed points. It is a vector subspace of V and
$$(G \times_H V) \bigcap W_{(H)} = \left\{[g, v] \in G \times_H V \mid G_{[g,v]} \in (H)\right\} = G \times_H F$$
is a subbundle of $G \times_H V$ and in particular a submanifold. □

The simplest example is that of type (G) orbits:

Corollary 2.2.2 *The set of fixed points of G is a submanifold of W.* □

Of course these submanifolds are not connected in general, and their components do not all have the same dimension.

Anyway the orbit types give a decomposition of W in submanifolds. This decomposition is *locally finite*:

Proposition 2.2.3 *If W is compact, there is only a finite number of orbit types.*

Proof. Let $n = \dim W$. If $n = 0$, the proposition is obviously true: W itself is a finite number of points. Suppose that the proposition is proved for G-manifolds of dimension $\leq n-1$. Thanks to the compacity of W, it is sufficient to show that each tube $N = G \times_H V$ contains only a finite number of orbit types.

Choose an H-invariant metric on the H-vector space V, call SV the unit sphere for that metric and write $SN = G \times_H SV$ (it is the *sphere bundle* of N). It is a G-manifold of dimension $n-1$, we can apply the induction hypothesis: in SN, there is only a finite number of orbit types.

Compare now the types in SN and in N: I claim that, as the H-action on V is *linear*, $\forall\lambda \neq 0$, the orbit type of $[g, \lambda v] \in N$ is the same as that of $[g, v] \in SN$ (exercise). Thus the types in N are: those in SN on the one hand, and that of the central orbit in G/H on the other hand. □

It is not only true that there are few orbit types, but that there is one which takes almost all the place:

Proposition 2.2.4 *If W/G is connected, there is an orbit type (H) in W for which $W_{(H)}$ is a dense open subset of W. Moreover, $W_{(H)}/G$ is a connected manifold.*

Definition 2.2.5 *Such orbits are called principal orbits.*

Proof. Assume G is connected and prove the result by induction on $n = \dim W$. If $n = 0$ and W/G is connected, there is only one orbit and the proposition is obvious. As previously, assume the result proved for G-manifolds of dimension $\leq n - 1$ and consider $N = G \times_H V$.

In order to apply the induction hypothesis to SN we would have to know that SN/G is connected. Nearly always, SN itself is connected (and thus *a fortiori* its quotient SN/G is): call π the projection of the vector bundle $N \longrightarrow G/H$; to connect two points a and b of SN by a path, choose a path from $\pi(a)$ to $\pi(b)$ in the connected orbit G/H and lift it to a path beginning at a in the boundary SN of the tube. The end b' of the lift is in the same fiber as b and to be able to connect it to b' (thus to a), it is sufficient that this fiber, which is a sphere in V, is connected.

The only possible problem appears in the case where the unit sphere in V has dimension 0. In this case $\dim V = 1$ and $SN \longrightarrow G/H$ is a twofold covering. If it not trivial, it is connected and we are done; if it is trivial, it means that H acts *trivially* on the "slice" $V = \mathbf{R}$ ... in which case all orbits contained in N are the same and we do not need the induction hypothesis to conclude that the proposition is true in N. In the other cases, the induction hypothesis applies to SN and there exists an orbit type (H) such that $(SN)_{(H)}$ is open and dense in SN. As in the proof of 2.2.3, it is easily seen that the same happens in N.

Thus the proposition is proved in each equivariant tube. Choose a locally finite covering of W by such tubes and conclude, with the help of the orbit space connectivity, that the type (H) is the same in all tubes. □

Example. If G is *commutative*, let $(H) = \{H\}$ be the type of the principal orbits. The subgroup H fixes all the points of an open dense subset of W, and thus fixes all the points of W. If the action is effective, principal orbits have type (1).

Definition 2.2.6 *If all orbits are principal, the action is called principal.*

Remarks.

- An orbit is principal if and only if the representation in V of its stabilizer is trivial.
- In this case, the quotient is a manifold.

Exercise 2.2.7 Which closed surfaces can be endowed with a *principal* S^1-action? Hints:

1. The quotient is also a circle S^1.
2. The equivariant tubular neighborhoods have the form $S^1 \times \mathbf{R}$.
3. It is possible to glue all these tubes, writing $S^1 = [0,1]/(0 \sim 1)$, and W is obtained from $S^1 \times [0,1]$ identifying $S^1 \times 0$ and $S^1 \times 1$ by an equivariant diffeomorphism φ.

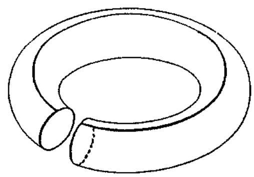

Figure 2

4. φ is a rotation $z \mapsto az$.
5. W is diffeomorphic, by an equivariant diffeomorphism, to $S^1 \times S^1$.

Remark. Principal orbits have maximal dimension among all orbits.

Definition 2.2.8 *An orbit is called exceptional if it is not principal but has the same dimension as principal orbits, singular if its dimension is strictly less.*

3 Examples: 2 and 3-dimensional S^1-manifolds

In this part, we shall give examples of classification theorems. The most interesting one is that of fixed point free S^1-actions on orientable 3-manifolds (Seifert manifolds), which will be used in the sequel to understand periodic hamiltonians on 4-manifolds. Both as an introduction and a preliminary exercise, consider first S^1-actions on surfaces.

3.1 S^1-actions on surfaces

Begin with a list of examples.

1. We have already seen S^1 acting on the torus $T^2 = S^1 \times S^1$ by multiplication on one summand: it is the unique principal S^1-action on a surface, as we already know.

2. S^1 also acts on the unit sphere S^2 of $\mathbf{R}^3$ as the group of rotations around a fixed axis. All orbits are principal, except for the two fixed points where the axis meets the sphere. The orbit space is a closed interval.

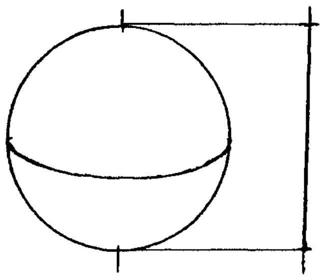

Figure 3

3. It also acts on the projective plane $\mathbf{P}^2(\mathbf{R})$. Here are three possible (but nevertheless equivalent) descriptions of that action:

   - If $\mathbf{P}^2(\mathbf{R})$ is the quotient of the sphere S^2 by the antipodic map $(x \mapsto -x)$, the action previously defined on the sphere descends to the quotient.
   - If $\mathbf{P}^2(\mathbf{R})$ is the "projective completion" of $\mathbf{R}^2$ (to which a line at infinity is added), then the linear $S^1 = SO(2)$-action on $\mathbf{R}^2$ may be extended to $\mathbf{P}^2(\mathbf{R})$.
   - If $\mathbf{P}^2(\mathbf{R})$ is the set of lines in $\mathbf{R}^3$ and is described by homogeneous coordinates $[x, y, z]$, then $S^1 = SO(2)$ acts by $A \cdot [x, y, z] = [A \cdot (x, y), z]$.

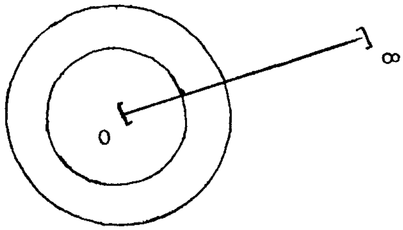

Figure 4

Exercise 3.1.1 Check that these three actions are the same.

In each of the given three descriptions[3] it is clear that all orbits are principal except for two of them. One is a fixed point and the other has a $\mathbf{Z}/2$ as stabilizer (it corresponds to the equatorial circle of S^2 for example). The orbit space is a closed interval, each of its ends corresponding to a non principal orbit.

4. S^1 also acts on the Klein bottle K. Write K as the quotient of $\mathbf{R}^2$ by the group of affine transformations generated by

$$\begin{array}{ccc} (x,y) & \overset{t}{\mapsto} & (x+1,y) \\ (x,y) & \overset{s}{\mapsto} & (-x,y+1) \end{array}$$

and make $\mathbf{R}$ act on $\mathbf{R}^2$ by $(x,y) \mapsto (x, y+a)$. It induces an action of $\mathbf{R}/\mathbf{Z} = S^1$ on K. The quotient map is identified with

$$\begin{array}{ccc} K & \longrightarrow & \mathbf{R}/(x \sim x+1, x \sim -x) \\ [x,y] & \longmapsto & [x] \end{array}$$

and this identifies the orbit space with $[0, 1/2]$.

All orbits are principal, except those corresponding to 0 and $1/2$ which have a $\mathbf{Z}/2$ as stabilizer.

Here is now a complete study, whose aim is of course to show that this list exhausts all possible examples. Assume W is a closed (*i.e.* compact and without boundary) connected surface.

The closed subgroups (and thus the possible orbit types) of S^1 are: $\{1\}, \mathbf{Z}/m$, and S^1.

[3]The reader wanting to know more about the projective plane is kindly requested to have a look at the nice book [2] by F. Apéry.

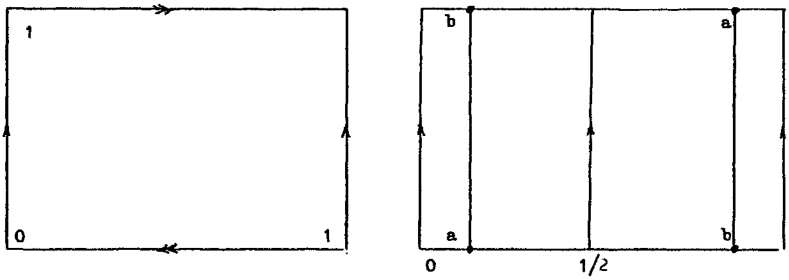

Figure 5

As S^1 is commutative and the action is assumed to be effective, the principal orbits must have type $\{1\}$. In the neighborhood of a principal orbit, according to the slice theorem 2.1.1, the action is the product over $S^1 \times \mathbf{R}$ of the multiplication and the trivial action. A neighborhood of such an orbit in the quotient space is thus homeomorphic to $\mathbf{R}$.

The type $\mathbf{Z}/m$ ($m \geq 2$) orbits are exceptional. That group must have an effective action on the normal space to the orbit, the dimension of which is 1. This can happen only if $m = 2$ the action being $x \mapsto -x$. In the neighborhood of such an orbit, a tube looks like:

$$S^1 \times_{\mathbf{Z}/2} \mathbf{R} = (S^1 \times \mathbf{R})/(z, u) \sim (-z, -u).$$

It is a Möbius strip, thus the existence of an exceptional orbit forces W to be nonorientable. In the neighborhood of the orbit, the quotient map is $[z, t] \mapsto | t |$ and the quotient space is a half line.

Singular orbits have type S^1: they are fixed points. Near such a point, the action linearises as the standard $S^1 = SO(2)$-action on $\mathbf{R}^2$.

We thus proved that the orbit space is a dimension 1 manifold with boundary, of course compact and connected as W is. It suffices now to investigate all possible cases.

1. If W/S^1 is a circle, the action is principal, and we saw that W is a torus, with the usual action.

2. If not, W/S^1 is a closed interval; over the open interval, we see $S^1 \times \mathbf{R}$ and each of the ends is the image, either of a fixed point, or of an exceptional orbit, with $\mathbf{Z}/2$ as stabilizer (*see* figure 6).

   - If there are two fixed points, W is obtained by adding two points to $S^1 \times \mathbf{R}$, and is a sphere.

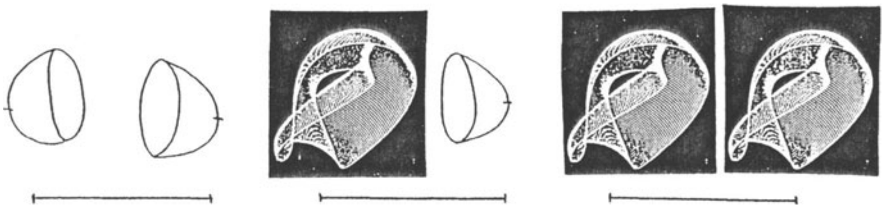

Figure 6

- If there is only one fixed point, and one exceptional orbit, we must glue a disc to a Möbius strip along the boundary (which is a principal orbit). We thus get a projective plane.
- If both ends correspond to exceptional orbits, we must glue two Möbius strips along their common boundary, thus getting a Klein bottle.

Exercise 3.1.2 Which surfaces may be endowed with (non trival!) $SO(3)$-actions? Hint: any subgroup of $SO(3)$ is conjugate either to $SO(2)$ or to $O(2)$ or is finite.

3.2 3-manifolds; the principal case

The study of 3-manifolds with fixed point free S^1-action was initiated by Seifert during the thirties [63]. The classification of these S^1-actions is due to F. Raymond [60]. He actually classified *topological* actions. Of course we shall restrict ourselves to the *differentiable* classification... but both are the same!

Le us start with principal actions. The quotient space is then a closed surface B and $\pi : W \longrightarrow B$ is a "principal S^1-bundle" over B, the local triviality of which is asserted by the slice theorem.

In addition to the diffeomorphism type of B, we shall exhibit another invariant, a number called the "Euler class". It is an integer when B is orientable (and an integer modulo 2 otherwise).

Assume the surface is orientable. Here is a description of the Euler class e. Choose a point in B and a disc D_0 around this point. The complement of the point has a wedge of $2g$ circles as deformation retract (*see* [5]).

Exercise 3.2.1 Show that, if the orbit space of a principal S^1-action is either a wedge of circles, or the complement of a point in a surface or a disc, then the fibration is trivialisable.

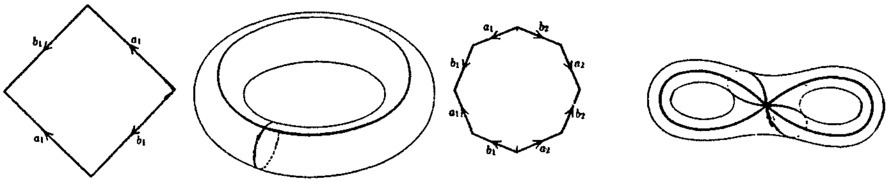

Figure 7

This allows to choose a section $\sigma : \overline{B - D_0} \longrightarrow \overline{W - \mathcal{V}_0}$. When restricted to D_0 the bundle is trivialisable as well, thus $\mathcal{V}_0 = \pi^{-1}(D_0)$ is a solid torus

$$\mathcal{V}_0 = D_0 \times S^1$$

and $W = (\overline{W - D_0}) \cup_\partial \mathcal{V}_0$.

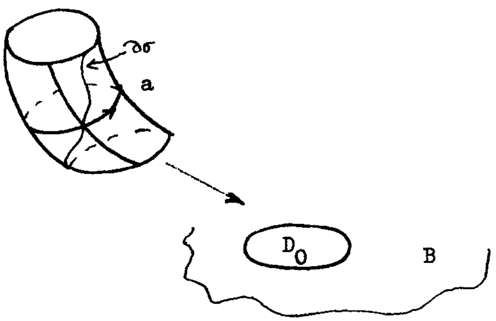

Figure 8

On the boundary $\partial\mathcal{V}_0$ we see two natural curves:

1. The meridian a. By definition, it is a closed simple curve which bounds a disc in $\mathcal{V}_0$. Its homotopy class is (up to sign) well defined by this property: a is a generator of the kernel of

$$\begin{array}{ccc} \pi_1(\partial\mathcal{V}_0) & \longrightarrow & \pi_1(\mathcal{V}_0) \\ \mathbf{Z}^2 & \longrightarrow & \mathbf{Z} \end{array}$$

   we can choose for a the boundary of a section of $\pi_{|D_0}$.

2. Any orbit b of the S^1-action. It is also a simple closed curve; with a chosen as above, the two curves meet at exactly one point.

As the surface B was supposed orientable and the action principal, the manifold W is orientable as well (*exercise*). Choose an orientation of W, the induced orientation on $\mathcal{V}_0$, the boundary orientation[4] on $\partial\mathcal{V}_0$. Orient b by the orientation of S^1 and choose the orientation of a in such a way that $a \cdot b = +1$.

If $\partial\sigma$ is oriented *oppositely* to the boundary orientation, it meets b in one point and positively and we may write

$$\partial\sigma \sim a - eb$$

for some $e \in \mathbf{Z}$. This integer is the *Euler class* of the principal S^1-action on W (or of the principal S^1-bundle $W \longrightarrow B$).

Proposition 3.2.2 *The so defined integer e is independant of all choices. Together with the diffeomorphism type of B, it determines the equivariant diffeomorphism type of W. For any orientable surface B and any integer e, there exists a manifold W endowed with a principal S^1-action whose orbit space is B and whose Euler class is e.*

Proof. One constructs, given B and e, a manifold W with an S^1-action whose invariants are B and W:

Choose a disc D_0 in B. Write $W_0 = (\overline{B - D_0}) \times S^1$, endow it with the product action, thus $\sigma(x) = (x, 1)$ is a section. Consider also a solid torus $D^2 \times S^1$ with the product action, and define an equivariant diffeomorphism

$$\varphi : \partial W_0 \longrightarrow \partial(D^2 \times S^1)$$

reversing orientations. Make the choice of φ precise by prescribing which curve in ∂W_0 is sent to a meridian (*i.e.* is homotopic to a constant in the solid torus). Take $\partial\sigma_0 + eb$, where b denotes the homotopy class of any orbit.

The constructed manifold has the right invariants. Conversely, if W is endowed with a principal S^1-action with these invariants, it is easy to write down a diffeomorphism of W with the above model. □

Remark. The Euler class e vanishes if and only if the fibration $W \longrightarrow B$ has a section, in other words is trivialisable.

Example. The S^1-action on S^3 by $t \cdot (z_1, z_2) = (tz_1, tz_2)$ is principal and the quotient is a sphere S^2 which we may consider here as the complex projective line $\mathbf{P}^1(\mathbf{C})$, the quotient mapping being the Hopf fibration:

$$\begin{array}{ccc} S^3 & \longrightarrow & \mathbf{P}^1(\mathbf{C}) \\ (z_1, z_2) & \longmapsto & [z_1, z_2] \end{array}$$

which associates with each unit vector the complex line it generates. The sphere decomposes into two solid tori (*see* figure 9) above which the fibration is trivialisable.

[4]The convention we use to orient boundaries is the one in which the *outward* normal followed by the orientation of the boundary gives the orientation of the whole manifold.

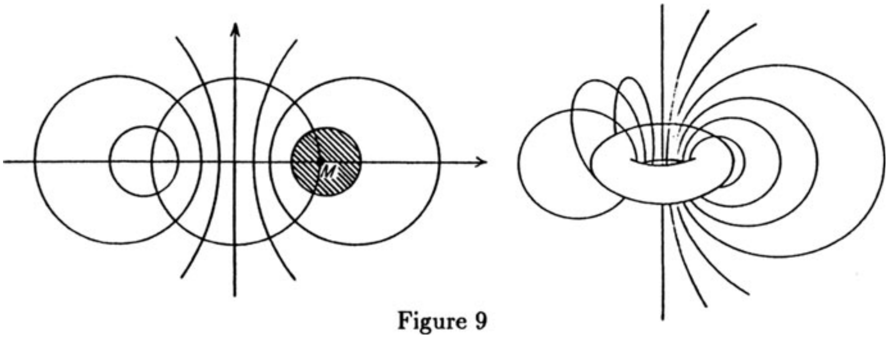

Figure 9

Exercise 3.2.3 Prove that the Euler class of the action giving the Hopf fibration is -1.

3.3 Seifert manifolds

We want now to generalise the previous study to the case of closed 3-manifolds endowed with an S^1-action, still without fixed points but with possibly exceptional orbits. I shall make restrictions on orientation later on. The proof we shall give of the classification theorem can be found in [59], another useful reference is [23].

There are principal orbits, of type $\{1\}$, and exceptional orbits, of type $\mathbf{Z}/m$ (for $m \geq 2$). As in the case of surfaces, we try to understand the local structure of the orbit space near an exceptional orbit with the help of the slice theorem 2.1.1.

3.3.1 Type $\mathbf{Z}/m$ orbits. The slice has dimension 2 and S^1 acts on $V = \mathbf{R}^2$ as a subgroup of $O(2)$. If $m \geq 3$, $\mathbf{Z}/m$ sits in $SO(2)$ as the subgroup of order m rotations, but if $m = 2$, there are two different ways to embed $\mathbf{Z}/2$: it can be generated by a reflection or by a rotation. Written with the help of a well-chosen complex coordinate in $\mathbf{R}^2$, with $\xi = e^{2i\pi/m}$:

$$\xi \cdot v = \xi v$$

and possibly, if m=2:

$$\xi \cdot v = \bar{v}$$

In that last case, the equivariant tube is

$$S^1 \times \mathbf{C} /(z, v) \sim (-z, \bar{v})$$

if one writes $v = x + iy$, one sees it is the product of a Mõbius strip and an $\mathbf{R}$. This can only happen if W is *non* orientable.

From now on, assume W is an *oriented* manifold. Thus all orbits have tubular neighborhoods of the form $S^1 \times_{\mathbf{Z}/m} \mathbf{C}$ where $\mathbf{Z}/m$ acts on $\mathbf{C}$ by order m rotations.

Remarks.

1. Exceptional orbits are isolated in W: in $S^1 \times_{\mathbf{Z}/m} \mathbf{C}$ all orbits are principal, except for the central ($v = 0$) one. In particular, as W is compact, there are only finitely many.

2. Near a type $\mathbf{Z}/m$ orbit, the orbit map is identified with

$$\begin{array}{ccc} S^1 \times_{\mathbf{Z}/m} \mathbf{C} & \longrightarrow & \mathbf{C}/(\mathbf{Z}/m) \\ [z, v] & \longmapsto & [v] \end{array}$$

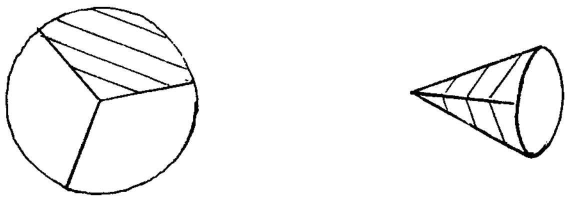

Figure 10

In particular, the quotient space is a closed topological surface B, with an additional structure inherited from the differentiable structure of W: the structure of a differentiable manifold except at a finite number of points, where B has a conical "singularity". One says that B is an orbifold[5] (*see* II-3.6.6). The additional structure is included in the S^1-action on W, in this chapter we shall therefore concentrate on the *topological* surface B.

We shall only need to consider the case where B is an *orientable* surface: when it will appear again in chapter IV, it will be symplectic.

In the sequel of this chapter, W is thus an oriented manifold and the quotient surface B is assumed to be orientable as well. The projection

$$W \longrightarrow B$$

is a *Seifert fibration* and the manifold W a *Seifert manifold.*

3.3.2 Seifert invariants of an exceptional orbit. Thanks to the slice theorem, a tubular neighborhood of an exceptional orbit looks like:

$$\mathcal{V} = S^1 \times_{\mathbf{Z}/m} D^2 \tag{2}$$

[5]previously called a *V-manifold.*

where $\mathbf{Z}/m$ acts as a subgroup on S^1 and acts in a linear fashion on the disc $D^2 \subset \mathbf{R}^2 = \mathbf{C}$. This action is by order m rotations, thus the quotient in (2) is given by

$$(z, v) \sim (e^{-2i\pi\beta/m} \cdot z, e^{2i\pi/m} \cdot v)$$

β being an integer prime to m (and defined only modulo m).

Definition 3.3.3 *The two relatively prime integers* (m, β) *with* $0 < \beta < m$ *are called the Seifert invariants of the exceptional orbit.*

Exercise 3.3.4 Let ν be an integer such that $\beta\nu = -1 \mod m$. Check that

$$\begin{array}{ccc} S^1 \times D^2 & \longrightarrow & S^1 \times D^2 \\ (z, v) & \longmapsto & (z^m, z^{-\nu} v) \end{array}$$

induces a diffeomorphism from $\mathcal{V}$ onto $S^1 \times D^2$, which is equivariant with respect to the S^1-actions given by

- on the left hand side by $t \cdot [z, v] = [tz, v]$
- on the right hand side by $t \cdot (Z, V) = (t^m Z, t^{-\nu} V)$

In particular the tube $\mathcal{V}$ is indeed a solid torus.

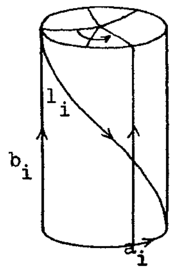

Figure 11

We thus see that any principal orbit turns m times around the exceptional orbit and thus meets any "meridian" at m points (*see* figure 11 in which $m = 5$ and $\nu = 2$, where the principal orbit is b_i, composed of m segments in the figure, and where a_i is a meridian).

3.3.5 Before giving the classification, let us investigate some examples.

Examples.

1. If m_1 and m_2 are relatively prime, we saw that the S^1-action on S^3 by

   (3) $$t \cdot (z_1, z_2) = (t^{m_1} z_1, t^{m_2} z_2)$$

   is effective. The orbits $(z_1, 0)$ and $(0, z_2)$ are exceptional (if m_1 and m_2 are at least 2), with respective types $\mathbf{Z}/m_1$ and $\mathbf{Z}/m_2$.

   A tubular neighborhood of the orbit $(z_1, 0)$ is a solid torus $S^1 \times D^2$ on which S^1 acts by (3).

   If α are β are chosen both by Bezout and by equation

   (4) $$\beta m_2 - \alpha m_1 = 1$$

   with $0 < \beta < m_1$, $0 < \alpha < m_2$, then the orbit invariants are (m_1, β) and $(m_2, m_2 - \alpha)$.

2. Let $\mathbf{Z}/p$ act on S^3 by
   $$\xi \cdot (z_1, z_2) = (\xi z_1, \xi^q z_2)$$
   where $\xi = e^{2i\pi/p}$ and q is prime to p.

   It is a *free* action of a finite group on a compact manifold. The quotient is a 3-manifold, denoted $L(p, q)$ and called *type* (p, q) *lens space*[6]. Then the circle acts on $L(p, q)$ by projecting any action on the sphere. For instance, from the principal action we get an action with one exceptional type $\mathbf{Z}/q$ orbit if $q \geq 2$.

Exercise 3.3.6 Find the quotient and Seifert invariants of the exceptional orbit in the previous example. Give a description of $L(p, q)$ by gluing of two solid tori.

3.3.7 Coming back to the classification problem: we have a finite number r of exceptional orbits in W, having $\mathbf{Z}/m_1, \ldots, \mathbf{Z}/m_r$ as stabilizers and respective Seifert invariants $(m_1, \beta_1), \ldots, (m_r, \beta_r)$. On the other hand, we have an orientable quotient surface B.

Remove r discs $D_1, \ldots, D_r$ from B around r points corresponding to exceptional orbits and also a disc D_0 (around a point corresponding to any principal orbit). This removes $r + 1$ solid tori $\mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_r$ from W.

Thus when restricted to $W - (\mathcal{V}_0 \cup \ldots \cup \mathcal{V}_r)$ the action is principal and has a section (even if $r = 0$). Choose one

$$\sigma : \overline{B - (D_0 \cup \ldots \cup D_r)} \longrightarrow \overline{W - (\mathcal{V}_0 \cup \ldots \cup \mathcal{V}_r)}.$$

Its boundary $\partial\sigma$ has a component over each ∂D_i, call it $\partial_i\sigma$. Consider the curve $\partial_i\sigma$. We know it meets b_i at exactly one point: thus $(\partial_i\sigma, b_i)$ is a basis of $\pi_1(\partial\mathcal{V}_i)$. Choose a meridian a_i as above and a "parallel" l_i in such a way that (a_i, l_i) is a basis of $\pi_1(\partial\mathcal{V}_i)$ oriented as $(\partial_i\sigma, b_i)$ (*see* figure 11).

[6]This is an example of a manifold whose fundamental group is $\mathbf{Z}/p$.

We have $a_i \cdot b_i = m_i$ et $l_i \cdot b_i = \nu_i$, and thus

$$\begin{cases} \partial_i \sigma &= x a_i + y l_i \\ b_i &= -\nu_i a_i + m_i l_i \end{cases}$$

that is

$$(\partial_i \sigma, b_i) = (a_i, l_i) \begin{pmatrix} x & -\nu_i \\ y & m_i \end{pmatrix}$$

where $x m_i + \nu_i y = 1$. Invert the matrix to get

$$(a_i, l_i) = (\partial_i \sigma, b_i) \begin{pmatrix} m_i & \nu_i \\ -y & x \end{pmatrix}$$

Hence

$$a_i = m_i \partial_i \sigma - y b_i$$

with $y \nu_i = 1 - x m_i$ that is $-y \equiv \beta_i \bmod m_i$.

The meridian a_i is thus written:

$$a_i = m_i \partial_i \sigma + n_i b_i \tag{5}$$

where m_i is actually the order of the central orbit stabilizer in $\mathcal{V}_i$ and n_i is congruent to $\beta_i \bmod m_i$.

We can now modify the section σ in such a way that n_i is exactly β_i: any map $\varphi : (B - \cup D_i) \to S^1$ can be used to modify σ just by multiplication

$$\sigma_\varphi(x) = \varphi(x) \cdot \sigma(x).$$

If the degree of $\varphi_{|\partial D_i}$ is d_i, then $\partial_i \sigma_\varphi \sim \partial_i \sigma + d_i b_i$ and

$$a_i = m_i \partial_i \sigma_\varphi + (n_i - d_i m_i) b_i.$$

To replace n_i by β_i in (5) it is thus sufficient to find a map $\varphi : B - \cup D_i \to S^1$, such that $\deg \varphi_{|\partial D_i} = d_i$ and $0 < n_i - d_i m_i < m_i$. Now use the lemma:

Lemma 3.3.8 *Let V be a connected surface whose boundary has $r+1$ (circle) components $\partial V = \bigcup_{i=0}^{r} C_i$. Let $\psi : \partial V \to S^1$ be a continuous map and let $d_i = \deg(\psi_{|C_i})$. There exists an extension of ψ to V, if and only if $\sum d_i = 0$.*

Proof. Stokes formula for instance shows the necessity of the condition: assume ψ is differentiable and there exists a smooth extension $\tilde{\psi} : V \to S^1$. If $d\theta$ denotes the volume form on S^1,

$$\sum_{i=0}^{r} d_i = \int_{\partial V} \psi^* d\theta = \int_V d(\tilde{\psi}^* d\theta) = 0.$$

Prove the reciprocal by induction on r. If $r = 0$ (the case where the boundary is connected) and $d_0 = 0$, then ψ is homotopic to the constant map. Use the homotopy to construct the extension $\tilde{\psi}$: take it to be constant outside a collar (*see* figure 12).

Proceed from r to $r+1$ in the same way, using the other part of the figure, details being left as an exercise. □

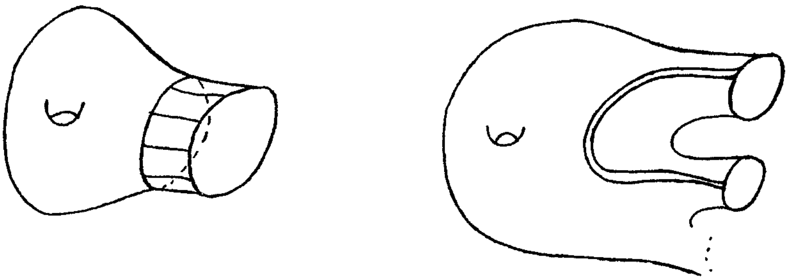

Figure 12

Using the lemma, one can find a map φ which can be used to modify the n_i's over all the ∂D_i's for $1 \le i \le r$. This forces $\deg \varphi_{|\partial D_0} = -(d_1 + \cdots + d_r)$ and at the same time forces an invariant β such that $\partial_0\sigma + \beta b_0$ is the meridian in $\mathcal{V}_0$ to appear and explains why we needed the "extra" tube $\mathcal{V}_0$.

We thus determined a list of (normalised) invariants:

$$(g \mid \beta, (m_1, \beta_1), \ldots, (m_r, \beta_r)).$$

Just as in 3.2.2:

Theorem 3.3.9 *If W is a fixed point free oriented S^1-manifold whose base space is a genus g oriented surface, then its diffeomorphism type is determined by the list:*

$$(g \mid \beta, (m_1, \beta_1), \ldots, (m_r, \beta_r))$$

where r is the number of exceptional orbits, β_i is prime to m_i and is chosen such that $0 < \beta_i < m_i$ and (m_i, β_i) are the Seifert invariant of the orbit numbered i. Reciprocally, given such a list one can find an S^1-manifold with these invariants. □

Up to order, the list is unique once we require the Seifert invariants to be normalised. Previous remarks (how to get normalised invariants) show moreover that the gluing data

$$(g \mid \beta, (m_1, \beta_1), \ldots, (m_r, \beta_r))$$

or

$$(g \mid 0, (1, -\beta), (m_1, \beta_1), \ldots, (m_r, \beta_r))$$

or even

$$(g \mid 0, (1, -\beta - (d_1 + \cdots + d_r)), (m_1, \beta_1 + d_1 m_1), \ldots, (m_r, \beta_r + d_r m_r))$$

define the same Seifert manifold.

3.4 Associated principal actions and Euler class

If W is a Seifert manifold, then for any integer n, one can make $\mathbf{Z}/n$ act on W as a subgroup of S^1.

Proposition 3.4.1 *If n is a common multiple of the orders of the exceptional orbits, then the manifold W' quotient of W by the $\mathbf{Z}/n$-action is endowed with an effective and principal action of $S^1/(\mathbf{Z}/n) \cong S^1$.*

Proof. Look at the neighborhood of an orbit (principal or exceptional), the method will proove at the same time that W' is a manifold.

Consider a tube $\mathcal{V}$, neighborhood of an orbit whose stabilizer is $\mathbf{Z}/m$ (m might be 1), and write $n = mk$. On $\mathcal{V} \cong S^1 \times D^2$, S^1 acts by

$$t \cdot (z, u) = (t^m z, t^{-\nu} u).$$

Take the quotient of the tube by $\mathbf{Z}/n \subset S^1$:

$$\begin{array}{ccc} S^1 \times D^2 & \longrightarrow & S^1 \times D^2 \\ (z, u) & \longmapsto & (z^k, u^{mk}) \end{array}$$

The quotient $\mathcal{V}'$ is a solid torus as well and the quotient map an n-fold branched covering ramified along the central orbit when it is exceptional, that is when $m \geq 2$.

Looking at each tube, we see that the quotient W' of W by $\mathbf{Z}/n$ is a manifold. Look now at the S^1-action. The diagram

$$\begin{array}{ccccc} \mathcal{V} & \longrightarrow & \mathcal{V}' & & \\ (z,u) & \longmapsto & (z^k, u^{mk}) & = & (Z, U) \\ \downarrow & & \downarrow & & \\ \mathcal{V} & \longrightarrow & \mathcal{V}' & & \\ (t^m z, t^{-\nu} u) & \longmapsto & (t^{mk} z^k, t^{-\nu mk} u^{mk}) & = & (t^n Z, t^{-\nu n} U) \end{array}$$

shows that the S^1-action on $\mathcal{V}$ (*resp.* W) induces an action of $S^1/(\mathbf{Z}/n)$ on $\mathcal{V}'$ (*resp.* W') which obviously is effective and principal: after identification of S^1 and $S^1/(\mathbf{Z}/n)$ by $t \mapsto t^n$ it can be written on tubes $(tZ, t^{-\nu} U)$. □

The quotient of W' by this principal action is the same surface B that was the quotient of W: each orbit of the action on W gives one and only one orbit in W'.

3.4.2 To define an *Euler class* for the S^1-action on W, one can try to divide by n the Euler class of the principal action just defined on W'. We need then to verify that the rational number obtained in this way does not depend on the choice of n. It is of course sufficient to understand what happens when a multiple n' of n is chosen:

Proposition 3.4.3 *Let $(t, x) \mapsto t \cdot x$ be a principal S^1-action on a 3-manifold W, and let W' be the quotient by the subgroup $\mathbf{Z}/k$ of S^1. Make $S^1 \cong S^1/(\mathbf{Z}/k)$ act on W'. The Euler class e' of the new action is $e' = ke$.*

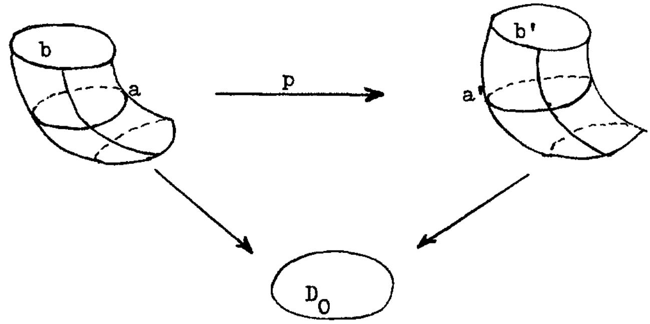

Figure 13

Proof. Use the same notations as in 3.2. Let p be the projection of W on W'. If σ is a section of $\overline{W - \mathcal{V}_0} \to \overline{B - D_0}$, then $p \circ \sigma$ is a section of $\overline{W' - \mathcal{V}'_0} \to \overline{B - D_0}$.

In figure 13, a and b represent meridian and orbit in $\partial\mathcal{V}_0$ and a' and b' represent analogous objects in $\partial\mathcal{V}'_0$. Now, $p_\star : \pi_1\partial\mathcal{V}_0 \to \pi_1\partial\mathcal{V}'_0$ sends a on a' and b on kb' and of course it sends the class of $\partial\sigma$ onto that of $\partial(p \circ \sigma)$. Thus:

$$a' - e'b' = [\partial\sigma'] = [\partial(p \circ \sigma)] = p_\star(a - eb) = a' - keb'.$$

□

Thus we get a well defined Euler class $e \in \mathbf{Q}$ for all (oriented, with oriented base space) Seifert fibrations.

Exercise 3.4.4 The Euler class of the Seifert manifold wose invariants are $(g \mid \beta, (m_1, \beta_1), \ldots, (m_r, \beta_r))$ is

$$e = \beta - \sum_{i=1}^{r} \frac{\beta_i}{m_i}.$$

Hint: both sides are the same in the case of a principal action.

Exercise 3.4.5 Show that the S^1-action on S^3 by

$$t \cdot (z_1, z_2) = (t^{m_1} z_1, t^{m_2} z_2)$$

(where m_1 and m_2 are relatively prime) has invariants: $(0 \mid 1, (m_1, \beta)(m_2, m_2 - \alpha))$ with $\beta m_2 - \alpha m_1 = 1$. Hint: the orbit invariants having been calculated in 3.3.5, it is easy to compute the Euler class, then everything follows from the previous exercise.

There exist other definitions of the Euler class of Seifert fibrations. We shall have the opportunity to meet one more. The interested reader may consult [23] for a rather complete list of definitions... and references.

Chapter II

Symplectic geometry

Before defining symplectic manifolds, recall a family of examples we shall use as a guide in this chapter. I am speaking of the $\mathcal{H}_\lambda$'s we have already met: $\mathcal{H}_\lambda$ is the set of all $n \times n$ hermitian matrices with given spectrum $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{R}^n$. We already know that $\mathcal{H}_\lambda$ is indeed a manifold, as an orbit of the compact group $U(n)$ (*see* I-1.3) acting by conjugation on the vector space $\mathcal{H}$ of all hermitian matrices. Moreover the $\mathcal{H}_\lambda$'s are *symplectic* manifolds.

1 Symplectic manifolds

1.1 Symplectic vector spaces

These are real vector spaces endowed with *nondegenerate* alternating bilinear forms.

Example. Consider $\mathbf{C}^n$ as a *real* $2n$-dimensional vector space, and the form

$$\omega(X, Y) = \operatorname{Im}\langle X, Y\rangle$$

where the brackets denote the standard hermitian form.

This example is in fact the *only* example: recall the

Proposition 1.1.1 *If ω is a nondegenerate alternating bilinear form on a finite dimensional vector space E, there exists a symplectic basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$,* i.e. *such that $\omega(e_i, f_j) = \delta_{i,j}$, and $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$.*

The proof is left as an exercise. □

In particular the dimension of E is even and is the only invariant of the isomorphism type of (E, ω).

Example of a symplectic basis. If $(e_1, \ldots, e_n)$ is a unitary basis in $\mathbf{C}^n$, and if we write $f_j = -ie_j$, then

$$\operatorname{Im}\langle e_j, f_k\rangle = -\operatorname{Im}\langle e_j, ie_k\rangle = -\operatorname{Im}(-i\langle e_j, e_k\rangle) = \langle e_j, e_k\rangle = \delta_{j,k}$$

and, analogously, $\omega(e_j, e_k) = \omega(f_j, f_k) = 0$, thus $(e_1, \ldots, e_n, -ie_1, \ldots, -ie_n)$ is a symplectic basis.

1.2 Symplectic manifolds, definition

We try now to understand what could be the definition of *symplectic* manifolds. One would first ask that each tangent space is endowed with the linear structure defined above: the manifold W would thus have a differential 2-form ω, in other words for each x an alternating bilinear form ω_x on T_xW. One asks moreover that ω_x is nondegenerate, which can be expressed by writing, if the dimension (obviously even) is denoted $2n$

$$\bigwedge^n \omega_x \neq 0 \in \bigwedge^{2n} T_x^\star W$$

for all x, that is to say that the $2n$-form $\omega^{\wedge n}$ is a volume form on W. Note that, in particular, W must have a volume form and thus be orientable.

In fact this will not be sufficient: we ask moreover that the calculus ω allows on W is locally the same as that on $\mathbf{C}^n$ using the "constant" form. Let us write that form as a differential form: if each complex number is denoted by $z_j = q_j + ip_j$, vectors in $\mathbf{C}^n$ are written $X = q + ip$, $Y = q' + ip'$ $(q, p \in \mathbf{R}^n)$, then

$$\begin{aligned}
\langle X, Y\rangle &= \langle q + ip, q' + ip'\rangle \\
\omega(X, Y) &= -\langle q, p'\rangle + \langle p, q'\rangle .
\end{aligned}$$

In other words

$$\omega = \sum_{i=1}^n dp_i \wedge dq_i = d(\sum_{i=1}^n p_i dq_i)$$

in particular, ω is an *exact* form.

One possible property to ask of a symplectic form is that it be exact... unfortunately, this will prevent the manifold W from being compact, which is a little embarrassing if one wants the $\mathcal{H}_\lambda$'s to be symplectic manifolds, because of:

Proposition 1.2.1 *If W is compact, there exists no 2-form which is both nondegenerate and exact on W.*

Proof. We saw that ω is nondegenerate if and only if $\omega^{\wedge n}$ is a volume form. As W is compact, we know that volume forms cannot be exact[1]. But, if ω was exact, $\omega = d\alpha$, then $\omega^{\wedge n} = d(\alpha \wedge \omega^{\wedge(n-1)})$. □

In fact, it was not a very good idea, in order to be able to calculate *locally* as in $\mathbf{C}^n$, to use a... global property such as of exactness of a differential form. The right "local exactness" condition is that of closedness... and now we are at the definitive

Definition 1.2.2 *A symplectic manifold is a pair (W, ω) where W is a manifold and ω a closed nondegenerate 2-form.*

Of course we shall very often call W a symplectic manifold (where no ambiguity is possible).

[1] It is now time for the beginners to learn a little de Rham cohomology in chapter VIII of [1].

Examples.

1. Of course, $\mathbf{C}^n$ with the constant form $\sum dp_i \wedge dq_i$, is a symplectic manifold.

2. If W is a manifold, consider the total space of its cotangent bundle, with the projection:
$$\pi : \mathrm{T}^\star W \longrightarrow W.$$
On $\mathrm{T}^\star W$, there is a canonical differential 1-form, the *Liouville* form, α, defined by:
$$\alpha_{(x,\varphi)}(X) = \varphi(\mathrm{T}_x\pi(X))$$
where $x \in W$, $\varphi \in \mathrm{T}^\star_x W$ and $X \in \mathrm{T}_{(x,\varphi)}(\mathrm{T}^\star W)$. It is easy to check that $\omega = d\alpha$ is nondegenerate (and it is even easier to show it is closed)! If $(q_1, \dots, q_n)$ are local coordinates on W, and if $(p_1, \dots, p_n)$ are the "dual" coordinates, then $(p_1, \dots, p_n, q_1, \dots, q_n)$ is a system of local coordinates in which $\alpha = \sum p_i dq_i$ et $\omega = \sum dp_i \wedge dq_i$. We shall often refer to ω as *the* symplectic form on the cotangent bundle[2]. The example of $\mathbf{C}^n$ above may be considered as the special case where $W = \mathbf{R}^n$, the imaginary coordinates playing the role of the cotangent ones.

3. If W is a surface, then the notion of symplectic form coincides with that of volume form: all orientable surfaces may thus be potentially considered as symplectic manifolds.

1.3 The Darboux theorem

Let us check that it is indeed possible to calculate locally in symplectic manifolds as in $\mathbf{C}^n$. First of all, a theorem which asserts that, locally, all symplectic forms are isomorphic.

Theorem 1.3.1 ([57][13]) *Let ω_0 et ω_1 be two symplectic forms on W which coincide at the point x. Then there exists a neighborhood U_0 of x in W and a map*
$$\psi : (U_0, x) \longrightarrow (W, x)$$
with $\psi^\star\omega_1 = \omega_0$.

Remark. The map ψ necessarily is a local diffeomorphism because $\omega_0^{\wedge n} = \psi^\star\omega_1^{\wedge n}$ and because these two $2n$-forms are volume forms.

Corollary 1.3.2 (the Darboux theorem) *Let x be a point in a symplectic manifold (W, ω). There exists a system of local coordinates $(p_1, \dots, p_n, q_1, \dots, q_n)$ centerd at x in which $\omega = \sum dp_i \wedge dq_i$.*

---

[2]One more example of an exact symplectic form, on a noncompact manifold.

Proof of corollary. Compare ω with the (constant) form on T_xW. The theorem does not allow this, the two forms not being defined on the same space. We must first use a local diffeomorphism $T_xW \to W$ and for instance the exponential of any riemannian metric will do.

Thus let $\varphi = \exp_x : V_0 \to U_0$, where $V_0 \subset T_xW$ and $V_0 \subset W$ are open subsets, so small that φ is a diffeomorphism. Define $\omega_0 = (\varphi^{-1})^\star\omega_x$. Now ω_0 and ω are two symplectic forms on U_0, which coincide at x by definition. We can now apply the theorem: if necessary by restricting U_0, there exists a diffeomorphism ψ such that $\psi^\star\omega = \omega_0$. If $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ are coordinates with respect to a symplectic basis of the vector space T_xW, the local chart $\psi \circ \varphi$ transforms them into a system of local coordinates in which $\omega = \sum dp_i \wedge dq_i$. □

Proof of theorem. Apply the path method due to J. Moser (*see* [57]). This means that we consider the form $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$. It is closed, and, as ω_0 and ω_1 coincide at x, it is symplectic (nondegenerate) at x and thus on a neighborhood of x.

As ω_0 and ω_1 are closed, so is $\omega_0 - \omega_1$ and it is thus possible to find, on a neighborhood of x, a 1-form φ such that $d\varphi = \omega_0 - \omega_1$. One may assume that $\varphi_x = 0$

The 2-form ω_t being symplectic, it defines a pairing between tangent and cotangent spaces, we thus get, for all t, a vector field X_t dual to φ. Consider now X_t as a vector field depending on time, vanishing at x for all t. Its flow f_t keeps x fixed, thus one can find a neighborhood U of x where f_t is defined and which f_t embeds in itself.

We thus get:

$$\frac{d}{dt}[f_t^\star\omega_t] = f_t^\star\left[\frac{d\omega_t}{dt} + \mathcal{L}_{X_t}\omega_t\right] = f_t^\star[\omega_1 - \omega_0 + \omega_0 - \omega_1] = 0$$

because $\mathcal{L}_{X_t}\omega_t = di_{X_t}\omega_t + i_{X_t}d\omega_t = d\varphi$ by Cartan and from the definition of φ. The form $f_t^\star\omega_t$ does not depend on t, it equals ω_0 for $t = 0$, we deduce the result with $\psi = f_1$. □

1.3.3 In case we need them later, notice both proofs become easily "equivariant proofs" if there is a Lie group G acting on W with ω_0 and ω_1 invariant and x fixed; this is the reason why we used $\varphi = \exp_x$ in the above proof instead of any local chart: it suffices now to choose an invariant metric to get equivariant local coordinates.

Remark. A statement analogous to 1.3.1 for riemannian metrics would be definitely false: in the riemannian case, there is a *local* invariant which distinguishes between the neighborhood of a point and the tangent space at that point, namely the curvature. We have just proved that there exist no local invariants in symplectic geometry.

1.3.4 A more general result can be obtained by essentially the same method [13]. It describes tubular neighborhoods of *isotropic* submanifolds (these on which ω vanishes) of W. It asserts in particular that if L is an isotropic submanifold of maximal dimension (then $\dim L = \frac{1}{2} \dim W$ and L is said to be *lagrangian*), a tubular neighborhood of L in W is (in a symplectic fashion) isomorphic to a neighborhood of the zero section in $\mathrm{T}^\star L$ (with the canonical symplectic form).

1.4 Examples: $U(n)$-orbits in hermitian matrices

Consider once again the group $U(n)$ acting by conjugation on the vector space $\mathcal{H}$ of hermitian matrices. We shall now check what was announced at the very beginning of this chapter: the $\mathcal{H}_\lambda$'s are symplectic manifolds. Begin by a little (bi)linear algebra in the space of all hermitian matrices.

Exercise 1.4.1 Show that

1. On $\mathcal{H}$, $(X,Y) \mapsto \mathrm{tr}(XY)$ is a *nondegenerate* symmetric bilinear form.
2. For $h \in \mathcal{H}$ define an alternating bilinear form ω_h on $\mathfrak{u}(n) = \mathrm{T}_1 U(n) = i\mathcal{H}$ (space of skew-hermitian matrices) by
$$\omega_h(X,Y) = \mathrm{tr}([X,Y]h).$$
Also $\omega_h(X,Y) = \mathrm{tr}(X(Yh - hY))$ and $Yh - hY \in \mathcal{H}$.
3. The kernel of ω_h is $K_h = \{Y \in \mathfrak{u}(n) \mid [Y,h] = 0\}$.
4. K_h is the Lie algebra of the stabilizer of h (hint: differentiate the relation $ghg^{-1} = h$).
5. the ω_h's induce nondegenerate 2-forms on the orbits $\mathcal{H}_\lambda$.
6. These 2-forms are closed.

Thus all the orbits $\mathcal{H}_\lambda$ under consideration are compact symplectic manifolds.

Examples.

- If all the eigenvalues $\lambda_1, \ldots, \lambda_n$ are distinct, put them in increasing (for example) order. Any element in $\mathcal{H}_\lambda$ defines a family of pairwise orthogonal lines in $\mathbf{C}^n$: its eigenspaces. One thus defines a bijection from $\mathcal{H}_\lambda$ onto the manifold of complete flags in $\mathbf{C}^n$:
$$\mathcal{D}(\mathbf{C}^n) = \{0 = P_0 \subset P_1 \subset \ldots \subset P_n = \mathbf{C}^n\}$$
associating to h the subspaces $P_i = l_1 \oplus \ldots \oplus l_i$ where l_i is the eigenspace corresponding to λ_i.
- On the other hand, if all eigenvalues are equal, there is only one point in the orbit. There is not much to say here.

- Suppose now that $\lambda_1 \neq \lambda_2 = \ldots = \lambda_n$. Then the eigenspace associated with λ_1 is a line and the one associated with λ_2 is the orthogonal hyperplane. The map: $\Psi_\lambda : \mathcal{H}_\lambda \to \mathbf{P}^{n-1}(\mathbf{C})$ which associates the λ_1-eigenspace with h is a diffeomorphism. We have thus exhibited a lot of symplectic forms on $\mathbf{P}^{n-1}(\mathbf{C})$, one for each pair of distinct real numbers.

1.5 Calibrated almost complex structures

Recall we decided to call $\mathbf{C}^n$ the unique example of a symplectic (real) vector space, and recall that all the manifolds we just studied are in fact *complex* manifolds. This is not accidental: on any symplectic manifold, there are almost complex structures.

Recall that a complex structure on a real vector space E is an endomorphism J of E such that $J^2 = -1$, and analogously, that an *almost complex* structure on a manifold W is a section J of the associated bundle $\operatorname{End}(\mathrm{T}W)$ such that, at each point $J_x^2 = -1$.

An almost complex structure J on a symplectic manifold (W, ω) will be said *calibrated*[3] if J is an "isometry" of ω and if the symmetric bilinear form $\omega(JX, Y)$ is positive-definite at each point.

Let us now show that calibrated complex structures exist on any symplectic vector space (E, ω). First choose a euclidean scalar product (X, Y). As $(\,,)$ and ω are nondegenerate, the relation $(X, AY) = \omega(X, Y)$ defines an isomorphism $A : E \to E$.

Write the polar decomposition of A : $A = BJ$, were B is symmetric positive definite, A and B commute and J is isometric.

Lemma 1.5.1 *A is skew-symmetric, J is an isometry for ω, $J^2 = -1$ and $JB = BJ$.*

Proof.

$$(X, AY) = \omega(X, Y) = -\omega(Y, X) = -(AX, Y)$$

thus A is skew-symmetric. If one writes $J = B^{-1}A$,

$${}^tJ = {}^tA\,{}^tB^{-1} = -AB^{-1} = -B^{-1}A = -J$$

as J is an isometry one gets ${}^tJJ = 1$, and thus J is indeed a complex structure. Moreover

$${}^t(BJ) = {}^tJ\,{}^tB = {}^tJB = -BJ$$

hence B and J commute thus A and J as well, and we are done:

$$\omega(JX, JY) = (JX, AJY) = (JX, JAY) = (X, AY) = \omega(X, Y)$$

□

Consider now the scalar product defined by B: $((X, Y)) = (BX, Y)$ which is also $\omega(JX, Y)$... this shows the existence, on every symplectic vector spece, of calibrated complex structures.

[3] implicitely: "by the symplectic form".

Remark. The form $((\,,\,)) + i\omega(\,,\,)$ is hermitian.

Exercise 1.5.2 The very same proof, starting with a symplectic manifold (W, ω) endowed with a riemannian metric, produces a calibrated almost complex structure J on W. Hint: use the fact that the map $A \mapsto B$ of the polar decomposition is smooth (use $B = \sqrt{A^t A}$).

As usual, there is an equivariant version of all this.

Remark. The symplectic form ω induces a *symplectic* form on any submanifold of W which is preserved by a calibrated almost complex structure. Indeed, if $Z \overset{j}{\subset} W$ is such a submanifold, $j^\star\omega$ is nondegenerate:

$$\forall X \in T_z Z - \{0\}, \exists Y \in T_z Z \quad \text{such that} \quad \omega(X, Y) \neq 0$$

because $Y = JX$ works.

1.5.3 Assume now that W is a *complex* manifold. That is to say that we have found an *integrable* complex structure J (changes of coordinates are J-holomorphic). Suppose that $\langle\,,\,\rangle$ is a hermitian metric on W. Its imaginary part is a (type $(1,1)$) nondegenerate 2-form ω. If it is symplectic, that is to say, if it is closed, one says that the metric is *Kähler*. The manifold W endowed with this metric and with the form ω is also called *Kähler*. By its very definition, the complex structure is calibrated by the Kähler form.

2 Hamiltonian vector fields and Poisson manifolds

2.1 Hamiltonian vector fields

Being nondegenerate, the symplectic form ω defines a pairing between the tangent and cotangent spaces of W (already used in the proof of 1.3.1). In particular, for any function $f : W \to \mathbf{R}$, there is a symplectic analogue of the gradient, namely a vector field X_f defined by

(1) $$i_{X_f}\omega = df.$$

We shall call X_f the symplectic gradient of f or the *hamiltonian vector field* associated with f.

If X is the hamiltonian vector field associated with f, we shall also say that f is a *hamiltonian* for X.

Remark. As ω is nondegenerate, the zeros of the vector field X_f are precisely the critical points of the function f.

Exercise 2.1.1 The function $f = \frac{1}{2}\sum |z_i|^2$, from $\mathbf{C}^n$ into $\mathbf{R}$ has

$$X_f = \sum_{j=1}^{n}(q_j \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial q_j})$$

as symplectic gradient.

Remark. If J is a calibrated almost complex structure, and if grad denotes the gradient for the riemannian metric $\omega(JX, Y)$, then $X_f = J\mathrm{grad}\, f$.

Definition 2.1.2 *A vector field X on W is hamiltonian if $i_X\omega$ is exact, locally hamiltonian if it is closed. One writes $\mathcal{H}(W)$ and $\mathcal{H}_{loc}(W)$ respectively for the spaces of hamiltonian and locally hamiltonian vector fields on W.*

Exercise 2.1.3 With the help of the map $X \mapsto i_X\omega$, define an exact sequence of real vector spaces:

$$0 \to \mathcal{H}(W) \longrightarrow \mathcal{H}_{loc}(W) \longrightarrow H^1(W; \mathbf{R}) \to 0.$$

Exercise 2.1.4 If α is a differential form and X, Y are two vector fields on W, show that

(2) $$\mathcal{L}_X i_Y \alpha - i_Y \mathcal{L}_X \alpha = i_{[X,Y]}\alpha.$$

One can check the formula when α is a 1-form[4], then by induction for decomposed k-formes $\alpha_1 \wedge \alpha_2$ where α_2 is a 1-form.

Suppose now ω is a symplectic form and X and Y two locally hamiltonian vector fields. Then

(3) $$i_{[X,Y]}\omega = \mathcal{L}_X i_Y \omega - i_Y \mathcal{L}_X \omega = (di_X + i_X d)i_Y\omega - i_Y(di_X + i_X d)\omega = di_X i_Y \omega$$

as ω, $i_X\omega$ and $i_Y\omega$ are closed forms. Thus the function:

$$i_X i_Y \omega = f_{X,Y} : \begin{array}{ccc} W & \longrightarrow & \mathbf{R} \\ x & \longmapsto & -\omega_x(X_x, Y_x) \end{array}$$

is a hamiltonian for $[X, Y]$: the bracket of two locally hamiltonian vector fields is (globally) hamiltonian.

Exercise 2.1.5 Show, for X any skew-hermitian matrix, that the fundamental vector field associated with $X \in \mathfrak{u}(n)$ on $\mathcal{H}$ (for the $U(n)$-action by conjugation) is $\underline{X}_h = [X, h]$[5]. Deduce that the function

$$f_X : \begin{array}{ccc} \mathcal{H}_\lambda & \longrightarrow & \mathbf{R} \\ h & \longmapsto & tr(iXh) \end{array}$$

is a hamiltonian for $\underline{X}$.

[4]It is always possible, in desperation, to prove this kind of formulae using local coordinates.

[5]*see* more generally 3.3.1.

2.2 The Poisson bracket on a symplectic manifold

We have just defined a map

$$\begin{array}{ccc} \mathcal{C}^\infty(W) & \longrightarrow & \mathcal{X}(W) \\ f & \longmapsto & X_f \end{array} \tag{4}$$

On the space $\mathcal{X}(W)$ of vector fields on W, there is a Lie algebra structure given by the bracket of vector fields. On a symplectic manifold, one can also define a Lie algebra structure on $\mathcal{C}^\infty(W)$.

Definition 2.2.1 *If (W,ω) is a symplectic manifold, the Poisson bracket*[6] *of two functions f and g is defined by*

$$\{f,g\} = \omega(X_f, X_g).$$

Thus $\{f,g\} = i_{X_f}\omega(X_g) = df(X_g) = X_g \cdot f$ and $\{f,\cdot\}$ is a derivation.

Exercise 2.2.2 Using (3), show that

$$X_{\{f,g\}} = -[X_f, X_g]$$

Proposition 2.2.3 *The Poisson bracket defines a Lie algebra sructure on $\mathcal{C}^\infty(W)$.*

Proof. The only thing to do is to check the Jacobi identity. Writing that ω is closed:

$$\begin{aligned} 0 = d\omega(X_1,X_2,X_3) &= X_1\cdot\omega(X_2,X_3) - X_2\cdot\omega(X_1,X_3) + X_3\cdot\omega(X_1,X_2) \\ &\quad -\omega([X_1,X_2],X_3) + \omega([X_1,X_3],X_2) - \omega([X_2,X_3],X_1) \end{aligned} \tag{5}$$

Suppose X_i is the symplectic gradient of the function f_i $(i = 1,2,3)$ then:

$$\begin{aligned} X_1\cdot\omega(X_2,X_3) &= X_1\cdot\{f_2,f_3\} \\ &= \{\{f_2,f_3\},f_1\} \\ &= \omega(X_{\{f_2,f_3\}},X_1) \\ &= \omega(X_1,[X_2,X_3]) \end{aligned}$$

from which it is easy to deduce that the relation (5) may also be written:

$$2(\{f_1,\{f_2,f_3\}\} + \{f_2,\{f_3,f_1\}\} + \{f_3,\{f_1,f_2\}\}) = 0$$

which is precisely the Jacobi identity. □

In particular the map (4) is (up to sign) a morphism of Lie algebras.

[6] *Translator's Note:* In French, Poisson=fish and the word for brackets means hook, which is the source of a lot of (unavoidable but) untranslatable jokes.

Exercise 2.2.4 For any nondegenerate (but not necessarly closed) 2-form ω on a manifold W one can define in the same way hamiltonian vector fields and a bracket $\{,\}$ on functions. Show that ω is closed if and only if $\{,\}$ satisfies the Jacobi identity[7].

2.2.5 Poisson manifolds. More generally, we will call *Poisson manifold* any manifold whose space of smooth functions has a Lie algebra structure $\{,\}$ such that $\{f,\cdot\}$ is a derivation (*i.e.* satisfies the Leibniz rule) is defined.

This notion was investigated by A. Lichnerowicz ([52]) and then by A. Weinstein ([65]).

It is strictly larger than the notion of symplectic manifold. Here is a family of examples of Poisson manifolds which are not symplectic.

2.2.6 The dual vector space to a Lie algebra. Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{g}^\star$ be the dual vector space. It is only a vector space: the Lie algebra structure of $\mathfrak{g}$ defines no canonical Lie algebra structure on $\mathfrak{g}^\star$; what it does define is a canonical Poisson structure, investigated mainly by A. A. Kirillov [10], B. Kostant [48] and J.-M. Souriau [12]:

For $f,g \in C^\infty(\mathfrak{g}^\star)$ and $\xi \in \mathfrak{g}^\star$ one writes:

$$\{f,g\}(\xi) = \langle \xi, [df_\xi, dg_\xi]\rangle$$

where $df_\xi : T_\xi\mathfrak{g}^\star = \mathfrak{g}^\star \to \mathbf{R}$ is identified to an element of $\mathfrak{g}$ by the joys of biduality. The Jacobi identity for $\{,\}$ is easily deduced from that of $[,]$ in $\mathfrak{g}$.

There is no reason why $\mathfrak{g}^\star$ should be a symplectic manifold: nothing prevents its dimension from being odd for example.

The perspicaceous reader will already have realised that there is a relationship between this general construction and that of the symplectic form on $\mathcal{H}_\lambda$ in 1.4. We considered there $G = U(n)$, $\mathfrak{g} = \mathfrak{u}(n)$, the dual being identified to $\mathcal{H} = i\mathfrak{u}(n)$ by the scalar product $\operatorname{tr}(XY)$. Modulo these identifications, the formal analogy with the definition of $\{,\}$ is only a particular case of a forthcoming result (3.3.5): when restricted to any orbit, the Poisson bracket is the one defined by the symplectic form of the orbit.

3 Symplectic and hamiltonian actions

3.1 Symplectic actions

One says that a G-action on the symplectic manifold (W,ω) is *symplectic* if any element g of G defines a diffeomorphism which preserves ω:

$$g^\star\omega = \omega \qquad (6)$$

Let us write the infinitesimal version of (6): let $X \in \mathfrak{g}$ and let $\underline{X}$ be the associated fundamental vector field. Let g_t be the flow of $\underline{X}$. If one differentiates (with respect

[7]This is one more good reason to insist that only *closed* 2-forms are *symplectic*.

to time) the equation $g_t^\star\omega = \omega$, one gets:

$$\mathcal{L}_{\underline{X}}\omega = \frac{d}{dt} g_t^\star \omega_{|t=0} = 0.$$

Thus the infinitesimal version[8] of (6) is: $\forall X \in \mathfrak{g}$, $\mathcal{L}_{\underline{X}}\omega = 0$. Use now both the Cartan formula

$$\mathcal{L}_X = d i_X + i_X d$$

and the fact that ω is closed, to get:

$$d i_{\underline{X}}\omega = 0.$$

Proposition 3.1.1 *If the G-action preserves the symplectic form ω then all the fundamental vector fields of the action are locally hamiltonian.* □

3.2 Hamiltonian actions

Consider now the following diagram

$$\begin{array}{ccccccc} C^\infty(W) & & \mathfrak{g} & & & & \\ \downarrow & & \downarrow & & & & \\ \mathcal{H}(W) & \hookrightarrow & \mathcal{H}_{loc}(W) & \rightarrow & H^1(W;\mathbf{R}) & \rightarrow & 0 \end{array}$$

Definition 3.2.1 *One says that the symplectic G-action on W is hamiltonian if there exists a morphism $\tilde{\mu}$ of Lie algebras $\mathfrak{g} \rightarrow C^\infty(W)$ making the diagram commute.*

In particular, it follows that fundamental vector fields are hamiltonian.

Remarks.

1. If the group is connected, each of its elements is a product of exponentials

$$g = \exp X_1 \exp X_2 \ldots \exp X_n$$

and thus the existence of μ, which always implies the infinitesimal "symplecticity" of the action, implies that here the action is actually symplectic. It is thus unuseful to require that the action is symplectic: this will be a consequence of the existence of the map $\tilde{\mu}$.

2. Some authors, among them some of the best ones (for instance J.-M. Souriau, who was one of the inventors of hamiltonian actions [12], *see* also [11]), call hamiltonian an action for which there exists a map $\tilde{\mu}$ making the diagram commute, without requiring that it is a Lie algebra morphism. Suppose that $X \mapsto \tilde{\mu}_X$ is only a map from $\mathfrak{g}$ to $C^\infty(W)$. Then $\tilde{\mu}_X$ is a function whose symplectic gradient field is $\underline{X}$, in particular $\{\tilde{\mu}_X, \tilde{\mu}_Y\}$ is a hamiltonian for $\underline{[X,Y]}$. Thus it is always true that $\{\tilde{\mu}_X, \tilde{\mu}_Y\} - \tilde{\mu}_{[X,Y]}$ is (locally) constant on W. We are just assuming here that the constant vanishes.

[8] For all the calculus used in this chapter, it may be convenient to read J. J. Duistermaat's book [9].

3. To finish with remarks, note that if G is commutative and W compact (the case we are mainly interested in), any lift $\tilde{\mu}$ is automatically a Lie algebra morphism: the Lie bracket in $\mathfrak{g}$ is trivial thus $\{\tilde{\mu}_X, \tilde{\mu}_Y\}$ is locally constant. The function $\tilde{\mu}_X$ must have a critical point on W, which we assumed to be compact, thus there exists a point at which $\{\tilde{\mu}_X, \tilde{\mu}_Y\}$ vanishes, and so it is zero everywhere.

3.2.2 The moment map. Associated to $\tilde{\mu}$ is its *moment map*:

$$\begin{array}{rccc} \mu : & W & \longrightarrow & \mathfrak{g}^\star = \operatorname{Hom}(\mathfrak{g}, \mathbf{R}) \\ & x & \longmapsto & (X \mapsto \tilde{\mu}_X(x)) \end{array}$$

Then the fundamental vector field $\underline{X}$ is the hamiltonian vector field of the function $x \mapsto \langle \mu(x), X\rangle$, and it is easily seen that $\tilde{\mu}$ is a Lie algebra morphism if and only if μ is a morphism of Poisson manifolds. Definition 3.2.1 can be reformulated in terms of μ instead of $\tilde{\mu}$.

Example. If $H : W \to \mathbf{R}$ is any function and if the hamiltonian vector field X_H is complete, its flow defines a hamiltonian $\mathbf{R}$-action whose moment map is H.

Exercise 3.2.3 If G is a Lie group acting on a manifold W, one defines a G-action on the cotangent bundle $\mathrm{T}^\star W$ by

$$\tilde{g}(x, \varphi) = (g(x), \varphi \circ \mathrm{T}_{g(x)} g^{-1})$$

where $x \in W$, $\varphi \in \mathrm{T}_x^\star W$ is a linear form $\mathrm{T}_x W \to \mathbf{R}$. Check it is indeed a (left) action. Show that the Liouville form α is invariant[9] ($\tilde{g}^\star \alpha = \alpha$) and conclude that the action is hamiltonian, with $\tilde{\mu}_X = -i_{\underline{X}}\alpha$.

3.2.4 Commutative groups. Suppose W is compact and the group is a torus T. Choose a basis $X_1, \ldots, X_k$ of $\mathfrak{t}$. The existence of the moment map is equivalent to that of a primitive $\tilde{\mu}_{X_i}$ of $i_{\underline{X_i}}\omega$ for any i. It is thus sufficient, in order to define $\tilde{\mu}$, to write $\tilde{\mu}_X = \sum_{i=1}^k \lambda_i \tilde{\mu}_{X_i}$ if $X = \sum_{i=1}^k \lambda_i X_i$.

A special case: if $H^1(W; \mathbf{R}) = 0$ (for example if W is simply connected), any symplectic T-action is hamiltonian. The moment μ is then well-defined up to the addition of a constant vector of $\mathfrak{t}^\star$.

Example. $T^n = \{(t_1, \ldots, t_n) \in \mathbf{C}^n \mid |t_i| = 1\}$ acts on $\mathbf{C}^n$ by

$$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (t_1 z_1, \ldots, t_n z_n).$$

This is a hamiltonian action with moment

$$\mu(z_1, \ldots, z_n) = \frac{1}{2}(|z_1|^2, \ldots, |z_n|^2) + \text{cste} \in \mathbf{R}^n \cong (\mathfrak{t}^n)^\star.$$

One should notice that the image of μ is the "first quadrant" ($x_1 \geq 0, \ldots, x_n \geq 0$) of $\mathbf{R}^n$, which is not unrelated to the convex polyhedra we shall encounter in the next chapter.

[9] I found this example in V. I. Arnold's book [8], which I take this opportunity to recommend.

Exercise 3.2.5 Which surfaces admit symplectic (*resp.* hamiltonian) S^1-actions? (Use the results of I-3.1.)

3.2.6 Semisimple groups. If $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, symplectic G-actions are hamiltonian[10]. Actually we saw in 2.2 that if X and Y are locally hamiltonian, then $[X, Y]$ is globally hamiltonian. But, if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, any element of $\mathfrak{g}$ may be writen as a sum of brackets, and thus defines a fundamental vector field which is globally hamiltonian.

This is the case, for example, for $G = SO(3)$ or $SU(2)$.

3.2.7 The moment μ was defined by a map $\tilde{\mu} : \mathfrak{g} \to C^\infty(W)$ such that $\tilde{\mu}_X$ is a primitive of $i_X\omega$. Thus it is not *a priori* unique. One can add locally constant functions c_X such that

$$\tilde{\mu}_{[X,Y]} + c_{[X,Y]} = \{\tilde{\mu}_X + c_X, \tilde{\mu}_Y + c_Y\} = \{\tilde{\mu}_X, \tilde{\mu}_Y\} = \tilde{\mu}_{[X,Y]}$$

(that is c vanishes on $[\mathfrak{g}, \mathfrak{g}]$) to $\tilde{\mu}_X$.

Thus μ is defined up to the addition of a constant linear form in the annihilator $[\mathfrak{g}, \mathfrak{g}]^\circ$ of $[\mathfrak{g}, \mathfrak{g}]$ and in particular of any constant if G is commutative; however if G is semisimple it is unique.

Before giving a list of examples, notice there is a simple procedure to construct examples:

Proposition 3.2.8 *If μ is the moment map for a hamiltonian action of a Lie group G on a symplectic manifold W, then for any Lie subgroup H of G, the composition of μ and the canonical projection:*

$$W \xrightarrow{\mu} \mathfrak{g}^\star \longrightarrow \mathfrak{h}^\star$$

is a moment map for the induced action. □

3.3 A machine producing examples: the coadjoint representation of a Lie group

3.3.1 Adjoint and coadjoint actions. Any Lie group G acts on itself by conjugation. The derivative at the identity of

$$\begin{array}{ccc} G & \longrightarrow & G \\ h & \longmapsto & ghg^{-1} \end{array}$$

is a map $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$. Letting g vary, one checks this defines a (left) action of G on its Lie algebra $\mathfrak{g}$, which is called the *adjoint* representation or action of G. It is the action which makes it possible to define the Lie bracket:

$$[X, Y] = \frac{d}{dt} \mathrm{Ad}_{\exp(tX)} Y_{|t=0}.$$

[10]Compact Lie groups whose Lie algebra satisfy $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ are *semisimple* .

Exercise 3.3.2 The fundamental vector field ${}^{\mathfrak{g}}\underline{X}$ associated with $X \in \mathfrak{g}$ by the adjoint representation is

$${}^{\mathfrak{g}}\underline{X}_Y = [X, Y].$$

To define the *coadjoint* action we merely have to transpose the adjoint G-action on $\mathfrak{g}^\star$:

$$\langle \mathrm{Ad}_g^\star \xi, X\rangle = \langle \xi, \mathrm{Ad}_{g^{-1}} X\rangle$$

where greek letters denote elements in $\mathfrak{g}^\star$, latin ones those in $\mathfrak{g}$ and brackets duality. Thus $\mathrm{Ad}^\star$ is a (left) G-action on $\mathfrak{g}^\star$.

Exercise 3.3.3 The fundamental vector field on $\mathfrak{g}^\star$ associated with $X \in \mathfrak{g}$ by the coadjoint action is defined by

$$(7) \qquad \langle {}^{\mathfrak{g}^\star}\underline{X}_\xi, Y\rangle = \langle \xi, [Y, X]\rangle.$$

A result which will be of great interest for us is the following proposition:

Proposition 3.3.4 *If G is a compact connected Lie group, the principal orbits of the coadjoint representation have a maximal torus of G as stabilizer.*

Sketch of the proof. If G is compact, it is easy to define an invariant scalar product on its Lie algebra $\mathfrak{g}$, and this allows us to identify $\mathfrak{g}$ with its dual. The proposition is then equivalent to:

The stabilizer of any $X \in \mathfrak{g}$ for the adjoint representation contains a maximal torus.

The infinitesimal analogue of this assertion, that is—the centralizer of any element in $\mathfrak{g}$ contains the Lie algebra of a maximal torus—is an easy consequence of the fact that all adjoint orbits meet the Lie algebra of any maximal torus (*see* [24]) : if $X \in \mathfrak{t}$, its centralizer contains $\mathfrak{t}$. □

Remark. M. Duflo and M. Vergne ([33]) have shown that the principal stabilizer for the coadjoint representation is always a commutative subgroup (even if G is not compact).

3.3.5 The symplectic form on the orbits. Just rewriting what we did for $\mathcal{H}_\lambda$ in 1.4, one defines for any $\xi \in \mathfrak{g}^\star$ an alternating bilinear form on $\mathfrak{g}$:

$$\omega_\xi(X, Y) = \langle \xi, [X, Y]\rangle.$$

Lemma 3.3.6 *The kernel of ω_ξ is the Lie algebra $\mathfrak{g}_\xi$ of the stabilizer of $\xi \in \mathfrak{g}^\star$ for the coadjoint representation.*

Proof. Let $X, Y \in \mathfrak{g}$. By the very definition of $\mathrm{Ad}^\star$, we have:

$$\langle \mathrm{Ad}^\star_{\exp(-tX)}\xi, Y\rangle = \langle \xi, \mathrm{Ad}_{\exp(tX)}Y\rangle.$$

Differentiating at 0 with respect to t:

$$\begin{aligned}\langle \tfrac{d}{dt}\mathrm{Ad}^\star_{\exp(-tX)}\xi_{|t=0}, Y\rangle &= \langle \xi, \tfrac{d}{dt}\mathrm{Ad}_{\exp(tX)}Y\rangle \\ &= \langle \xi, [X,Y]\rangle \\ &= \omega_\xi(X,Y)\end{aligned}$$

thus the kernel of ω_ξ is $\mathfrak{g}_\xi$. □

In particular, ω_ξ defines a nondegenerate alternating bilinear form on $\mathfrak{g}/\mathfrak{g}_\xi$ which we know from I-1.2 is identified with $\mathrm{T}_\xi(G\cdot\xi) \subset \mathfrak{g}^\star$. We only need now to solve the following exercise to show that there exists a canonical symplectic structure on each coadjoint orbit, induced by the canonical Poisson structure of $\mathfrak{g}^\star$.

Exercise 3.3.7 Show that ω_ξ defines a closed 2-form on the orbit of ξ in $\mathfrak{g}^\star$. Hint: the tangent space to the orbit being generated by fundamental vector fields, it is a straightforward consequence of the Jacobi identity in $\mathfrak{g}$.

3.3.8 Canonical moment map. Let $G\cdot\xi$ be an orbit in $\mathfrak{g}^\star$. We thus have a Lie group G acting on a symplectic manifold $G\cdot\xi$ and (obviously!) preserving the symplectic form and everything is so canonical that the following exercise is almost tautological:

Exercise 3.3.9 The inclusion $G\cdot\xi \subset \mathfrak{g}^\star$ is a moment map for the G-action on its orbit.

This construction generates a lot of examples, using 3.2.8. For instance we saw (3.3.4) that the principal coadjoint orbits have as type the conjugation class of any maximal torus. It is thus especially interesting to study the action of a fixed maximal torus $T \subset G$ on these orbits[11]. We deduce a lot of examples of *torus* actions on symplectic manifolds... thus even if one is only interested in commutative groups, this excursion into noncommutativity was not unuseful.

3.3.10 The $U(n)$ case. Let us make the torus of *diagonal* unitary matrices act on the orbits $\mathcal{H}_\lambda$.

Exercise 3.3.11 The map $\mathcal{H}_\lambda \to \mathbf{R}^n$, which associates with any hermitian matrix its diagonal entries, is a moment map for the T-action.

[11] For instance, fixed points will be the points in the orbit whose stabilizer is precisely T and not one of its conjugates.

Example. Put $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_2)$. We saw in 1.4 that $\mathcal{H}_\lambda$ was a complex projective space $\mathbf{P}^{n-1}(\mathbf{C})$. We thus exhibited a hamiltonian T^n-action on $\mathbf{P}^{n-1}(\mathbf{C})$.

Exercise 3.3.12 Show that the composite map

$$\mathbf{P}^{n-1}(\mathbf{C}) \xrightarrow{\Psi_\lambda^{-1}} \mathcal{H}_\lambda \xrightarrow{\mu} \mathfrak{t}^* = \mathbf{R}^n$$

(that is the moment map for the T-action on $\mathbf{P}^{n-1}(\mathbf{C})$ endowed with the canonical symplectic form of $\mathcal{H}_\lambda$) is

$$[z_1, \ldots, z_n] \longmapsto (\lambda_1 - \lambda_2)(|z_1|^2, \ldots, |z_n|^2) + (\lambda_2, \ldots, \lambda_2)$$

where $(z_1, \ldots, z_n)$ is of unit length. Describe its image in $\mathbf{R}^n$.

3.3.13 The $SO(3)$ case. The Lie algebra $\mathfrak{so}(3)$ is that of 3×3 skew-symmetric matrices. Identify it with $\mathbf{R}^3$ in such a way that the Lie bracket becomes the vector cross product and with its dual $\mathfrak{so}(3)^*$ with help of the euclidean scalar product of $\mathbf{R}^3$.

Exercise 3.3.14 The adjoint and coadjoint actions are then identified to the usual $SO(3)$-action on $\mathbf{R}^3$ (by rotations).

The coadjoint orbits are thus the point 0 and the concentric S^2's (with $SO(2)$ stabilizer).

3.4 Some properties of moment maps

With μ we have a map from W into a vector space which will allow us to investigate the group action.

Example. In the case of an S^1-action with fundamental vector field $\underline{X}$ (for X a basis of the Lie algebra $\mathfrak{s}^1 \cong \mathbf{R}$), the map $\mu : W \to \mathbf{R}$ is a primitive of $i_{\underline{X}}\omega$. Thus the fixed points of the S^1-action are the critical points of a function... which makes life easier: we already know for example that if W is compact (and the action is nontrivial) there will be at least two components in the fixed point set[12] (corresponding to *extrema* of μ).

3.4.1 The tangent map

$$\mathrm{T}_x W \xrightarrow{\mathrm{T}_x\mu} \mathfrak{g}^*$$

is by definition the transpose of

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{{}^t\mathrm{T}_x\mu} & \mathrm{T}_x^* W \\ X & \longmapsto & (i_{\underline{X}}\omega)_x \end{array}$$

[12]This is not necessarly the case for a general S^1-action as the reader may check by making S^1 act on S^4 as an *ad hoc* subgroup of $SO(5)$.

In particular, $\operatorname{Im} \mathsf{T}_x\mu$ is the annihilator in $\mathfrak{g}^\star$ of $\operatorname{Ker} {}^t\mathsf{T}_x\mu$ that is the annihilator of

$$\left\{X \in \mathfrak{g} \mid (i_{\underline{X}}\omega)_x = 0\right\} = \{X \in \mathfrak{g} \mid \underline{X}_x = 0\} = \mathfrak{g}_x$$

Thus the rank of $\mathsf{T}_x\mu$ is the dimension of the orbit of x and in particular

Proposition 3.4.2 *μ is a submersion at x if and only if G_x is discrete.*

This property is indeed equivalent to the nullity of $\mathfrak{g}_x$. □

Corollary 3.4.3 *If G is a commutative group acting in an effective way on the symplectic manifold W with moment map $\mu : W \longrightarrow \mathfrak{g}^\star$ then μ is a submersion on the open dense subset of principal orbits in W.*

Proof. We already know that the principal orbits' stabilizer is the discrete subgroup $\{1\} \subset G$. □

If G is not commutative, it may happen that μ is nowhere submersive, even if the action is effective:

Example. $SO(3)$ acts on $S^2 \times W$ by the usual (effective) action on S^2 and by the trivial action on W. It preserves the "product" symplectic form $\omega_1 \oplus \omega_2$, and the moment map

$$\mu : S^2 \times W \longrightarrow \mathfrak{so}(3)^\star = \mathbf{R}^3$$

has a sphere S^2 as image (and in particular contains no open dense subset of $\mathfrak{so}(3)^\star$) and μ is nowhere submersive.

Exercise 3.4.4 Show that $\operatorname{Ker} \mathsf{T}_x\mu$ is the orthogonal (for ω_x) of the tangent space to the orbit.

3.4.5 Equivariant moment map. The moment map μ sends a G-manifold W into another G-manifold $\mathfrak{g}^\star$, it is thus natural to ask whether it is equivariant.

Let us consider any $X \in \mathfrak{g}$ and let us show:

$$\mathsf{T}_x\mu({}^W\underline{X}_x) = {}^{\mathfrak{g}^\star}\underline{X}_{\mu(x)} \tag{8}$$

Thanks to relation (7), on the one hand:

$$\begin{aligned} \langle {}^{\mathfrak{g}^\star}\underline{X}_{\mu(x)}, Y\rangle &= \langle \mu(x), [Y, X]\rangle \\ &= -\tilde{\mu}_{[X,Y]}(x) \end{aligned}$$

and on the other hand:

$$\begin{aligned}
\langle \mathrm{T}_x\mu({}^W\underline{X}_x), Y\rangle &= \langle {}^W\underline{X}_x, {}^t\mathrm{T}_x\mu(Y)\rangle \\
&= \langle {}^W\underline{X}_x, (i_{W\underline{Y}}\omega)_x\rangle \\
&= \omega_x({}^W\underline{Y}, {}^W\underline{X}) \\
&= -\{\tilde{\mu}_X, \tilde{\mu}_Y\}(x) \\
&= -\tilde{\mu}_{[X,Y]} \qquad \square
\end{aligned}$$

We have thus proved the infinitesimal version of:

Proposition 3.4.6 *Let μ be the moment map for the hamiltonian action of the connected group G on the symplectic manifold W. Then μ is an equivariant map.* $\square$

This will endow hamiltonian actions with very specific properties, for example because 3.4.6 forces $G_x \subset G_{\mu(x)}$. Here is an example:

Proposition 3.4.7 *Let W be a symplectic manifold endowed with a nontrivial symplectic action of $G = SO(3)$ or $SU(2)$. Then the stabilizer of the principal orbits is a commutative subgroup of G.*

Proof. We saw in 3.2.6 that because G is semisimple, the action is hamiltonian. Let $\mu : W \to \mathfrak{g}^\star$ be the (unique) moment map for this action.

Let x be a point in W. Then $G_x \subset G_{\mu(x)}$ and the orbit stabilizers in $\mathfrak{g}^\star$ are G or the circle $SO(2) \subset SO(3)$ or $SU(2)$ (*see* 3.3.13).

If G_x is not contained in $SO(2)$, we have $G_{\mu(x)} = G$, or $\mu(x) = 0$. If this was the case for all points of principal orbits, μ would vanish on an open dense subset of W, thus everywhere. But the operation would then be trivial. $\square$

In the same way, one can show more generally (using 3.3.4) that, if μ is a submersion at at least one point, then "the" stabilizer of the principal orbits is commutative and discrete.

Exercise 3.4.8 Let $x \in W$, then μ sends the orbit $G \cdot x$ into $G \cdot \mu(x) \subset \mathfrak{g}^\star$. Let ω be the symplectic form of the orbit $G \cdot \mu(x)$. What can be said of the 2-form $\mu^\star\omega$ on $G \cdot x$?

3.5 Noether type theorems

Consider now the "levels" of the moment map[13] $\mu : W \to \mathfrak{g}^\star$ of a hamiltonian G-action on the symplectic manifold (W, ω). The most classical form of E. Noether's theorem seems to be stated nowadays as:

[13] I shall call *level* ξ the inverse image $\mu^{-1}(\xi)$ even if ξ is not a number.

Theorem 3.5.1 *Let H be a function on W which is invariant by the G-action. Then μ is constant on the trajectories of the hamiltonian vector field X_H.*

Proof. Indeed, if $\gamma(t)$ is a trajectory of X_H, one can write, for any $X \in \mathfrak{g}$:

$$\begin{aligned}
\frac{d}{dt} < \mu \circ \gamma(t), X > &= < \mathsf{T}_{\gamma(t)}\mu(X_H(\gamma(t))), X > \\
&= < X_H(\gamma(t)), {}^t\mathsf{T}_{\gamma(t)}\mu(\underline{X}) > \\
&= < X_H(\gamma(t)), (i_X\omega)_{\gamma(t)} > \\
&= \omega(\underline{X}, X_H)_{\gamma(t)} \\
&= -dH_{\gamma(t)}(\underline{X})
\end{aligned}$$

But H is invariant and $\underline{X}$ a fundamental vector field of the action, and thus:

$$H(\exp(sX) \cdot \gamma(t)) = H(\gamma(t))$$

which, when differentiated at $s = 0$, is:

$$dH(\underline{X}(\gamma(t))) = 0.$$

□

The field X_H is thus tangent to the levels $\mu^{-1}(\xi)$.

3.5.2 Periodic hamiltonians. Begin by noticing that even the most trivial examples are already interesting:

Example. $G = S^1$ and $H = \mu$: then the fundamental vector field is tangent to the hypersurfaces of constant "energy" H.

Such a function is called a *periodic hamiltonian.* We see that all regular levels of a periodic hamiltonian are oriented submanifols endowed with a fixed points free S^1-action. For example in dimension 4, the regular levels are Seifert manifolds. We shall see later how it is possible to use the constructions of chapter I to study periodic hamiltonians.

3.5.3 Let us return to the general case. For $\xi \in \mathfrak{g}^\star$ a regular value of μ, call $V_\xi = \mu^{-1}(\xi)$. As μ is equivariant, the subgroup $G_\xi \subset G$ keeps the set V_ξ invariant: if $x \in V_\xi$ and $g \in G_\xi$, then $\mu(g \cdot x) = g \cdot \mu(x) = g \cdot \xi = \xi$.

Look now at what is happening to the symplectic form when we restrict it by the inclusion map $j_\xi : V_\xi \hookrightarrow W$:

Lemma 3.5.4 *$j_\xi^\star\omega$ is a form whose kernel at x is $\mathsf{T}_x(G_\xi \cdot x)$, its rank is constant (i.e. does not depend on x) and equals* $2\dim V_\xi + \dim(G \cdot \xi) - \dim W$.

Proof. By definition, $\operatorname{Ker}(j_\xi^\star\omega)_x = \mathsf{T}_xV_\xi \cap (\mathsf{T}_xV_\xi)^\circ$; on the other hand, $\mathsf{T}_xV_\xi = \operatorname{Ker}\mathsf{T}_x\mu$ and we saw in 3.4.1 that $\operatorname{Ker}\mathsf{T}_x\mu = (\mathsf{T}_x(G\cdot x))^\circ$, thus

$$\operatorname{Ker}(j_\xi^\star\omega)_x = \operatorname{Ker}\mathsf{T}_x\mu \cap \mathsf{T}_x(G\cdot x).$$

The tangent space $\mathsf{T}_x(G\cdot x)$ to the orbit is generated by the fundamental vector fields ${}^W\underline{X}$. According to (8)

$${}^W\underline{X} \in \operatorname{Ker}\mathsf{T}_x\mu \Leftrightarrow {}^{\mathfrak{g}^*}\underline{X}_{\mu(x)} = 0 \Leftrightarrow X \in \mathfrak{g}_{\mu(x)}$$

thus $\operatorname{Ker}(j_\xi^\star\omega)_x$ is generated by the fundamental vector fields coming from $\mathfrak{g}_\xi$. Hence $\operatorname{Ker}(j_\xi^\star\omega)_x = \mathsf{T}_x(G_\xi\cdot x)$ and its dimension is:

$$\dim G_\xi - \dim G_x = \dim G_\xi$$

(because ξ is a regular value so that G_x is discrete) and its rank:

$$\begin{aligned} \operatorname{rg}(j_\xi^\star\omega)_x &= \dim V_\xi - (\dim G_\xi - \dim G_x) \\ &= \dim V_\xi \underbrace{- \dim G_\xi + \dim G}_{\dim G\cdot\xi} - \dim W + \dim V_\xi \\ &= 2\dim V_\xi - \dim W + \dim G\cdot\xi \end{aligned}$$

is constant on V_ξ. □

Exercise 3.5.5 Show that this equality remains true if one merely assumes that V_ξ is a submanifold whose tangent space is $\operatorname{Ker}\mathsf{T}_x\mu$ instead of ξ being a regular value. Hint: show that in this case $\dim G_x = \dim G - \dim W + \dim V_\xi$.

These results allow to characterise the hamiltonian actions of abelian groups:

Proposition 3.5.6 *If G is connected and the action is effective, the three following properties are equivalent:*

1. *G-orbits are isotropic*
2. *μ is constant on any orbit*
3. *G is commutative.*

In this case, $\dim W \geq 2\dim G$

Proof. $G\cdot x$ is isotropic if and only if $\mathsf{T}_x(G\cdot x) \subset \mathsf{T}_x(G\cdot x)^\circ = \operatorname{Ker}\mathsf{T}_x\mu$. In this case, for any $X, Y \in \mathfrak{g}$, the map $x \mapsto -\omega_x(\underline{X}_x, \underline{Y}_x)$ is identically zero. But we saw (relation (3)) that its differential is $i_{[\underline{X},\underline{Y}]}\omega$. Thus $i_{[\underline{X},\underline{Y}]}\omega = 0$, from which it follows that $[\underline{X},\underline{Y}] = 0$ and, thanks to I-(1), that $[X,Y] = 0$ for any $X, Y \in G$, and so G is commutative.

Reciprocally, if G is commutative, $G_\xi = G$ for all ξ in $\mathfrak{g}^\star$ and thus $\operatorname{Ker} j_\xi^\star\omega$ is the whole tangent space to the orbit which is thus isotropic. □

The inequality in 3.5.6 is easily generalised:

Proposition 3.5.7 *Let W be a symplectic manifold endowed with a hamiltonian action of a compact connected Lie group G, with moment map $\mu : W \to \mathfrak{g}^\star$. Let T be a maximal torus in G. If μ is submersive in at least one point in W, then*

$$\dim G + \dim T \leq \dim W. \tag{9}$$

Proof. It is sufficient to observe that the rank of $(j_\xi^\star \omega)_x$ is a nonnegative integer, and to compute:

$$\begin{aligned} \operatorname{rg} j_\xi^\star \omega &= 2 \dim V_\xi - \dim W + \dim G \cdot \xi \\ &= 2(\dim W - \dim G) - \dim W + \dim G \cdot \xi \\ &= \dim W - 2 \dim G + \dim G - \dim G_\xi \end{aligned}$$

thus $\dim W \geq \dim G + \dim G_\xi \geq \dim G + \dim T$, thanks to 3.3.4. □

For example a manifold endowed with a symplectic $SO(3)$-action whose moment map is submersive at one point has dimension at least 4. The hypothesis "submersion at one point" is necessary: for example the $SO(3)$-action on the unit sphere S^2 by isometries preserves the volume form and is thus symplectic.

3.6 Symplectic reduction, examples

3.6.1 The essence of what is called symplectic reduction is contained in a rather obvious algebraic lemma. Recall that we call *coisotropic* any vector subspace (*resp.* submanifold) of a symplectic vector space (*resp.* manifold) whose symplectic orthogonal is *isotropic*: $F \subset E$ is coisotroic if and only if $F^\circ \subset F$.

Lemma 3.6.2 (Symplectic reduction) *Let E be a vector space endowed with a (constant) symplectic form ω. Let F be a coisotropic subspace. Then:*

1. *the form ω induces a symplectic form on F/F°.*
2. *if G is any isotropic subspace transverse to F, then the composite map $G \cap F \subset F \to F/F^\circ$ is the inclusion of an isotropic subspace.*

The proof is easy enough to be left as an exercise. □

3.6.3 Straightaway, we have an application. Consider a torus T acting effectively on a compact symplectic manifold (W, ω) with moment map $\mu : W \to \mathfrak{t}^\star$. We know that in this case there must exist regular values. Let ξ such a value. For $x \in V_\xi$, we have:

$$\mathsf{T}_x V_\xi = \operatorname{Ker} \mathsf{T}_x \mu = (\mathsf{T}_x (T \cdot x))^\circ.$$

Moreover, the tangent space to the orbit, being generated by fundamental vector fields, is isotropic since T is commutative (3.5.6). Thus $\mathsf{T}_x V_\xi$ is coisotropic and its orthogonal is the tangent space to the orbit. Hence ω_x induces a nondegenerate alternating bilinear form on $\mathsf{T}_x V_\xi / \mathsf{T}_x(T \cdot x)$.

The T-action on W defines by restriction a T-action on the level V_ξ, of which we know that all its stabilizers are finite: $\mathsf{T}_x\mu$ is surjective thus $\mathfrak{t}^\star$ is the annihilator of $\mathfrak{t}_x$.

If moreover all these stabilizers are trivial, in other words if the T-action on the regular level V_ξ is free, then the quotient V_ξ/T is a manifold B_ξ (with dimension $\dim W - 2\dim T$) and $\mathsf{T}_x V_\xi/\mathsf{T}_x(T \cdot x)$ may be identified to its tangent space at point $[x] = T \cdot x$. Thus:

Proposition 3.6.4 *If the torus T acts freely on the regular level V_ξ of the moment map $\mu : W \to \mathfrak{t}^\star$, then the orbit space V_ξ/T is a manifold naturally endowed with a symplectic form σ_ξ called the reduced symplectic form.*

Proof. The form defined by ω_x sur $\mathsf{T}_x V_\xi/\mathsf{T}_x(T \cdot x)$ does not depend on the point x we chose in the orbit $T \cdot x$, thanks to the invariance of ω. In the diagram

$$\begin{array}{ccc} V_\xi & \overset{j_\xi}{\hookrightarrow} & W \\ \downarrow{\scriptstyle p_\xi} & & \\ B_\xi & & \end{array}$$

σ_ξ is defined by

(10) $$p_\xi^\star \sigma_\xi = j_\xi^\star \omega$$

from which it follows that $p_\xi^\star d\sigma_\xi = 0$ and that σ_ξ is closed. □

Exercise 3.6.5 Consider the S^1-action on $\mathbf{C}^n$ by multiplication

$$t \cdot (z_1, \ldots, z_n) = (tz_1, \ldots, tz_n).$$

Show that it is hamiltonian with moment map

$$\mu(z_1, \ldots, z_n) = \frac{1}{2} \sum_{i=1}^{n} |z_i|^2$$

Check that all non-zero real numbers are regular values of μ. What are the reduced symplectic manifolds obtained by symplectic reduction of the regular levels?

3.6.6 Symplectic orbifolds. In general, the torus action may well have exceptional (*i.e.* with finite stabilizer) orbits in regular levels. It is then no longer absolutely correct that the orbit space B_ξ is a symplectic *manifold*. We have already met this kind of problem in I-3.3.1 where the *topological* structure of the quotient B_ξ happened to be enough for what we needed. Now we have to look carefully at what is preserved of the differentiable structure near singular points in order to be able to speak of a symplectic form on the quotient.

The right notion here is that of orbifold, invented by Satake [61] under the name of V-manifold. The idea of a symplectic form on an orbifold was made precise by A. Weinstein in [64].

Locally, orbifolds are the open subsets of the quotient spaces $\mathbf{R}^n/\Gamma$, where Γ is a finite group equipped with a representation ρ in $GL(n, \mathbf{R})$ whose fixed point set has codimension at least two. The local isomorphisms $U_1 \longrightarrow U_2$ are the pairs (φ, γ), where $\gamma : \Gamma_1 \longrightarrow \Gamma_2$ is a group homomorphism and $\varphi : \tilde{U}_1 \longrightarrow \tilde{U}_2$ is a diffeomorphism of the saturated open subsets of $\mathbf{R}^n$, over U_1 et U_2, and all the diagrams:

$$\begin{array}{ccc} \tilde{U}_1 & \xrightarrow{\varphi} & \tilde{U}_2 \\ \uparrow{\scriptstyle \rho_1(g)} & & \uparrow{\scriptstyle \rho_2(g)} \\ \tilde{U}_1 & \xrightarrow{\varphi} & \tilde{U}_2 \end{array}$$

are commutative.

Definition 3.6.7 *An orbifold is a topological space modeled on the local structure just described.*

A *differential form* on an orbifold is of course given by the prescription, on each open local chart U, of a differential form (that is a Γ-invariant differential form on $\tilde{U}$).

A *symplectic* form is a 2-form, which, in each local chart, is closed (local notion!) and nondegenerate.

Example. $\Gamma = \mathbf{Z}/m$ acts by rotations on $\mathbf{R}^2$. Then $\mathbf{R}^2/\Gamma$ is an orbifold. The surfaces B, quotients of Seifert manifolds we discussed in I-3.3.1 are orbifolds.

Proposition 3.6.8 *If V_ξ is a regular level of the map $\mu : W \longrightarrow \mathfrak{t}^\star$, then the orbit space $B_\xi = V_\xi/T$ is an orbifold, naturally endowed with a reduced symplectic form σ_ξ.*

Proof. We saw that for any point x, ω_x induces a nondegenerate alternating bilinear form on $\mathrm{T}_x V_\xi/\mathrm{T}_x(T \cdot x) = E_x$ which is a slice at x for the T-action on V_ξ. A neighborhood of $T \cdot x$ in V_ξ has the form $T \times_{T_x} E_x$, and thus a neighborhood of x in B_ξ has the form E_x/T_x. Now T_x is a finite group of isomorphisms of E_x, and we may assume that it preserves a complex structure (calibrated by the form induced from ω_x). The dimension of its fixed point set is both even and positive, thus is ≥ 2.
□

Example. If H is a periodic hamiltonian on a compact 4-dimensional symplectic manifold, its regular levels are Seifert manifolds, with oriented topological surfaces as base spaces (one should check that symplectic orbifolds are oriented). We shall see in chapter IV that all the Seifert manifolds we met in I-3.3 appear in this framework.

Chapter III

Morse theory for hamiltonians

The most famous and spectacular theorem we shall prove in this chapter is the convexity theorem of Atiyah [18] and Guillemin-Sternberg [38] which asserts that the image of a compact symplectic manifold under the moment map of a hamiltonian torus action is a convex polyhedron (*see* 4.2.1).

In the case of the hamiltonian action of a torus, the fixed points are the critical points of a *function*. This is why the study of hamiltonian actions is in some sense easy: one can use Morse theory. We shall not avoid it here, so we shall begin by proving that the functions under consideration have very convenient properties from that point of view (Frankel's theorem [36], here 2.2.1, from which we shall derive, like Atiyah, the basic theorem 3.2.1 announced in the introduction, then the convexity theorem itself and some applications).

Morse theory will also be used, in a rather nice way, in chapter IV. It also plays a role in more difficult problems, for instance in the results analogous to 4.2.1 for noncommutative groups, which for a rather long period were not completely solved. Then F. Kirwan succeeded in [47] with the help of the function $\|\mu\|^2$, which is rather degenerate, but has nevertheless enough of the properties of Morse functions to be handled and give results [46][1].

In this chapter, we thus consider a symplectic manifold endowed with the hamiltonian action of a *torus* T, with moment map $\mu : W \to \mathfrak{t}^\star$.

1 Critical points of almost periodic hamiltonians

1.1 Almost periodic hamiltonians

We know that the fixed points of T correspond to certain critical points of μ. It would be more convenient to have a single function (with values in $\mathbf{R}$) allowing to study the whole T-action. It is not very difficult to find: choose any $X \in \mathfrak{t}$ which generates T in the sense that the one parameter subgroup $\exp(tX)$ is dense in T.

The associated hamiltonian

$$W \xrightarrow{\mu} \mathfrak{t}^\star \longrightarrow (\mathbf{R}X)^\star$$

[1]It seems that there is an alternative way to prove this theorem [29].

has the fixed points of T as critical points.

This leads us to the

Definition 1.1.1 *A function $H : W \to \mathbf{R}$ is an almost periodic hamiltonian if the flow of its symplectic gradient X_H generates a subgroup of the group of all diffeomorphisms of W the closure of which is a torus.*

Remark. The closure of the subgroup generated by any vector is a *commutative* subgroup; it is thus equivalent to require that the subgroup generated by X_H is compact.

1.2 Critical points

Let us now investigate the fixed points of an almost periodic hamiltonian H. Call T the torus "generated by H". The critical points of H are the zeros of X_H, or the fixed points of T. We saw (I-2.2.2) that this set Z is a submanifold in W.

The symplectic form ω is preserved by T and by definition[2]. With the help of an invariant riemannian metric, one derives a calibrated almost complex structure J and a hermitian metric (*see* II-1.5).

Let $z \in Z$. Then the torus T acts on the complex vector space $\mathsf{T}_z W$ preserving J and the hermitian form, that is to say, as a subgroup of $U(n)$. Notice first that all these transformations are diagonalisable: the exponential map $\mathfrak{t} \to T$ is onto and the elements of $\mathfrak{t}$, sitting in $\mathfrak{u}(n)$ (the skew-hermitian matrices) cannot avoid being diagonalisable. As T moreover is commutative, there is a basis of $\mathsf{T}_z W$ in which all the elements of T are diagonal. We can thus write:

$$\mathsf{T}_z W = V_0 \oplus V_1 \oplus \dots \oplus V_k \tag{1}$$

where V_0 is the subspace of fixed points of T, in other words $\mathsf{T}_z Z \subset \mathsf{T}_z W$, and each V_j is T-invariant.

Let us look at how $\exp X_H$ acts on this decomposition: on each V_j, it acts as multiplication by some scalar $\exp(i\lambda_j)$ where λ_j is real ($X_H \in \mathfrak{t} \subset \mathfrak{u}(n)$), and nonzero if $j \neq 0$ as X_H generates the whole torus T. Remark by the way:

Proposition 1.2.1 *Each component of the set Z of zeros of the almost periodic vector field X_H is a symplectic submanifold.*

Actually we just saw that it is almost complex for a structure calibrated by ω.
□

At each fixed point z, the above gives the second derivative of H:

Proposition 1.2.2 *The second derivative of H at the critical point z, is, in the above notations, the hermitian form*

$$\frac{1}{2}\sum_{j=1}^{k} \lambda_j \, |v_j|^2 .$$

[2]This is one more zeugma.

If $v_0^1, \ldots, v_0^r, v_1, \ldots, v_k$ are local coordinates corresponding to the decomposition (1) and if $v_j = q_j + ip_j$, it follows from above that:

$$X_H = \sum_{j=1}^{k} \lambda_j \left(q_j \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial q_j} \right)$$

or

$$dH = \sum_{j=1}^{k} \lambda_j (p_j dp_j + q_j dq_j)$$

and in these coordinates

$$H = \frac{1}{2} \sum_{j=1}^{k} \lambda_j (p_j^2 + q_j^2) + o(|v|^2).$$

□

Also remark that the second derivative is nondegenerate in the direction transverse to the fixed submanifold. This leads us to the next notion.

2 Morse functions (in the sense of Bott)

2.1 Definitions

A function f on a manifold W is called a *Morse function* if its critical set is a submanifold of W, and if its second derivative is a nondegenerate quadratic form in the transverse directions.

Example. If the critical points are isolated, it means that the second derivative is nondegenerate and f is a Morse function (in the sense of Morse!)

One uses a Morse lemma "with parameter":

Proposition 2.1.1 *There exist local coordinates (x, y) in the neighborhood of the point z of the critical submanifold Z in which*

1. *The submanifold Z is described by $y = 0$.*
2. *The function f may be written*

$$f(x, y) = f(z) + q_x(y)$$

where q_x is a quadratic form, nondegenerate in the y variables (transverse to Z).

Proof. All the constructions in the classical proofs of Morse lemma (*see* [14] for example) can be parametrised without any difficulty. □

The index of the second derivative (number of "negative squares") is locally constant along the critical submanifold. Its value on a connected component Z_j will be called the *index* of the critical submanifold Z_j and denoted by $\lambda(Z_j)$.

Thanks to the Morse lemma, one can construct subbundles (all isomorphic) of rank $\lambda(Z_j)$ of the normal bundle to Z_j on which the second derivative is negative. Such a subbundle is called "the" *negative normal bundle.*

2.2 Frankel's theorem

The previous section may be resumed in

Theorem 2.2.1 ([36]) *Let H be an almost periodic hamiltonian on a symplectic manifold (W,ω). Then H is a Morse function, all critical submanifolds of which have an even index.* □

2.3 Perestroïka

In the case of isolated critical points, one can *reconstruct*[3] the manifold with the help of the critical points and their indices (*see* [14]). When he introduced these more general Morse functions, Bott proved that that was also the case for them. Let, for $a \in \mathbf{R}$, $W_a = \{x \in W | f(x) \leq a\}$ and $V_a = \{x \in W | f(x) = a\}$ the level a. The two main results of the theory are

Theorem 2.3.1 *If W is compact and if the interval $[a,b] \subset \mathbf{R}$ contains no critical value of f, then W_b is diffeomorphic to W_a.*

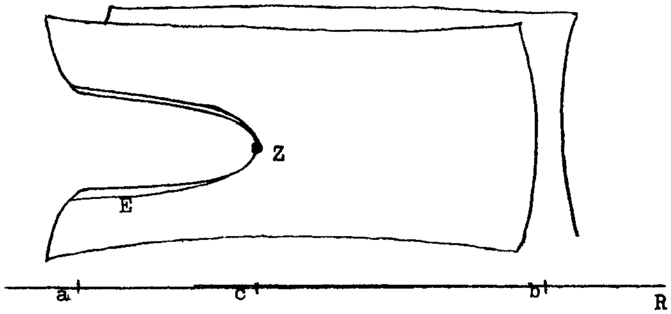

Figure 1

Hence the topology does not change if we are not going through a critical level; here is what happens when we cross one:

Theorem 2.3.2 *If $c \in [a,b]$ is the unique critical value of f in this interval, the homotopy type of W_b is described by the addition to W_a of the negative normal bundle of the critical submanifold at level c.*

In figure 1, the critical submanifold is a point, and its negative normal bundle an interval.

[3]This reconstruction has always been called перестройка by Russian geometers.

Sketch of a proof of theorem 2.3.1 (*see* [14]). Choose a riemannian metric on W, thus getting a gradient vector field grad f, which does not vanish on $f^{-1}[a,b]$. Modify it with the help of a differentiable function ρ on W which takes the value

$$\frac{1}{\|\mathrm{grad}\, f\|^2} \quad \text{on} \quad f^{-1}[a,b]$$

and which vanishes outside a neighborhood, as follows:

$$X = \rho \,\mathrm{grad}\, f.$$

Calling φ_t the flow of X, it is easily checked that φ_{b-a} sends W_b onto W_a. □

Proof of theorem 2.3.2. Let Z be the critical submanifold corresponding to c and $\mathcal{U}_\varepsilon$ be a small enough tubular neighborhood of Z in W. Let us now describe the homotopy type of the pair

$$(W_c, \overset{\circ}{W}_c) \cap \mathcal{U}_\varepsilon$$

Let furthermore N be the normal bundle to Z in W and E the negative normal bundle along Z. Identify $\mathcal{U}_\varepsilon$ to a part of N, thus getting a projection $p : \mathcal{U}_\varepsilon \to Z$, and call A_ε the image of E :

$$A_\varepsilon = \left\{ v \in \mathcal{U}_\varepsilon | v \in E_{p(v)} \right\}$$

and A_ε^- the complement of Z in A_ε (*see* figure 2).

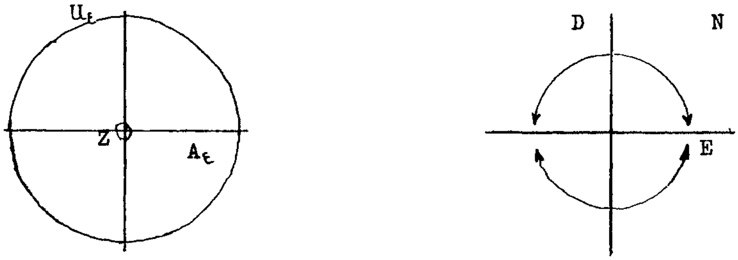

Figure 2

Choosing a supplementary bundle D of E in N, it is now very easy to write down a retraction $N - D \to E$, and it is not more difficult to prove the

Lemma 2.3.3 *For ε small enough, there exists a retraction (by deformation)*

$$(W_c, \overset{\circ}{W}_c) \cap \mathcal{U}_\varepsilon \longrightarrow (A_\varepsilon, A_\varepsilon^-).$$

□

□

3 Connectivity of the fibers of the moment map

Let us begin by stating and proving the key result of [19].

3.1 Connectivity of levels

Theorem 3.1.1 *If f is a nondegenerate function (in the sense of Bott) which has no index 1 or $n-1$ critical submanifold, then it has a unique local minimum and a unique local maximum. Moreover, all of its non empty levels are connected.*

Proof. Use theorem 2.3.2: the homotopy type of W_a can change only by crossing a critical level, in which case it changes by adding to W_a the negative normal bundle of the critical submanifold. If it has index zero, that is if it is a local minimum, we add a connected component. To connect all pieces later on, as W is connected, we must go through a new critical level, for which the sphere bundle of the negative bundle must be connected... but this is impossible except if the index of the submanifold is 1. Thus it is seen that there can be only one local minimum, and, applying the result to $-f$, only one local maximum

Moreover W_a, and for the same reasons $\overline{W - W_a} = \{x | f(x) \geq a\}$ are connected. Assume $V_c = f^{-1}(c)$ is a nonconnected level. Any component of V_c thus defines a nontrivial element in $H_{n-1}(W_c)$ (any coefficients). But this group is zero: indeed, if c is strictly contained between the minimum and the maximum of f, the critical sublmanifolds of critical levels lower than c all have negative normal bundle of dimension $\leq n-2$ which cannot create H_{n-1}. □

3.2 Case of almost periodic hamiltonians

From theorems 2.2.1 et 3.1.1, one easily deduces:

Corollary 3.2.1 *Let (W, ω) be a compact connected symplectic manifold and let $H : W \to \mathbf{R}$ be an almost periodic hamiltonian. All levels $H^{-1}(t)$ of H are empty or connected.* □

4 Application to convexity theorems

4.1 Proof of the Hausdorff-Ginsburg theorem

Recall its statement from the introduction:

Theorem 4.1.1 *Let A be any $n \times n$ complex matrix, and let $\lambda \in \mathbf{R}^n$. The image of*

$$\begin{array}{rccc} f_A : & \mathcal{H}_\lambda & \longrightarrow & \mathbf{C} \\ & X & \longmapsto & tr(AX) \end{array}$$

is a convex set in $\mathbf{C}$.

Begin by proving a direct application of 3.2.1:

Lemma 4.1.2 *If (W,ω) is a symplectic manifold and if f_1 and f_2 are two functions, whose hamiltonian vector fields generate a relatively compact subgroup in the diffeomorphism group of W, then the image of $f_1 \times f_2$ is convex in $\mathbf{R}^2$.*

Proof of the lemma. Let $H : W \to \mathbf{R}$ be any projection of $f_1 \times f_2$. Its hamiltonian vector field generates a relatively compact subgroup, thus H is almost periodic. Its fibers are connected... but all these fibers, when H varies, are all the intersections of the image of $f_1 \times f_2$ with the straight lines in $\mathbf{R}^2$. Hence this image meets any straight line in an interval, thus it is convex. □

Proof of the theorem. Apply the lemma to

$$W = \mathcal{H}_\lambda, \quad f_1 = \operatorname{Re} f_A, \quad f_2 = \operatorname{Im} f_A.$$

The only thing we need to check is that the hamiltonian fields of f_1 and f_2 generate a relatively compact subgroup in the whole diffeomorphism group of $\mathcal{H}_\lambda$... it is true because they are fundamental vector fields for the action of $U(n)$ on $\mathcal{H}_\lambda$, the generated subgroup then being a subgroup of the compact group $U(n)$: indeed if one writes $A = U + iV$ with U and V hermitian, then $f_1(h) = tr(Uh)$, $f_2(h) = tr(Vh)$, and f_1 is a hamiltonian for the vector field $i\underline{U}$, f_2 a hamiltonian for the vector field $i\underline{V}$, as we saw in II-2.1.5. □

4.2 Convexity of the image of the moment map

Imitating the proof of lemma 4.1.2, we get Atiyah's proof of the Atiyah [19] and Guillemin-Sternberg [38] convexity theorem.

Theorem 4.2.1 *Let (W,ω) be a compact connected symplectic manifold, and consider n functions $f_1, \ldots, f_n$ on W. Assume the flows of their hamiltonian vector fields generate a subgroup in the whole diffeomorphism group of W whose closure is a torus. Then the image of W by $f = (f_1, \ldots, f_n)$ is a convex subset of $\mathbf{R}^n$. Moreover if $Z_1, \ldots, Z_N$ are the connected components of the set of common critical points of the f_j's, then $f(Z_j)$ is a point c_j and $f(W)$ is the convex hull of the points c_j.*

Remark. In particular, if μ is the moment map for a hamiltonian action of the torus T^n on W, the choice of a basis of T^n identifies $\mathfrak{t}^\star$ to $\mathbf{R}^n$ and the components of μ to n functions $f_1, \ldots, f_n$ satisfying the hypothesis of the theorem. The image of μ cannot avoid satisfying its conclusion as well: it is a convex polyhedron in $\mathfrak{t}^\star$, the convex hull of the images of the fixed points of T^n.

Proof of the theorem. Following Atiyah, call respectively A_n et B_n the statements:

- *$f^{-1}(t)$ is empty or connected for all $t \in \mathbf{R}^n$*
- *$f(W)$ is convex*

in which n is the number of components functions of f.

First consider the diagram:

$$\begin{array}{ccc} W & \xrightarrow{f=(f_1,\dots,f_{n+1})} & \mathbf{R}^{n+1} \\ & & \downarrow \pi \\ & & \mathbf{R}^n \end{array}$$

where π is any linear projection. The hypothesis applies to $g = \pi \circ f$: any connected closed subgroup of a torus is a torus. As in the proof of 4.1.2, all $f(W) \cap \pi^{-1}(t) = f(g^{-1}(t))$ describe, when π and t vary, the intersections of $f(W)$ with all straight lines in $\mathbf{R}^{n+1}$. By A_n, $g^{-1}(t)$ is empty or connected, likewise its image by f, thus $f(W)$ is convex.

Let us now show by induction on n that A_n holds. Theorem 3.2.1 is exactly A_1. Thus assume A_n to be true. Let $f_1, \dots, f_{n+1} : W \to \mathbf{R}$ be $n+1$ functions satisfying the hypothesis and let $\xi = (\xi_1, \dots, \xi_{n+1}) \in \mathbf{R}^{n+1}$. We want to show that

$$f^{-1}(\xi) = f_1^{-1}(\xi_1) \cap \dots \cap f_{n+1}^{-1}(\xi_{n+1})$$

is empty or connected. By continuity, we may assume ξ is a regular value of f: if f has no regular value, one of the df_i is a linear combination of the others[4], we may forget it and apply A_n.

Then $N = f_1^{-1}(\xi_1) \cap \dots \cap f_n^{-1}(\xi_n)$ is a submanifold of W. By A_n, we know it is connected.

Let us now check that $f_{n+1|N}$ satisfies the hypothesis of theorem 3.1.1. Thus consider its critical points on N, in other words the points of W where (on W) df_{n+1} is a linear combination of the df_i's:

$$df_{n+1} = \sum_{i=1}^{n} \lambda_i df_i.$$

Let $x \in N$ be such a critical point. The λ_i's are constants near x. Put $F = f_{n+1} - \sum \lambda_i f_i$ and let Z be the component of critical points of F (on W) which contains x. The function F is an almost periodic hamiltonian on W as well, to which we cannot refrain from applying Frankel's theorem (2.2.1).

Lemma 4.2.2 *Z and N are transversal at x.*

Assuming the lemma to hold, then $F|_N$ has $Z \cap N$ as a critical submanifold with even index; but, restricted to N, $F = f_{n+1} - \sum \lambda_i \xi_i$, (and thus f_{n+1}) has the same property. We may then apply 3.1.1 to $f_{n+1}|_N$, from which it follows that

$$(f_{n+1}|_N)^{-1}(\xi_{n+1}) = N \cap f^{-1}(\xi_{n+1}) = f^{-1}(\xi)$$

is connected, hence A_{n+1} holds.

[4]Any linear relation $\sum \lambda_j df_j = 0$ is the translation of a relation $\sum \lambda_j X_j = 0$ in the Lie algebra of the torus generated by the f_j's: the λ_j's are constants.

Proof of the lemma. We must show that $df_1(x), \dots, df_n(x)$ are still independant when restricted to Z. Let $X_1, \dots, X_n$ be the hamiltonian vector fields associated to $f_1, \dots, f_n$. They are tangent to Z because the Poisson brackets $\{f_i, F\}$ vanish, but Z is symplectic, thus for all $(\alpha_1, \dots, \alpha_n)$, there exists $Y \in T_x Z$ such that $\omega(\sum \alpha_i X_i, Y) \neq 0$, which may be equally well written $\sum \alpha_i df_i(Y) \neq 0$. □

To finish with the proof of the convexity theorem, we have only to prove the last assertion. Let Z_j be a connected component of the fixed points of the torus T generated by the hamiltonian vector fields $X_1, \dots, X_n$. On Z_j, all the X_i's vanish, thus all f_i's are constant, this explaining why $f(Z_j)$ is a point c_j in $\mathbf{R}^n$. Consider real numbers $\lambda_1, \dots, \lambda_n$ generic enough in order that the one parameter subgroup generated by $X = \sum \lambda_i X_i$ is dense in T and let $\varphi = \sum \lambda_i f_i$ be an associated hamiltonian. Then Z is the critical set of φ, in particular the maximum of φ occurs on Z: hence the restriction of the linear form $\sum \lambda_i \xi_i$ on $\mathbf{R}^n$ to $f(W)$ reaches its maximum at one of the c_j's... and this true for almost any λ_i, thus the image of W is contained in the convex hull of points c_j. We already know that it is convex and it contains c_j by definition, so we are done. □

Remark. Hence the vector fields $X_1, \dots, X_n$ have common zeros (in other words the torus action does have fixed points): it follows from the theorem that Z is not empty, moreover, there may be a lot of those fixed points, as the following corollary shows.

Corollary 4.2.3 *Let T be an n-dimensional torus acting in an effective and hamiltonian way on a compact symplectic manifold (W, ω). The action has at least $n + 1$ fixed points.*

Actually, we noticed in II-3.4.3, as the action is effective, that there exists at least one point at which the moment map μ is submersive, and thus open. Hence the image $\mu(W)$ is a convex polyhedron with non empty interior in $\mathbf{R}^n$, thus has at least $n + 1$ vertices, which cannot avoid being the images of at least as many fixed points. □

This corollary generalises the remark we made for $n = 1$ at the beginning of II-3.4.

4.2.4 Orbits, stabilizers and the image of the moment map. The polyhedra we get as images of moment maps are rather special ones. Applying the equivariant version of Darboux's theorem, we shall now investigate them and especially look at the "equations" of their faces.

Let z be a fixed point of the hamiltonian action of torus T^n on the symplectic manifold W. The group T^n acts in a linear way on $\mathsf{T}_z W$ as in (1) :

$$\mathsf{T}_z W = V_1 \oplus \dots \oplus V_m$$

where each V_i is a complex line on which T^n acts by multiplication:

$$t \cdot v_i = t^{\alpha_i} v_i$$

with $t = (t_1, \ldots, t_n) \in T^n = S^1 \times \ldots \times S^1 \subset \mathbf{C}^n$, $\alpha_i = (a_i^1, \ldots, a_i^n) \in \mathbf{Z}^n$ is a multi-exponent.

Call moreover α_i the element of $\mathfrak{t}^*$ defined by

$$\langle \alpha_i, e_j \rangle = a_j^i$$

where $(e_1, \ldots, e_n)$ is the basis of $\mathfrak{t}$ defined by the above decomposition.

More intrinsically, the T-action on V_i is defined by its *character* $\alpha_i : T \to S^1$; differentiate, transpose, thus getting a map $\mathbf{R} \to \mathfrak{t}^*$, and α_i is the image of 1, the *weight* of the representation.

Proposition 4.2.5 *Let $p = \mu(z) \in \mathfrak{t}^*$. There exists a neighborhood $\mathcal{U}$ of z in W and a neighborhood $\mathcal{V}$ of p in $\mathfrak{t}^*$ such that*

$$\mu(\mathcal{U}) = \mathcal{V} \cap C_p(\alpha_1, \ldots, \alpha_n)$$

where $C_p(\alpha_1, \ldots, \alpha_n)$ denotes the convex cone with vertex p generated by $(\alpha_1, \ldots, \alpha_n)$ in $\mathfrak{t}^*$.

Proof. Apply first Darboux theorem in its equivariant form (II-1.3.3) near the fixed point z: the exponential of any invariant riemannian metric

$$\exp : \mathsf{T}_z W \longrightarrow W$$

conjugates the T-action on W and the linear action on $\mathsf{T}_z W$ described above. Moreover it sends a neighborhood $\mathcal{U}_0$ of 0 in $\mathsf{T}_z W$ on a neighborhood $\mathcal{U}$ of z in W. If $\mathcal{U}_0$ is small enough, one finds a T-equivariant map

$$\psi : (\mathcal{U}_0, 0) \longrightarrow (\mathsf{T}_z W, 0)$$

such that $\psi^* \exp^* \omega = \omega_0$.

We now have two moment maps:

$$\mu' : \mathsf{T}_z W \supset \mathcal{U}_0 \xrightarrow{\exp} \mathcal{U} \xrightarrow{\mu} \mathfrak{t}^*$$

on the one hand, and $\mu_0 : \mathsf{T}_z W \to \mathfrak{t}^*$ associated to the linear action on the other hand. They differ only by a constant vector, we may thus fix $\mu_0(0) = \mu'(0) = \mu(z) = p$ in order that they agree everywhere.

Of course we have:

$$\mu_0(v_1, \ldots, v_n) = p + \frac{1}{2}\Big(\sum_{i=1}^{m} a_1^i \, |v_i|^2, \ldots, \sum_{i=1}^{m} a_n^i \, |v_i|^2\Big)$$

and hence the image of $\mathcal{U}$ by μ (as that of $\mathcal{U}_0$ by μ_0) is the cone $p + \sum_{i=1}^{m} u_i \alpha_i$ ($u_i \geq 0$). □

Assume the action to be effective, thus the interior of the polyhedron P under consideration is non-empty (μ being a submersion at some point).

Corollary 4.2.6 *The hyperplanes which delimit the polyhedron P in $\mathfrak{t}^*$ have integral equations.* □

This statement may be understood, either in coordinates, or, in a more intrinsic way, as follows: any hyperplane equation in $\mathfrak{t}^*$ may be writen as $\langle X, \xi \rangle = a$ for some $X \in \mathfrak{t}$ and $a \in \mathbf{R}$; and in our case we can assume that X generates a *circle* S_X in T ($\exp X = 1$).

Choose one of these hyperplanes, say H, and such an "integral" vector X. The vector field $\underline{X}$ is periodic and is the symplectic gradient of the function

$$x \longmapsto \langle X, \mu(x) \rangle = \tilde{\mu}_X(x).$$

Let $F \subset H$ be the face of P which is contained in H. Consider $\mu^{-1}(F)$. By the very definition, $\tilde{\mu}_X$ is constant on $\mu^{-1}(F)$, that is to say that the circle S_X fixes all points of $\mu^{-1}(F)$, which is thus included in the union of the fixed submanifolds of S_X, and hence is a *symplectic submanifold* of W.

As the T-action on W is effective, T/S_X is a torus with an effective action on $\mu^{-1}(F)$. The open set consisting of principal orbits corresponds to those points x of $\mu^{-1}(F)$ for which $\operatorname{Im} T_x\mu = H$ (it is always included in H). In this case, $(\operatorname{Im} T_x\mu)^\circ = \mathfrak{s}_X$ is indeed the Lie algebra of the stabilizer and in particular $\dim \mu^{-1}(F) = \dim H - 1$. The moment map of the T/S_X-action on $\mu^{-1}(F)$ is obtained from

- the exact sequence of groups

$$S_X \hookrightarrow T \longrightarrow T/S_X$$

- which may be differentiated, giving the exact sequence of Lie algebras

$$0 \to \mathfrak{s}_X \longrightarrow \mathfrak{t} \longrightarrow \mathfrak{t}/\mathfrak{s}_X \to 0$$

- and then transposed to

$$0 \to (\mathfrak{t}/\mathfrak{s}_X)^* \longrightarrow \mathfrak{g}^* \longrightarrow \mathfrak{s}_X^* \to 0$$

which exhibits $(\mathfrak{t}/\mathfrak{s}_X)^*$ as a hyperplane of $\mathfrak{g}^*$ (parallel to H!). The moment map μ, which takes its values in this hyperplane (up to translation), is the moment map for the action of the quotient group.

4.3 Application: a theorem of Schur on hermitian matrices

We shall apply the convexity theorem to the action by conjugation of the torus of diagonal unitary matrices on hermitian matrices. Thanks to II-3.3.10, we already know that the moment map of this action is:

$$\begin{array}{rcl} \mathcal{H}_\lambda & \longrightarrow & \mathbf{R}^n \\ h & \longmapsto & (h_{1,1}, \ldots, h_{n,n}) \end{array}$$

and the convexity theorem says:

Corollary 4.3.1 ([62], and also [49,18,38]) *Let h be a hermitian matrix with spectrum $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{R}^n$ and let $S = \left\{(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}) | \sigma \in \mathfrak{S}_n\right\}$. Then the diagonal of h is in the convex hull $\hat{S}$ of S. Reciprocally, any point of $\hat{S}$ is the diagonal of a hermitian matrix with spectrum λ.*

Proof. The only thing we have to check is that the points in S are the fixed points of T on $\mathcal{H}_\lambda$. It is thus enough to prove that an element h of $\mathcal{H}_\lambda$ is fixed by T if and only if it is diagonal, but this is clear. □

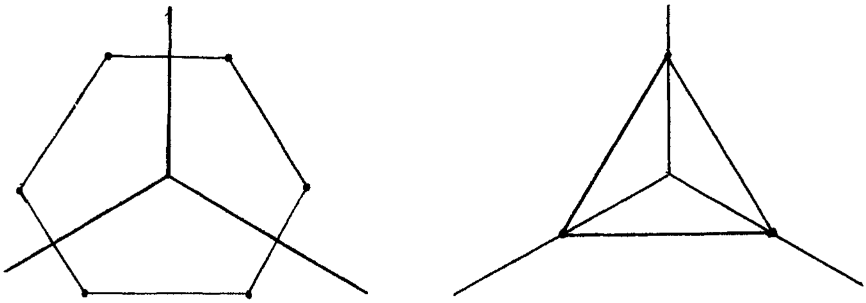

Figure 3

Remark. We have $h_{1,1} + \cdots + h_{n,n} = \operatorname{tr} h = \lambda_1 + \cdots + \lambda_n$ constant on $\mathcal{H}_\lambda$, in particular, its image is included in a hyperplane and the action is not effective.

In the case $n = 3$, figure 3 shows some examples.

In the first picture, the three eigenvalues are distinct, $\mathcal{H}_\lambda$ is the manifold of complete flags in $\mathbf{C}^3$ and the image is a hexagon. In the second one, there are only two distinct eigenvalues, $\mathcal{H}_\lambda$ is a complex projective plane and the image a triangle.

4.4 Application: a theorem of Kushnirenko on monomial equations

We now want to discuss the Kushnirenko theorem, stated in the introduction. Consider a finite set $S \subset \mathbf{Z}^n$ of multiexponents and the system of n equations and n unknowns:

(2) $$\sum_{\alpha \in S} c^j_\alpha z^\alpha = 0$$

where $1 \leq j \leq n$, the parameters c^j_α's are complex and the unknown z sits in the complex torus $(\mathbf{C}^\star)^n$.

Theorem 4.4.1 *The number of solutions of* (2) *for a general enough choice of the coefficients* c^j_α *is*

$$N(S) = n! Vol(\hat{S})$$

where $\hat{S}$ *denotes the convex hull of* S *in* $\mathbf{R}^n$.

I shall only give a sketch of a proof due to Khovanski and Arnold, and explained by Atiyah in [19].

Let $N = \operatorname{card} S$, and let $\alpha_1, \ldots, \alpha_N$ be the elements of S. The complex torus $(\mathbf{C}^*)^n$ acts by

(3) $$t \cdot (z_1, \ldots, z_N) = (t^{\alpha_1} z_1, \ldots, t^{\alpha_N} z_N)$$

on $\mathbf{C}^n$ and by the same formulas on $\mathbf{P}^{N-1}(\mathbf{C})$.

Consider the orbit X_S of the point $[1, \ldots, 1]$ of $\mathbf{P}^{N-1}(\mathbf{C})$. It is parametrised by

$$z_1 = t^{\alpha_1}, \ldots, z_N = t^{\alpha_N} \quad t \in (\mathbf{C}^*)^n.$$

Its closure $\bar{X}_S$ is a complex algebraic (in particular symplectic) submanifold of $\mathbf{P}^{N-1}(\mathbf{C})$.

The (real) torus $T^n \subset (\mathbf{C}^*)^n$ acts on $\bar{X}_S$ with moment map

$$\mu : \bar{X}_S \subset \mathbf{P}^{N-1}(\mathbf{C}) \xrightarrow{\mu_0} \mathbf{R}^N \xrightarrow{\text{proj}} \mathbf{R}^n$$

where

$$\mu_0([z_1, \ldots, z_N]) = \frac{1}{\sum |z_i|^2} (|z_1|^2, \ldots, |z_N|^2)$$

is the moment map for the linear (diagonal) T^N-action and proj the (integral) projection $\mathbf{R}^N \to \mathbf{R}^n$ defined by the transposition of the inclusion $T^n \subset T^N$ (*see* poposition II-3.2.8): the matrix of proj is the matrix of the N column vectors $(\alpha_1, \ldots, \alpha_N)$ of S. In particular the points $\alpha_1, \ldots, \alpha_n$ of $\mathbf{Z}^n \subset \mathbf{R}^n$ are the images of the points $[1, 0, \ldots, 0], \ldots, [0, \ldots, 0, 1]$ of $\mathbf{P}^{N-1}(\mathbf{C})$ by $\text{proj} \circ \mu_0$.

In the case of $\mathbf{P}^{N-1}(\mathbf{C})$ and more generally on any *Kähler* manifold, Atiyah's paper [18] contains a theorem more precise than 4.2.1, which says here that the N points under consideration are in $\bar{X}_S$ and are exactly the fixed points of the T^n-action on $\bar{X}_S$. It is not very hard to check it directly in our case (exercise).
Thus $\mu(\bar{X}_S) = \hat{S}$, the convex hull of these points.

Consider in the same framework the system (2). Its solutions are the intersection points of X_S and of the projective subspace V with equations

$$\sum_{i=1}^{N} c^j_{\alpha_i} z_i = 0 \quad (1 \leq j \leq n).$$

If the c^j_α's are generic, the subspace V has codimension n, is transverse to X_S, and does not meet $\bar{X}_S - X_S$ (which has dimension $< n$). Thus (2) has a finite number of solutions, and this number is the intersection number of V and $\bar{X}_S$ (at least if the $(\alpha - \beta)_{\alpha,\beta \in S}$ generate the lattice $\mathbf{Z}^n$), which is also the *degree* of the algebraic submanifold $\bar{X}_S$.
If ω is the standard symplectic form of $\mathbf{P}^{N-1}(\mathbf{C})$, it is both classical and easily checked that

$$N(S) = \deg \bar{X}_S = \int_{\bar{X}_S} \omega^{\wedge n} = n! \int_{\bar{X}_S} \frac{\omega^{\wedge n}}{n!} = n! \text{Vol}\,(\bar{X}_S)$$

(*see* the calculation of the cohomology ring of $\mathbf{P}^{N-1}(\mathbf{C})$ in chapter V).

But the moment map $\mu : X_S \rightarrow \overset{\circ}{\hat{S}}$ is a fibration, with fiber T^n having volume 1, thus Vol $(\bar{X}_S)$ = Vol $(\hat{S})$ and we get the desired result.

In order to make this proof correct without using Atiyah's theorem which we did not prove, we must be able to reduce to the case where

1. $\bar{X}_S$ is smooth,
2. the $\alpha - \beta$ generate $\mathbf{Z}^n$,

both conditions which we implicitely assumed. Note first that if the $\alpha - \beta$'s generate a lattice of dimension $< n$, the two sides of the equality we want to prove vanish. On the other hand, if they generate a finite index lattice in $\mathbf{Z}^n$, we may reduce to the case where they actually generate $\mathbf{Z}^n$ by using a finite branched covering of $\bar{X}_S$... the same branched covering making $\bar{X}_S$ smooth[5].

[5]Actually $\bar{X}_S$ is the *toric variety* associated with the polyhedron $\hat{S}$ (*see* chapter VI).

Chapter IV

About manifolds of this dimension

My aim in this chapter is to present some of the many recently proven results about 4-dimensional manifolds.

The first classification theorem is due to P. Iglesias [43], who made a list of all the compact symplectic manifolds of dimension 4 which have an $SO(3)$-action which is symplectic[1]. The $SU(2)$ case is handled in almost the same way (and is even easier, *see* appendix A). Then, T. Delzant [31] worked out the case of hamiltonian T^2-actions, as a special case of a rather general result which we shall come back to in chapter VI. In both cases, the inequality in II-(9) is an equality: the dimension of the manifold is as small as possible.

The case of S^1-actions is somewhat different. All S^1-actions on 4-manifolds were classified by Fintushel [35] a long time ago. Of course the symplectic (and even more: hamiltonian) character of the actions under consideration here changes the lanscape, mainly because, as we have already said, one can use Morse theory to study the fixed points. Free (principal) actions were investigated in detail by A. Bouyakoub [25]. It is reasonable to think that fixed point free actions may be handled in *grosso modo* the same pattern. Of course these actions do not have the ghost of a chance to be hamiltonian (all manifolds under consideration are compact). Actually, a very beautiful application of the fiber connectivity theorem (III-3.2.1) and of other tricks was given recently by D. McDuff [55]: in dimension 4, an S^1-action is hamiltonian if and only if it has fixed points. These methods allowed the author to easily give a classification of those 4-manifolds which may be given a symplectic form invariant by an S^1-action with at least one fixed point [21][2].

I shall begin by explaining the proof of the theorem of D. McDuff [55], then I shall explain some of the results just mentionned.

All the S^1-manifolds under consideration are endowed with invariant metrics and calibrated almost complex structures.

[1] It is necessarily hamiltonian according to II-3.2.6.

[2] A longer and less geometrical proof of some of the results in [21] has since written down in a preprint [17].

1 Characterisation of those circle actions which are hamiltonian

1.1 Statement of the theorem

In the paper [36] which we already met in chapter III, Frankel also showed, using Hodge theory, that, on a *Kähler* manifold[3], a circle action preserving the Kähler form is hamiltonian if and only if it has a fixed point. Here is the very beautiful theorem of D. McDuff proven in [55]:

Theorem 1.1.1 *Let (W, ω) be a compact symplectic manifold of dimension 4, endowed with an S^1-action which preserves the symplectic form. Then, either the action has no fixed points and W is the total space of a bundle over S^1, or it has at least one and it is hamiltonian.*

Remark. A. Bouyakoub had shown, in the case where the action is free and in [25], that the manifold is a bundle over the circle.

1.2 Proof

Sketch of the proof.

1. It suffices to investigate the case where the cohomology class of the form $[\omega]$ is integral.

2. In this case, there exists a map h, taking values in S^1, which generalises the moment map and has all the properties of a periodic hamiltonian (from the Morse theoretical viewpoint).

3. Assuming the action not to be hamiltonian, one deduces, by an argument similar to III-3.2.1, that h has no local extrema.

4. By a nice argument using Euler classes of Seifert manifolds (I-3.4), one shows that there are no critical points at all.

Proof.

Proof of 1. The set of all invariant symplectic forms η such that $i_X\eta$ is exact is a closed subset in the set of all invariant symplectic forms. On the other hand, as close to ω as one wants, there is an invariant symplectic form whose cohomology class is rational. Hence we get the existence of an invariant rational symplectic form ω', such that, if $i_X\omega$ is not exact, then $i_X\omega'$ is not either. As ω' is rational, multiply it by a large enough integer N making it integral (it stays symplectic). Of course $i_X(N\omega')$ is as exact as $i_X\omega'$ is. Hence there exists a hamiltonian for ω if and only if there exists one for $N\omega'$. Rename $N\omega'$, giving it the sweet name of ω. □

[3]*see* the definition in II-1.5.3.

Proof of 2. As ω is integral, the closed 1-form $i_X\omega$ is integral as well: if C is any 1-cycle, then φC, got by pushing C with the S^1-action is a 2-cycle, and

$$\int_C i_X\omega = \int_{\varphi C} \omega.$$

For any x_0 (fixed) in W, the function $h(x) = \int_{x_0}^{x} i_X\omega$ is thus well defined modulo $\mathbf{Z}$ and defines a map $h : W \to S^1$ with $h^\star d\theta = i_X\omega$.

The map h has the very same properties as a periodic hamiltonian: restricted to any $W - h^{-1}(\theta)$, it *is* a periodic hamiltonian. In particular, Frankel's theorem (III-2.2.1) applies in this (slightly more general) situation. □

Proof of 3. Suppose that the action is *not* hamiltonian, in particular that h is surjective. From Frankel's theorem, one derives, similarly to the proof of III-3.2.1, that the number of connected components of the fibers of h is constant: it may only change by critical submanifolds of index 1 or $n-1$... which do not appear here, because all have even indices. If h is surjective, there is no local extremum. □

Proof of 4. Let us now use the dimension hypothesis, enumerating possibilities for the critical submanifolds, their dimensions and indices. They are either surfaces or isolated points. The index of any critical surface, being even, may only be 0 or 2, so the critical surfaces are extrema, but we just exclued that case. We still have only isolated critical points, the indices of which cannot be 0 nor 4... in short: the map h has, at most, isolated index 2 critical points.

Near such a point, the action may be linearised as usual:

$$t \cdot (x, y) = (t^p x, t^{-q} y) \tag{1}$$

where $t \in S^1$, $(x, y) \in \mathbf{C}^2$, and p and q are nonnegative integers which we shall assume to be relatively prime (reducing to the case of an effective action).

Consider two regular values a and b of h between which there is only one critical value c. The fibers V_a and V_b are 3-dimensional submanifolds of W, to which the vector field X is tangent without vanishing, so they are Seifert manifolds, exactly as in the case of a genuine hamiltonian.

The trick to end the proof is the following: one shows that the Euler class of V_b is strictly larger than that of V_a, hence, if some critical points actually appear, we get a nondecreasing function... defined on the circle, which is absurd: after a while, one has to come back to V_a. □

It suffices thus to prove the

Lemma 1.2.1 *With notations as in* (1), *we have:*

$$e(V_a) - e(V_b) = -\frac{1}{pq}$$

Proof of the lemma. We study the surgery which allows us to go from V_a to V_b. Consider $D_x^2 \times D_y^2 \subset \mathbf{C}^2$.

$$V_a \cap (D_x^2 \times D_y^2) \sim D_x^2 \times S_y^1$$

using notations I hope to be clear, where one can see a type $\mathbf{Z}/q$ fiber (which I shall consider as exceptional even if $q = 1$). Also

$$V_b \cap (D_x^2 \times D_y^2) \sim S_x^1 \times D_y^2$$

where now $y = 0$ is the exceptional (at least if $p \geq 2$) type $\mathbf{Z}/p$ fiber.

Hence, for any critical point at the level c, described by (1), surgery replaces an exceptional type $\mathbf{Z}/q$ orbit by a type $\mathbf{Z}/p$ orbit.

Let u and v be the smallest positive integers such that

$$pv - qu = 1. \tag{2}$$

Then by definition, (*see* I-3.3.1), the Seifert invariants of the orbit $x = 0$ in V_a are (q, v), those of $y = 0$ in V_b are (p, u). The other possibly exceptional orbits are the same in V_b and V_a, and, calculating as in I-3.4.4:

$$e(V_a) - e(V_b) = -\sum \frac{\beta_i}{m_i} - \frac{v}{q} - (-\sum \frac{\beta_i}{m_i} - \frac{u}{p}) = -\frac{1}{pq}.$$

□

□

2 Symplectic reduction of the regular levels for a periodic hamiltonian

Let H be a periodic hamiltonian on a 4-manifold. We know that the regular levels of H are Seifert manifolds V_a and that enough is kept over from the smooth structure of V_a after forming the quotient to enable us to say that the quotient B_a is a symplectic orbifold (*see* II-3.6.6).

The previous section gave us a hint of what was happening to V_a when going through a critical value of H and we shall now concentrate on the adventures of B_a during the same operation.

2.1 What happens near an extremum

2.1.1 Near an extremum reached on a surface B. Call L the normal bundle of B in W, considered as a complex line bundle over B. Near each point z of B, both the action and the symplectic form are conjugate to the linear ones

$$\begin{array}{rcl} \mathsf{T}_z W & = & \mathsf{T}_z B \;\oplus\; L_z \\ t \cdot (u, v) & = & (u \quad , \quad t^m v) \end{array}$$

where m equals ± 1 because of effectiveness (the sign depending on the kind of extremum we are dealing with). The V_a's (for a close to the critical value) are the circle bundles of L, the action is principal, and the quotient is identified with B.

2.1.2 Near an extremum reached at an isolated point. With the very same method, we find ourselves in $\mathbf{C}^2$, with

$$t \cdot (u, v) = (t^m u, t^n v)$$

where m and n have the same sign and are relatively prime. The hamiltonian may be written $H = \frac{1}{2}(m|u|^2 + n|v|^2)$. The levels close to the critical one are ellipsoids (topologically S^3). If m or $n \geq 2$, there are one or two exceptional orbits. Considering

$$\begin{array}{ccc} V_a & \longrightarrow & \mathbf{P}^1(\mathbf{C}) \\ (u,v) & \longmapsto & [u^n, v^m] \end{array}$$

one sees that the quotient surface of these regular levels is a sphere S^2.

2.2 What happens when going through a critical value

As we are able to say that we are going through it, it follows that the critical level corresponds to some index 2 critical point.

One more very beautiful remark due to Dusa McDuff is that the topology of the quotient surface does not change:

Lemma 2.2.1 **([55])** *Let a and b be two regular values of H. There exists a smooth map*

$$\pi : H^{-1}([a,b]) \longrightarrow B_b$$

such that

1. $\pi_{|V_b} : V_b \to B_b$ *is the quotient map.*
2. *For any regular value t of H, there exists a homeomorphism φ_t making the following diagram*

$$\begin{array}{ccc} V_t & \subset & H^{-1}([a,b]) \\ \downarrow & & \downarrow \pi \\ B_t & \xrightarrow{\varphi_t} & B_b \end{array}$$

commute.

Proof. The lemma is obviously true when there is no critical value between a and b, because we merely need to push using the gradient flow of H to get

$$H^{-1}([a,b]) \xrightarrow{\text{push}} V_b \xrightarrow{\text{proj}} B_b$$

with the desired properties.

Then, it is enough to prove the lemma in the case where there is only one critical value between a and b. For this, we shall try to show, as above, that the topology of V_a does not change (!) and to understand where we are wrong.

Let us thus try to prove that it is possible to retract $H^{-1}([a,b])$ on V_b by pushing along the gradient flow.

If we push by the gradient up to the level b, any point of $H^{-1}([a,b])$ has an image $\varphi(m)$ which is well defined in V_b... except those which fell into the hole,

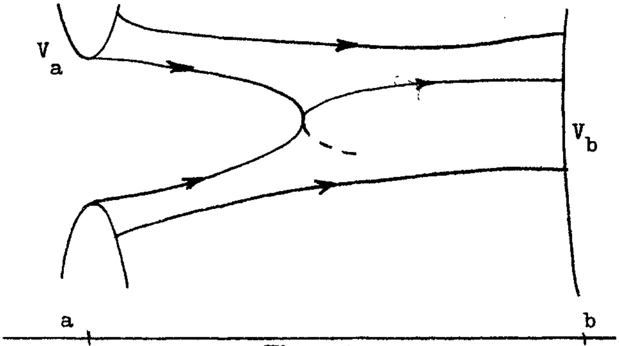

Figure 1

more precisely those of the stable manifold of a critical point at the level under consideration.

If the action is written $(t^p x, t^{-q} y)$ in the neighborhood of a critical point we identified with $\mathbf{C}^2$, the points of the y-axis are those which are forced to stay on the critical level. Associate with them the "points of the x-axis which are on V_b". Of course, this does not define a map into V_b... but notice that all these points lie in the same S^1-orbit in V_b, so that we have a perfectly well defined map into the quotient B_b, which is easily checked to have the required properties. □

Exercise 2.2.2 Let H be a periodic hamiltonian on a compact symplectic manifold W. Consider a neighborhood of a critical submanifold Z with signature $(2, 2n)$, that is, such that, in the directions transverse to Z, the action may be linearised as

$$t \cdot (x, y_1, \ldots, y_n) = (t^p x, t^{-q_1} y_1, \ldots, t^{-q_n} y_n).$$

where p and the q_i's are > 0. Show that there exists, for small enough ε and for all t such that $0 < |t| < \varepsilon$, a commutative diagram of maps

$$\begin{array}{ccc} V_t & \subset & H^{-1}([-\varepsilon, \varepsilon]) \\ \downarrow & & \downarrow \\ B_t & \xrightarrow{\varphi_t} & B_\varepsilon \end{array}$$

2.3 First applications

If the maximum of H is reached on a surface B_{max}, nothing prevents us, in the previous proof, from following the gradient until the end.

2.3.1 If the minimum and the maximum of H are both reached on a surface, we get in the very same way a smooth map

$$\pi : W \longrightarrow B_{max}$$

and homeomorphisms

$$\varphi_t : B_t \longrightarrow B_{max}$$

(as long as t is not the image of an index 2 critical point) making the diagram

$$\begin{array}{ccc} V_t & \subset & W \\ \downarrow & & \downarrow \pi \\ B_t & \overset{\varphi_t}{\rightarrow} & B_{max} \end{array}$$

commute. In particular, B_{min} and B_{max} are two copies of the same surface.

2.3.2 If one of the extrema is reached at an isolated fixed point, then the other one is reached either on a sphere, or at an isolated critical point.

2.3.3 A consequence of a previous lemma. If the action is semifree (*i.e.* without any exceptional orbit) it has at least four fixed points.

This result is actually obvious if the minimum is reached on a surface. Suppose it is reached at an isolated point. As the action is semifree, it may be written near this point: $t \cdot (x, y) = (tx, ty)$. In particular, the Euler class is -1 (it is the Hopf fibration *see* I-3.2.3). There must be an index 2 critical point, otherwise, with an isolated maximum, W would be a sphere S^4, which would prevent it from being symplectic. At the first index 2 critical point, and again because the action is semifree, the integers p and q both equal 1. According to lemma 1.2.1, the Euler class is zero, but it is then impossible that the maximum is the only critical point left, as there is no action of S^1 on S^3 with a zero Euler class. We have shown more precisely that if both extrema are isolated, the action has exactly 4 fixed points (*see* more generally V-3.2.3).

2.4 What happens when there are only two critical values

2.4.1 One possibility is that one is reached at an isolated point, and then the other one is reached on a sphere (and not at a point, as S^4 is still not symplectic). Then all the regular levels are S^3's, with, when looked at from the isolated critical point, an action as $(t^m x, t^n y)\ldots$ but on the other side they are principal bundles over S^2, thus $m = n = \pm 1$, and each bundle over S^2 is the Hopf one.

In this case the manifold W is $\mathbf{P}^2(\mathbf{C})$ and the action that of S^1 by $t \cdot (x, y) = [tx, y, z]$. This is a special case of a result due to T. Delzant [31].

2.4.2 The other possibility is that the two critical values are reached on two copies of the same surface B. Each regular level is the circle bundle of the normal bundle L of B_{min} in W.

The manifold W is obtained by compactifying the vector bundle L, that is by adding a section at infinity, B_{max}.

Look more carefully at this example to be convinced that there actually exists a symplectic manifold with a hamiltonian S^1-action, and which looks like what we have just described.

In order to add a hyperplane at infinity to a vector space E, the easiest thing we can do is to projectify $E \oplus \mathbf{C}$. Here we do this in each fiber, which means that

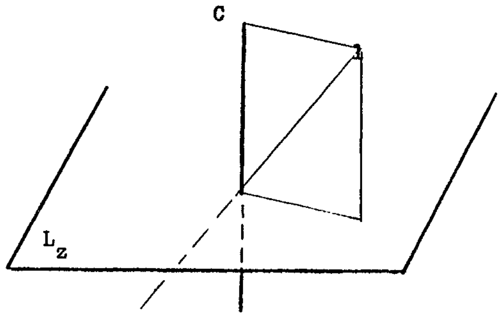

Figure 2

we consider

$$\mathbf{P}(L \oplus \mathbf{1}) = \{(z,l) | z \in B, l \subset L_z \oplus \mathbf{C}\}$$

where $\mathbf{1}$ denotes the trivial complex line bundle over B, and, for any $z \in B$, L_z is the fiber of L at z.

The map $(z,l) \mapsto z$ defines a projection onto B, which fiber is $\mathbf{P}^1(\mathbf{C})$, moreover there are two inclusions (sections) of B in $\mathbf{P}(L \oplus \mathbf{1})$:

- the "zero section" $z \mapsto (z, 0 \oplus \mathbf{C})$,
- and the "section at infinity" $z \mapsto (z, L_z \oplus 0)$.

There is also a natural S^1-action on $\mathbf{P}(L \oplus \mathbf{1})$:

$$t \cdot (z, [v,u]) = (z, [tv, u])$$

(in "homogeneous coordinates") which admits both sections $v = 0$ and $u = 0$ as fixed points submanifolds. We must still construct a *symplectic form*. There are a lot of different possible ways to do this but the one I shall give was suggested to me by P. Iglesias (and is analogous to a classical construction we shall meet again in V-3.1.1).

If L is trivialisable, $\mathbf{P}(L \oplus \mathbf{1}) \cong B \times \mathbf{P}^1(\mathbf{C})$ and one can take a product symplectic form. Suppose then that L is *not* trivialisable. Endow it with a hermitian metric, which will allow us to consider its unit circle bundle $\pi : S(L) \to B$ with the circle acting by rotations in the fibers as an S^1-principal bundle. Let $e \in \mathbf{Z}$ be its Euler class. Fix a volume form η on the oriented surface B. One can find a unique real number λ such that $e = \lambda \int_B \eta$, in such a way that λ is zero exactly when the bundles L and $S(L)$ are trivialisable: we thus assume that $\lambda \neq 0$.

We shall see in chapter V (exercise 2.4.3) that it is then possible to find an invariant 1-form α on $S(L)$ such that

- if X is the fundamental vector field of the S^1-action, $i_X \alpha \equiv 1$.
- $d\alpha = \pi^*(\lambda\eta)$.

Consider now $\mathbf{P}^1(\mathbf{C})$ with the S^1-action $t \cdot [a, b] = [a, tb]$ and the usual symplectic form ω_0. Choose a moment map H for this action and put:

$$\tilde{\omega} = d(H\alpha) + \omega_0.$$

This is a closed 2-form on $S(L) \times \mathbf{P}^1(\mathbf{C})$.

Look at the kernel of $\tilde{\omega}$. Consider a point $(u, v) \in S(L) \times \mathbf{P}^1(\mathbf{C})$, and two tangent vectors

$$(U, V), (U', V') \in \mathsf{T}_u S(L) \times \mathsf{T}_v \mathbf{P}^1(\mathbf{C}).$$

Then (U, V) is in the kernel if and only if

$$\forall (U', V'), \quad (d(H\alpha) + \omega_0)((U, V), (U', V')) = 0.$$

But

$$\begin{aligned}(d(H\alpha) + \omega_0)((U, V), (U', V')) &= dH(V)\alpha(U') - dH(V')\alpha(U) \\ &\quad + H(v) d\alpha(U, U') + \omega_0(V, V') \\ &= (-\alpha(U) dH + i_V \omega_0)(V') \\ &\quad + (dH(V)\alpha + H(v) i_U d\alpha)(U').\end{aligned}$$

Hence (U, V) is in $\operatorname{Ker} \tilde{\omega}$ if and only if

$$(3) \qquad \left\{ \begin{array}{rcl} -\alpha(U) dH + i_V \omega_0 &=& 0 \\ dH(V)\alpha + H(v) i_U d\alpha &=& 0 \end{array} \right.$$

That is: $i_V \omega_0 = i_{\alpha(U)X} \omega_0$ calling X the fundamental vector field on $\mathbf{P}^1(\mathbf{C})$ as well, for the first equation, which is equivalent to $V = \alpha(U)X$; while the second equation in (3) gives, as $dH(X) = 0$, $H(v) i_U d\alpha = 0$.

If we were careful enough to have chosen a function H which does not vanish on $\mathbf{P}^1(\mathbf{C})$, the second equation gives $i_U d\alpha = 0$.

From the definition of α (where η is a volume form, in particular nondegenerate), we know that the kernel of $d\alpha$ is generated by the vector field X hence $U = \mu X$ and $V = \alpha(U)X = \mu X$ for some real number μ.

We are thus led to consider the diagonal S^1-action on $S(L) \times \mathbf{P}^1(\mathbf{C})$, the fundamental vector field of which is $X = (X, X)$.

Exercise 2.4.3 Show that $i_X \tilde{\omega} = 0$ and that $\mathcal{L}_X \tilde{\omega} = 0$ (a formally identical assertion will be proved in V-3.1.1).

One deduces that $\tilde{\omega}$ descends to the quotient as a *symplectic* form ω on $S(L) \times_{S^1} \mathbf{P}^1(\mathbf{C})$ (as long as H does not vanish).

Exercise 2.4.4 Show that

1. The map
$$\begin{array}{ccc} S(L)\times \mathbf{P}^1(\mathbf{C}) & \longrightarrow & \mathbf{P}(L\oplus \mathbf{1}) \\ ((z,v),[a,b]) & \longmapsto & (z,[av,b]) \end{array}$$
induces a diffeomorphism
$$S(L)\times_{S^1}\mathbf{P}^1(\mathbf{C})\longrightarrow \mathbf{P}(L\oplus \mathbf{1})$$

2. The symplectic form ω so defined on $\mathbf{P}(L\oplus\mathbf{1})$ is invariant by the action we considered above $(t\cdot(z,[v,u])=(z,[tv,u]))$.

3 Blowing up fixed points; creation of index 2 critical points

3.1 Blowing up 0 in $\mathbf{C}^2$

Consider the space one gets by replacing, in $\mathbf{C}^2$, the point 0 by the set of all straight lines through it.

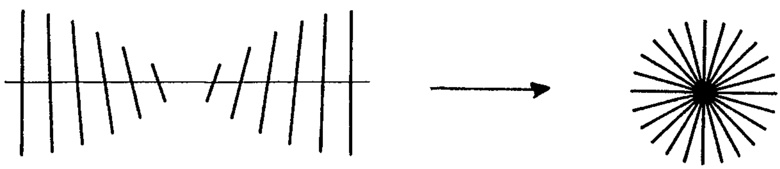

Figure 3

In other words consider
$$\widetilde{\mathbf{C}^2}=\{(v,l)|v\in l\}\subset \mathbf{C}^2\times\mathbf{P}^1(\mathbf{C}).$$

This is the *tautological* line bundle over $\mathbf{P}^1(\mathbf{C})$, but we are looking at it from the viewpoint of its projection onto $\mathbf{C}^2$.
$$\begin{array}{ccc} \widetilde{\mathbf{C}^2} & \xrightarrow{\pi} & \mathbf{C}^2 \\ (v,l) & \longmapsto & v \end{array}$$

If $v\neq 0$, $\pi^{-1}(v)$ contains only one point: v defines one line; but if $v=0$, there is a whole $\mathbf{P}^1(\mathbf{C})$ above. It is called the *exceptional divisor.*

The space $\widetilde{\mathbf{C}^2}$ is a complex manifold: it may be described by the algebraic equation
$$\widetilde{\mathbf{C}^2}=\{((x,y),[a,b])|ay-bx=0\}$$
it is also easy to find local coordinates for it

- if $b \neq 0$, put $u = \frac{a}{b}$, then

$$\begin{array}{ccc} \mathbf{C}^2 & \longrightarrow & \widetilde{\mathbf{C}^2} \\ (y,u) & \longmapsto & ((uy,y),[u,1]) \end{array}$$

is a diffeomorphism onto its image.

- one would write just as easily local coordinates where $a \neq 0$.

It has symplectic forms, induced by those of $\mathbf{C}^2 \times \mathbf{P}^1(\mathbf{C})$. It is moreover possible to construct symplectic forms inducing the standard form of $\mathbf{C}^2$ outside any given disc centered at 0 (more generally *see* [54]).

This allows, *via* Darboux, to blow up any point in any 4-dimensional symplectic manifold and to obtain new symplectic manifolds.

The following exercise gives a way to construct such a symplectic form.

Exercise 3.1.1 Let ω_1 be the standard form on $\mathbf{C}^2$, ω_2 that on $\mathbf{P}^1(\mathbf{C})$. Consider

$$\widetilde{\mathbf{C}^2} \overset{j}{\hookrightarrow} \mathbf{C}^2 \times \mathbf{P}^1(\mathbf{C})$$

and define, for any $r > 0$, $\Omega_r = j^\star(\omega_1 + r\omega_2)$. Check that it is a symplectic form on $\widetilde{\mathbf{C}^2}$, which gives volume r to the exceptional divisor.

Let D_r be the radius r ball in $\mathbf{C}^2$ ($|x|^2 + |y|^2 \leq r$) and let

$$\begin{array}{cccc} f_r : & \mathbf{C}^2 - D_r & \longrightarrow & \mathbf{C}^2 - 0 \\ & v & \longmapsto & \left(\frac{\|v\|-r}{\|v\|}\right)v \end{array}$$

be the radial diffeomorphism. Identify $\mathbf{C}^2 - 0$ with $\widetilde{\mathbf{C}^2} - \pi^{-1}(0)$ (*via* π). Show that

$$f_r^\star \Omega_r = \omega_{1|\mathbf{C}^2 - D_r}.$$

Here is now, in the spirit of 2.2, an other way to describe the blow-up. Let us come back to the situation of exercise 2.2.2 and suppose $W = \mathbf{C}^3$, $Z = 0$, $n = 2$, and $p = q_1 = q_2 = 1$; in other words, we want to study the S^1-action on $\mathbf{C}^3$ by

$$t \cdot (x,y,z) = (tx, \bar{t}y, \bar{t}z).$$

Choose the hamiltonian H such that $H(0) = 0$.

Exercise 3.1.2

1. Show that the regular levels

$$V_- = H^{-1}(-\varepsilon^2), \quad V_+ = H^{-1}(\varepsilon^2)$$

are respectively diffeomorphic to $S^3 \times \mathbf{C}$ and to $S^1 \times \mathbf{C}^2$.

2. Write down the S^1-action in these models and check that the quotients are $B_- = \widetilde{\mathbf{C}^2}$ et $B_+ = \mathbf{C}^2$.

Using the gradient of H as in 2.2 and exercise 2.2.2 we get a map

$$\mathbf{C}^3 \longrightarrow B_+$$

which

- may be restricted to $V_- \to B_+$ and descends to quotient as a map $B_- \to B_+$ which is precisely the blow up $\widetilde{\mathbf{C}^2} \to \mathbf{C}^2$.
- may be restricted in $V_+ \to B_+$ which is the S^1-quotient.

Exercise 3.1.3

1. Show that the quotient B_0 of the critical level V_0 is *smooth* and symplectic.
2. Consider the reduced symplectic form on $B_t = H^{-1}(t)/S^1$ (with $t < 0$ in such a way that $B_t = \widetilde{\mathbf{C}^2}$). What is the volume of the exceptional divisor?

When t varies staying < 0, we thus get a one parameter family of copies of $\widetilde{\mathbf{C}^2}$ with an exceptional divisor having smaller and smaller volume... until it disappears for $t \geq 0$ where $\widetilde{\mathbf{C}^2}$ becomes a $\mathbf{C}^2$. This remark together with the results of exercises 2.2.2, 3.1.2 and 3.1.3 are particular cases of results to be discussed in V-4.

3.2 Extension of an S^1-action

Consider the circle action near a fixed point, as linearised by Frankel, that is:

$$t \cdot (x, y) = (t^p x, t^{-q} y)$$

for some relatively prime integers p, q to be made precise. Blow up the fixed point $(0,0) \in \mathbf{C}^2$. It is clear that the S^1-action can be extended to $\tilde{\mathbf{C}}^2$ by

$$t \cdot ((x, y), [a, b]) = ((t^p x, t^{-q} y), [t^p a, t^{-q} b]).$$

Remarks:

- Outside the exceptional divisor ($x = y = 0$), nothing new happens.
- In the case where the blown up fixed point lay on a fixed surface, assuming that $p = 1$ et $q = 0$ (so the hamiltonian action is effective), the *strict transform* of the fixed points surface $x = 0$ is a fixed points surface as well (the $((0, y), [0, 1])$).

  We added the new fixed point $((0,0), [1,0])$, near which, in local coordinates (x, v) (where $v = b/a$ and $y = vx$), the action is written

  $$t \cdot (x, v) = (tx, \bar{t}v)$$

  where we see that it is an *index* 2 fixed point.

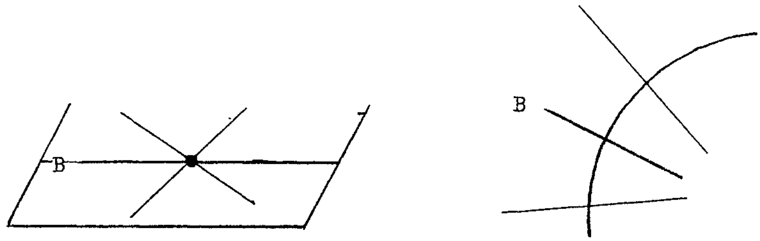

Figure 4

- If $p > 0$ and $q > 0$, the blown up fixed point is an isolated index 2 critical point for H, which we shall call *critical point* or *type* (p, q) *fixed point.* The blow-up replaces it with two fixed points: $((0,0),[0,1])$ of type $(p+q, q)$ and $((0,0),[1,0])$ of type $(p, p+q)$ (as is easily seen writing the action in standard local coordinates) connected by a gradient manifold[4] (the exceptional divisor).

- If p and q have opposite signs, then the blown up fixed point was an isolated extremum of the hamiltonian. Changing signs if necessary, the action may be written:
$$t \cdot (x, y) = (t^m x, t^n y)$$
where m and n are > 0 and relatively prime (the fixed point is a minimum with these signs). The action on $\tilde{\mathbf{C}}^2$ is:
$$t \cdot ((x, y), [a, b]) = ((t^m x, t^n y), [t^m a, t^n b]).$$
  - When $m = n = 1$, the whole exceptional divisor consists of fixed points; we replaced an isolated fixed point with a fixed $\mathbf{P}^1$.
  - Otherwise, one may assume that $m > n > 0$. Then the point $((0,0),[0,1])$ is a minimum, near which the hamiltonian action is written $(t^{m-n}u, t^n y)$, and the point $((0,0),[1,0])$ is an isolated index 2 critical point of type $(n, m-n)$.

3.3 Gradient manifolds and exceptional divisors

The blowing up of an index 2 critical point gives us the following situation:

- Before blowing up (figure 5) the gradient manifolds are the y-axis, stable manifold of the fixed point, the points of which have stabilizer $\mathbf{Z}/q$, and the x-axis, unstable manifold of the fixed point, the points of which have stabilizer $\mathbf{Z}/p$.

[4]That is: the closure of the stable or unstable manifold of some critical point.

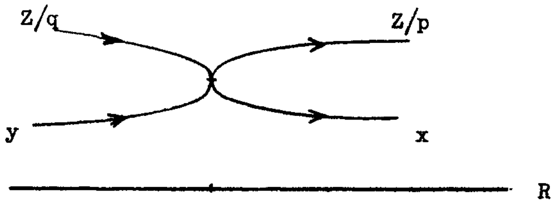

Figure 5

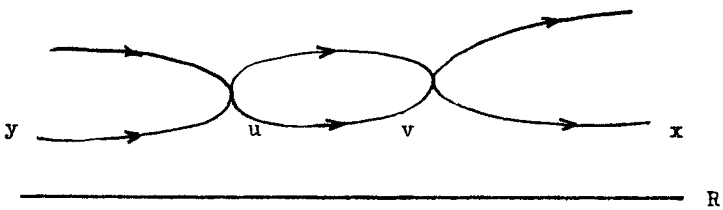

Figure 6

- After having blown up (figure 6) we have two critical points, the u-axis (on the x side) and the v-axis (on the y side) are the two discs which form the exceptional divisor. The latter is actually the gradient manifold connecting the two fixed points.

This is of course a general fact:

Proposition 3.3.1 *The gradient manifolds of H joining an index 2 critical point to another index 2 critical point or to a point of a critical surface are all symplectic (actually almost complex with tangent space generated by X and JX) smooth spheres S^2.* □

4 4-manifolds with periodic hamiltonians

We now have introduced all we need in order to understand the topology of the manifolds under consideration: they will be organised (*reconstructed*) around the S^2 spheres which are the gradient manifolds.

4.1 Description

4.1.1 In 2.4, we constructed examples of compact symplectic manifolds endowed with periodic hamiltonians: all the $\mathbf{P}(L \oplus \mathbf{1})$ (with two surfaces as the set of fixed points), and $\mathbf{P}^2(\mathbf{C})$ with an isolated fixed point and a fixed sphere.

On $\mathbf{P}^2(\mathbf{C})$, there are plenty of other periodic hamiltonians, for example the most famous Morse function in the world (because it is investigated at the beginning of [14]) is one of them. This is the function

$$H([x, y, z]) = \frac{m\,|x|^2 + n\,|y|^2}{|x|^2 + |y|^2 + |z|^2};$$

it is a hamiltonian for the S^1-action

$$t \cdot [x, y, z] = [t^m x, t^n y, z]$$

which is effective as soon as m and n are relatively prime, and which has three fixed points when m and n are distinct and nonzero.

Similarly, when the base space B of the bundle L is a sphere, we may combine an S^1-action on B and the action in the fibers to get more complicated actions on $\mathbf{P}(L \oplus \mathbf{1})$. Consider the "Hirzebruch surfaces"

$$W = \left\{([a, b], [x, y, z]) \in \mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^2(\mathbf{C}) | a^k y = b^k x\right\}$$

with the induced symplectic form and the S^1-action

$$t \cdot ([a, b], [x, y, z]) = ([t^m a, b][t^{mk} x, y, t^n z])$$

Exercise 4.1.2

1. The projection $\mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^2(\mathbf{C}) \to \mathbf{P}^1(\mathbf{C})$ actually makes W a $\mathbf{P}^1(\mathbf{C})$-bundle on $\mathbf{P}^1(\mathbf{C})$ (from chapter V (appendix B), it will be clear that W may be called $\mathbf{P}(\mathcal{O}(k) \oplus \mathbf{1})$).
2. Show that in general the action just described has four isolated fixed points; the case where $m = 0$ and $n = \pm 1$ is a special case of 2.4.2.

Now that we have got all these examples, which we shall call the *basic examples*, we are able to state the results we want to prove.

Theorem 4.1.3 *Let W be a closed 4-manifold, endowed with an S^1-action whose fundamental vector field is denoted X and on which there exists a symplectic form ω and a function H satisfying $i_X\omega = dH$ (in other words such that the action is hamiltonian for some symplectic form). Then there exists a finite sequence*

$$W_0 \xleftarrow{\pi_1} W_1 \longleftarrow \cdots \xleftarrow{\pi_m} W_m = W$$

where each W_i is a manifold having the same properties, each π_i is the blow-up of a fixed point of the S^1-action on W_{i-1}, and W_0 is one of the basic examples.

From the method of the proof, one can also derive the two following amusing results:

Theorem 4.1.4 *Any (oriented) Seifert manifold (with oriented basis) is a level of some periodic hamiltonian on a compact 4-dimensional symplectic manifold.*

Theorem 4.1.5 *Let W be a compact symplectic manifold of dimension 4 endowed with a hamiltonian S^1-action. Then there exists a hamiltonian T^2-action extending the given action (for some way of embedding S^1 as a subgroup in T^2) if and only if both following conditions hold:*

1. *one of the regular levels has a sphere S^2 as symplectic reduction*
2. *no level of H meets more than two gradient manifolds.*

The T^2-action is then unique.

Remark. Both conditions are necessary as soon as the image of W under the moment map of the T^2-action is a convex polyhedron (*see* III-4.2.1).

In the sequel of this chapter, I shall prove theorems 4.1.3 and 4.1.5 in the special case where the S^1-action has two surfaces of fixed points, and theorem 4.1.4. With a little care, the very same method makes possible to prove the results in complete generality, as the reader may convince herself or himself by looking at [21].

The fundamental construction is that of *plumbing.*

5 Plumbing

The submanifolds on which the extrema are reached, and the gradient manifolds as well, are (almost complex) surfaces with normal bundles which describe their neighborhoods in W. Plumbing was created precisely to handle this kind of situation (*see* for example [59,42]).

5.1 Plumbing of disc bundles

Let $E_1 \to B_1$ and $E_2 \to B_2$ be the disc bundles of two complex line bundles on surfaces. Choose two discs D_1 and D_2 *resp.* in B_1 and B_2. We know that the bundles E_1 and E_2 are trivialisable, over D_1 (*resp.* D_2) as well as over $\overline{B_1 - D_1}$ (*resp.* $\overline{B_2 - D_2}$). Choose trivialisations

$$\begin{aligned} E_{1|D_1} &\cong D_1 \times B^2 \cong B^2 \times B^2 \\ E_{2|D_2} &\cong D_2 \times B^2 \cong B^2 \times B^2 \end{aligned}$$

using which it is possible to glue... inverting factors.

There is a small problem in defining the structure of a smooth manifold on the space we get this way: one has to smooth corners, but this is not very difficult (*see* [42]).

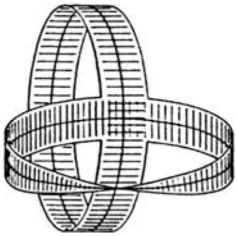

Figure 7

We thus get an almost complex 4-manifold with boundary, which contains the two surfaces B_1 and B_2, with normal bundles E_1 and E_2, the diffeomorphism type of which is well defined by that of B_1, B_2 and by the bundles.

Nothing prevents us from iterating, even reiterating the processus. For example, any graph defines a manifold said to be *plumbed along the graph*: the vertices correspond to the bundles to be plumbed and one has to plumb two such bundles when there is an edge connecting the two corresponding vertices.

5.2 Equivariant plumbing along star-shaped graphs

When the graph has the shape of a star, it is possible to endow the manifold plumbed along it with an S^1-action.

5.2.1 S^1-action on pieces. If the basis B of the disc bundle E is not a sphere, endow it with the trivial S^1-action, then the circle acts by rotation in each fiber of E.

Otherwise, if B is an S^2, write

$$S^2 = S^2_+ \cup S^2_-$$

where

$$S^2_+ = \{z \in \mathbf{C} \mid |z| \leq 1\},$$

$$S^2_- = \{z \in \mathbf{C} \cup \infty \mid |z| \geq 1\} \cong \{z \in \mathbf{C} \mid |z| \leq 1\}$$

and

$$E = (S^2_+ \times B^2) \cup_\varphi (S^2_- \times B^2)$$

with

$$\begin{array}{cccc} \varphi : & S^1 \times B^2 & \longrightarrow & S^1 \times B^2 \\ & (z,u) & \longmapsto & (\bar{z}, \varphi_z(u)) \end{array}$$

where $S^1 = S^2_+ \cap S^2_-$ and $\varphi_z \in SO(2)$ for all z. The isomorphism class of the bundle is well defined by the homotopy class of

$$\begin{array}{ccc} z & \mapsto & \varphi_z \\ S^1 & \to & SO(2) \end{array}$$

which we may assume to have the form $\varphi_z(u) = \bar{z}^k u$ for some integer k. The bundle obtained this way will be denoted $E(k)$.

Exercise 5.2.2 Define in an analogous way a bundle $E(k)$ on any surface B. The boundary is endowed with a principal S^1-action (rotation in the fibers) the quotient space of which is B. Show that the Euler class of this action (in the sense[5] of chapter I) is k.

Let S^1 act on each piece by

$$t \cdot (z, u) = (t^{m_1} z, t^{n_1} u)$$

on $S^2_+ \times B^2$ and by

$$t \cdot (z, u) = (t^{m_2} z, t^{n_2} u)$$

on $S^2_- \times B^2$.

Both actions must agree *via* the gluing map φ, that is: $m_2 = -m_1$ and $n_2 = -km_1 + n_1$.

In conclusion:

1. The bundle $E(k)$ is defined by the gluing map $\varphi : S^1 \times S^1 \to S^1 \times S^1$, the matrix of which (on π_1) is:

$$\varphi_\star = \begin{pmatrix} -1 & 0 \\ -k & 1 \end{pmatrix}$$

2. If the S^1-action is given, on the S^2_+ side by the integers (m_1, n_1), it will be given, on the S^2_- side by (m_2, n_2) with

$$\begin{pmatrix} m_2 \\ n_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -k & 1 \end{pmatrix} \begin{pmatrix} m_1 \\ n_1 \end{pmatrix}$$

5.2.3 Weighted star-shaped graphs. We shall actually describe stars as if they were comets; this might be silly from the astronomical viewpoint, but it will nevertheless be very convenient (figure 8).

1. The head of the comet represents a bundle $E(-b)$ over a genus g surface B.

2. The other vertices represent bundles $E(-b_{i,j})$ over spheres.

[5] We shall meet other versions of Euler classes in the next chapter.

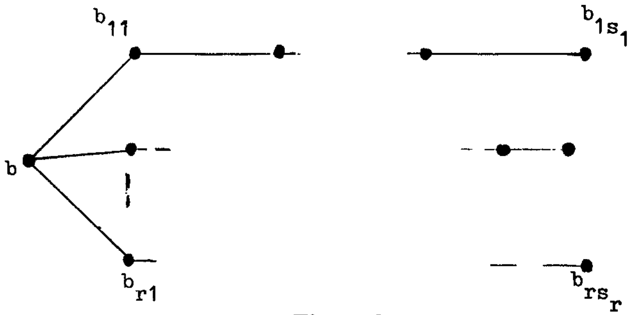

Figure 8

3. If two vertices are connected by an edge, we plumb the corresponding bundles.

The weighted graph $\Gamma = (g|b,(b_{1,1},\ldots,b_{1,s_1}),\ldots,(b_{r,1},\ldots,b_{r,s_r}))$ thus describes an almost complex manifold with boundary $W(\Gamma)$. It is endowed with an S^1-action which is perfectly well defined once we prescribe it to be trivial on the surface associated to the head, and to be the complex multiplication in the fibers of the corresponding bundle: it is determined on all pieces by the above method.

The boundary $\partial W(\Gamma)$ is thus endowed with a fixed point free S^1-action, in other words it is a Seifert manifold, and we shall begin with its identification:

Theorem 5.2.4 *If $\Gamma = (g|b,(b_{1,1},\ldots,b_{1,s_1}),\ldots,(b_{r,1},\ldots,b_{r,s_r}))$ and if all the reduced fractions of the continued fractions*

$$b_{j,1} - \cfrac{1}{b_{j,2} - \cfrac{1}{\ddots}} = [b_{j,1},\ldots,b_{j,s_j}] = \frac{\alpha_j}{\alpha_j - \beta_j}$$

have a non-zero rational value, then

$$\partial W(\Gamma) = (g|\beta,(\alpha_1,\beta_1),\ldots,(\alpha_r,\beta_r)).$$

Remarks.

1. The invariants so obtained are not necessarily normalised.
2. The determination of β in terms of b and $b_{i,j}$ is left as an exercise.
3. A good general reference on continued fractions is the book by Hardy and Wright [40]. The fact that we have signs does not make things very different. If we require that all b_i's are ≥ 2, any rational number $\frac{n}{q} > 1$ can be written $[b_1,\ldots,b_s]$ as one can easily show using a slight modification of the usual Euclidean algorithm. The reduced fractions we get here are nonincreasing, and with self-evident notations (I hope), we have

$$N_n D_{n+1} - N_{n+1} D_n = 1$$

for any n, which one is allowed to find more convenient than the analogous relation for continued fractions with "plus" signs.

The proof of the theorem will use the following lemma:

Lemma 5.2.5 *If* $\Gamma = (b_1, \ldots, b_s)$, *then the manifold* $\partial W(\Gamma)$ *is the lens space* $L(p,q)$ *where* $p/q = [b_1, \ldots, b_s]$.

Remark. This is an assertion about the manifold $\partial W(\Gamma)$ and not about any S^1-action.

Proof. We must glue the bundles $E(-b_1), \ldots, E(-b_s)$. One checks by an easy induction on 's that this is the same as gluing the two solid tori $S_{1+} \times S^1$ (where S_{1+} is the first hemisphere in the first sphere S_1) and $S_{s-} \times S^1$ by the map

$$\partial S_{1+} \times S^1 \to \partial S_{s-} \times S^1$$

with matrix

$$A_s = \begin{pmatrix} -1 & 0 \\ b_s & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ b_1 & 1 \end{pmatrix}.$$

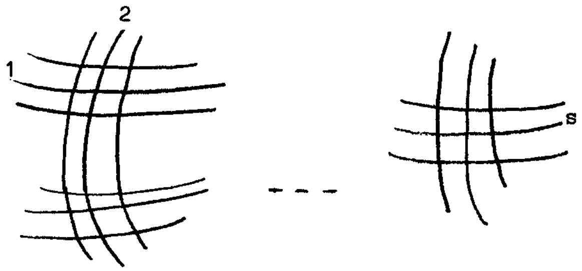

Figure 9

Using reduced fractions, we can write

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_s = \begin{pmatrix} b_s & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} N_s & D_s \\ -N_{s-1} & -D_{s-1} \end{pmatrix}$$

where

$$\frac{N_i}{D_i} = [b_1, \ldots, b_i]$$

as checked by an easy induction.

Then

$$A_s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} N_s & D_s \\ -N_{s-1} & -D_{s-1} \end{pmatrix} = \begin{pmatrix} -N_{s-1} & -D_{s-1} \\ N_s & D_s \end{pmatrix}$$

To use less cumbersome notation (there is a limit even to the joy of LaTeX), put: $N_s = p$, $D_s = q$ (the same as in the statement), $N_{s-1} = u$, $D_{s-1} = v$ (hence $qu - pv = 1$), in such a way that we have

$$A_s = \begin{pmatrix} -u & -v \\ p & q \end{pmatrix}$$

and that the gluing map is:

$$\begin{array}{ccc} \partial S_{1+} \times S^1 & \longrightarrow & \partial S_{s-} \times S^1 \\ (a,z) & \longmapsto & (a^{-u}z^{-v}, a^p z^q). \end{array}$$

We recognise the lens space $L(p,q)$ (*see* I-3.3.5). □

Proof of the theorem. Apply the lemma in each branch of the graph:

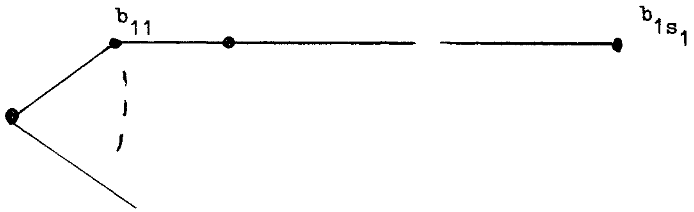

Figure 10

We know, as the action is trivial at the "head", that we have to glue the lens space $L(p,q)$ with the trivial action in the fibers over S_{1+}: the action on $S_{1+} \times B^2$ is

$$t \cdot (a,z) = (ta, z).$$

At the end of this branch, we know that the S^1-action is

$$t \cdot (a,z) = (t^{m'_s}a, t^{n'_s}z)$$

with

$$\begin{pmatrix} m'_s \\ n'_s \end{pmatrix} = A_s \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -u & -v \\ p & q \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -u \\ p \end{pmatrix} = \begin{pmatrix} -N_{s-1} \\ N_s \end{pmatrix}$$

The exceptional orbit is the central ($a = 0$) one, the order of its stabilizer is $n'_s = N_s = p$. To get the Seifert invariants (p, β) it is enough, up to normalisation, to find some β such that $m'_s\beta = +1 \mod n'_s$. But $m'_s = -N_{s-1} = -u$, we thus have to solve $-u\beta = 1 \mod p$. As we have $pu - qv = 1$, $\beta = p - q$ works. □

5.3 Periodic hamiltonians and plumbing

Proposition 5.3.1 *If the minimum of H is attained on a surface and if a is a regular level, then W_a is obtained by plumbing along a star-shaped graph.*

The "head" of the comet represents the minimum surface, it is weighted by the Euler class of its normal bundle; the other surfaces plumbed are the gradient manifolds (spheres S^2), the edges thus corresponding to the index 2 critical points.
□

5.3.2 Relations between the weights of the fixed points and those of the plumbing. Consider a branch of the graph, with vertices weighted by $(b_1, \dots, b_s)$. The edge connecting the two vertices weighted b_i and b_{i+1} corresponds to a type (p,q) fixed point. Thus, from the proof of lemma 1.2.1 and from theorem 5.2.4, we know that

$$\begin{aligned}[b_1, \dots, b_i] &= \tfrac{q}{q-v} \\ [b_1, \dots, b_{i+1}] &= \tfrac{p}{p-u}\end{aligned}$$

where $pv - qu = 1$, (q,v) are the Seifert invariants of the levels just before the critical point and (p,u) those just after.

To simplify put $v' = q - v$, $u' = p - u$, hence $qu' - pv' = 1$. If we subtract this relation from

$$N_{i-1}v' - qD_{i-1} = 1$$

we see that $b_{i+1} = \frac{p+N_{i-1}}{q} \left(= \frac{u'+D_{i-1}}{v'}\right)$

5.3.3 Specific properties of the weights in these graphs.

- *a priori*, branches have weights in $\mathbf{Z}$. As here *all* the regular levels are *Seifert manifolds*, all the reduced fractions of the continued fraction associated with any branch are in $\mathbf{Q} - 0$.

  It follows for instance that we never have two consecutive 1's nor a sequence $2, 1, 2$, etc...

- The weights on the branches are nonnegative: this is easily seen by induction looking at the construction of b_{i+1} above.

5.4 Description of W up to the maximum: finishing graphs

Consider now a regular level a very close to the maximum (in the sense that there is no critical value of H between a and the maximum). If we assume that the maximum is reached on the surface, say B, we know that V_a is a *principal* bundle over B (2.1). In other words, there is no exceptional orbit in V_a and W_a is obtained by plumbing along a weighted graph

$$\Gamma = (g|b, \dots, (b_{i,1}, \dots, b_{i,s_i}), \dots 1 \le i \le r)$$

where

$$[b_{i,1},\ldots,b_{i,s_i}] = \frac{1}{\gamma_i}$$

is the inverse of some integer for all i.

We say that such a weighted graph Γ is *finishing*. One derives, from the associated plumbed manifold, a closed 4-manifold by simply gluing the disc bundle of a complex line bundle over B (the one with Euler class $b - \sum \beta_i$ as one may check).

Example. If $\Gamma = (g|b)$, W_a is a disc bundle of the line bundle with Euler class $-b$ on the genus g surface B. Glue a disc bundle of the line bundle with Euler class b to get the manifold $\mathbf{P}(L \oplus \mathbf{1})$ (with $e(L) = \pm b$).

Proof of theorem 4.1.3. Let us begin with the investigation of the relationship between plumbing and blowing up. We saw that to blow up a type $(t^p x, t^{-q} y)$ fixed point was to replace it with two fixed points, of types $(t^{p+q} x, t^{-q} y)$ and $(t^p x, t^{-(p+q)} y)$.

The effect on the continued fraction is

$$[\ldots b_n, b_{n+1}, b_{n+2}\ldots] = [\ldots b_n, b_{n+1} + 1, 1, b_{n+2} + 1, \ldots]$$

or

$$b - \frac{1}{a - \frac{1}{x}} = b + 1 - \frac{1}{1 - \frac{1}{a+1-\frac{1}{x}}}$$

Contract in the graph all the vertices which are weighted by 1. By the remarks in 5.3.3, after a finite number of such operations, all branches will be weighted by integers ≥ 2... or will have disappeared. Now the following exercise is easy

Exercise 5.4.1 $b_1, \ldots, b_n \geq 2 \Leftrightarrow [b_1, \ldots, b_n] > 1$.

Thus, the branches which are weighted by integers ≥ 2 can only give inverses of integers, and so all branches disappear... that is to say that the graph obtained by contraction represents a manifold $(g|b)$ and that the manifold W is obtained from it by blowing up points. □

This proof is in some sense algorithmic. The algorithm is especially obvious in the *semifree* case, in which there are no levels containing exceptional orbits. Thus:

$$\forall i, j \quad [b_{i,1}, \ldots, b_{i,j}] = \frac{1}{\gamma_{i,j}}$$

It is easy to solve the equation "$[b_1, \ldots, b_j]$ inverse of an integer for all j": one finds of course first that $b_1 = 1$, then

$$b_1 - \frac{1}{b_2} = \frac{b_1 b_2 - 1}{b_2} \Rightarrow b_2 = 2$$

and by induction

$$b_1 = 1, b_2 = \ldots = b_j = 2, \quad [b_1, \ldots, b_j] = \frac{1}{j}$$

thus any branch can be contracted to give a branch having the same weights, but being shorter, in other words, all blow-ups are done at points of the minimum:

Theorem 5.4.2 *In the case of a semifree action, each π_i is the blow-up at some points of the surface on which the minimum is reached in W^{i-1}.*

□

Using these methods, it is not harder to deduce a proof of theorem 4.1.4:

Proof. We write the Seifert invariants as a continued fraction as theorem 5.2.4 allows us to do and we express all these manifolds as boundaries of manifolds plumbed along graphs which are weighted by integers which we may even assume to be ≥ 2. The only thing we must check is that such a graph may be extended as a *finishing* graph. This verification is done in each branch. From the arithmetical viewpoint, we have to show:

$$\forall b_1, \ldots, b_n, \quad \exists b_{n+1}, \ldots, b_{n+m} \| [b_1, \ldots, b_{n+m}] = \frac{1}{a}$$

for some integer a.

Thus fix an integer $a \geq 1$ and write:

$$\begin{cases} u &= aN_{n-1} &- D_{n-1} \\ v &= aN_n &- D_n \end{cases}$$

Hence

$$uN_n - vN_{n-1} = N_{n-1}D_n - N_nD_{n-1} = 1$$

$$uD_n - vD_{n-1} = a \geq 1$$

and

$$\frac{\frac{u}{v}N_n - N_{n-1}}{\frac{u}{v}D_n - D_{n-1}} = \frac{1}{a}$$

Developping u/v as a continued fraction $[b_{n+1}, \ldots, b_{n+m}]$ is then sufficient to give a solution. □

We still have to prove theorem 4.1.5, but we shall only do it here in the case where the extrema are reached along surfaces, for then it can be written:

Proposition 5.4.3 *Let*

$$\Gamma = (g \mid b, (b_{1,1}, \ldots, b_{1,s_1}), \ldots, (b_{r,1}, \ldots, b_{1,s_r}))$$

a finishing graph. The compact symplectic manifold $W(\Gamma)$ may be given a hamiltonian T^2-action extending the S^1-action if and only if $g = 0$ and $r \leq 2$.

Proof. Let us first prove that the condition is sufficient. If the number r of branches of Γ is 0, W is some $\mathbf{P}(L\oplus\mathbf{1}) \to \mathbf{P}^1(\mathbf{C})$ and it is easy to extend the given hamiltonian action. Otherwise we saw that W is the result of a sequence of blow-ups beginning with some $\mathbf{P}(L \oplus \mathbf{1})$ (4.1.3). As the star has at most two branches, we began by blowing up (at most) two points in B_{min}... but the latter is a $\mathbf{P}^1(\mathbf{C})$ with the standard action: there are actually two fixed points (for T^2) which we may blow up in such a way that the action may be extended. We can then proceed. Note that the S^1-action determines the T^2-action. □

As regular levels for a hamiltonian in this situation, we thus find all oriented Seifert manifolds with basis S^2 and at most two exceptional orbits. It is well known (*see* [59]) that these are exactly the lens spaces.

Corollary 5.4.4 *A 3-manifold is a regular level of a periodic hamiltonian induced by a hamiltonian T^2-action on a compact symplectic 4-manifold if and only if it is a lens space.* □

Remark. The S^1-manifolds which are plumbed along the finishing graphs we consider do have invariant symplectic forms, as they are obtained by blowing up symplectic manifolds. Thus, theorem 4.1.3 is actually a classification theorem for the S^1-equivariant diffeomorphism type of compact 4-manifolds having invariant symplectic forms (with hamiltonian)... but it is not a classification theorem for compact symplectic manifolds etc... as we did not take care of the invariant symplectic forms on these manifolds.

A Appendix: compact symplectic $SU(2)$-manifolds of dimension 4

In this appendix, we shall give a classification of the compact symplectic 4-manifolds with an action of $SU(2)$. It is, in essence, due to P. Iglesias [43] who actually considerd $SO(3)$ but the $SU(2)$ case is a little easier and a little less technical, that is why we chose it (and decided to present it with a lot of exercises). As in the S^1 case, we shall not worry about of the symplectic forms, but we shall concentrate instead on the manifolds with action.

We consider $SU(2)$ as the group of matrices

$$SU(2) = \left\{ \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \mid |x|^2 + |y|^2 = 1 \right\}$$

from which description it is easy to recognise that $SU(2)$ is S^3.

A.1 A list of examples

A.1.1 $SU(2)$-actions on complex line bundles over $\mathbf{P}^1(\mathbf{C})$. Consider $E(m) = S^3 \times_{S^1} \mathbf{C}$ where S^1 acts on $S^3 \times \mathbf{C}$ by

$$t \cdot ((x, y), u) = ((tx, ty), t^m u).$$

This is a complex line bundle over $\mathbf{P}^1(\mathbf{C})$ by the projection

$$[(x,y),u] \longmapsto [x,y].$$

Exercise A.1.2 Show that the bundle $E(m)$ is actually the same we defined in 5.2.1 (in particular, its Euler class is m, *see* V-B).

The group $SU(2)$ acts in a natural way on $S^3 \times \mathbf{C}$ with the help of its linear action on $\mathbf{C}^2$ by

$$g \cdot ((x,y),u) = (g \cdot (x,y),u).$$

Exercise A.1.3 Show that

1. this action descends to quotient, thus defining an $SU(2)$-action on $E(m)$,
2. the type of the points in the zero section is S^1, that of the other points is $\mathbf{Z}/m$,
3. the action is effective if and only if m is odd. If m is even, it defines an effective $SO(3)$-action.

A.1.4 Compactification. We compactify the fibers of $E(m)$ and get $\mathbf{P}(E(m)\oplus\mathbf{1})$ (*see* 2.4.2).

Exercise A.1.5 Make S^1 act on $S^3 \times \mathbf{P}^1(\mathbf{C})$ by

$$t \cdot ((x,y),[u,v]) = ((tx,ty),[t^m u,v]).$$

1. Show that the quotient may be identified with $\mathbf{P}(E(m) \oplus \mathbf{1})$.
2. Endow $\mathbf{P}(E(m)\oplus\mathbf{1})$ with an $SU(2)$-action extending the one defined on $E(m)$ above. Describe its orbit types.
3. Imitating the construction in 2.4.2, construct an $SU(2)$-invariant symplectic form on $\mathbf{P}(E(m) \oplus \mathbf{1})$.

For any odd m, we thus have an effective action on $\mathbf{P}(E(m)\oplus\mathbf{1})$ and an invariant symplectic form. Actually the diffeomorphism type of $\mathbf{P}(E(m) \oplus \mathbf{1})$ depends only on m being odd or even (the action really depends on $|m|$ as those who solved the exercises know: it is the order of the principal stablizer). The diffeomorphism type is rather simple:

Exercise A.1.6

1. Show that the algebraic submanifold (Hirzebruch surface, *see* 4.1.1)

$$X_m = \left\{([a,b],[x,y,z]) \in \mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^2(\mathbf{C}) \mid a^m y - b^m x = 0\right\}$$

may be identified with $\mathbf{P}(E(m) \oplus \mathbf{1})$. Deduce, for example, that $\mathbf{P}(E(1) \oplus \mathbf{1})$ is $\mathbf{P}^2(\mathbf{C})$ blown up at one point.

2. In the local chart $b \neq 0$ in $\mathbf{P}^1(\mathbf{C})$, put $u = a/b$ ($\in \mathbf{C}$) and $f(t) = \frac{1}{1+t^{-m}}$. Show that

$$\begin{array}{rccc} \varphi: & \mathbf{C} \times \mathbf{P}^1(\mathbf{C}) & \longrightarrow & X_m \\ & (u,[v,w]) & \longmapsto & ([u,1],[f(|u|)\bar{u}^m v, f(|u|)v, w]) \end{array}$$

defines a trivialisation of $X_m \to \mathbf{P}^1(\mathbf{C})$ in this local chart. Similarly write a trivialisation ψ in "the other" local chart ($a \neq 0$), in such a way that

$$\psi^{-1} \circ \varphi(u,[v,w]) = \left(u, \left[\frac{\bar{u}^m}{|u|^{-m}} v, w\right]\right).$$

Thus X_m is obtained by gluing two copies of $D^2 \times \mathbf{P}^1(\mathbf{C})$ by

$$\begin{array}{rccc} g: & S^1 \times \mathbf{P}^1(\mathbf{C}) & \longrightarrow & S^1 \times \mathbf{P}^1(\mathbf{C}) \\ & (z,[v,w]) & \longmapsto & (z,[\bar{z}^m v, w]). \end{array}$$

3. Show that $z \mapsto ([v,w] \mapsto [\bar{z}^m v, w])$ defines a loop in $SO(3)$, homotopic to the constant loop when m is odd.

4. Applying the result to $m-n$, show that X_m is diffeomorphic to X_n if $m \equiv n \mod 2$. Deduce that the manifold $\mathbf{P}(E(m) \oplus \mathbf{1})$ is diffeomorphic to $S^2 \times S^2$ when m is even and to $\mathbf{P}^2(\mathbf{C})$ blown up at one point when m is odd.

In addition to this list, we are able to make $SU(2)$ act linearly on $\mathbf{P}^2(\mathbf{C})$, as a subgroup of $U(3)$. We shall now prove that (up to a small hypothesis), the list exhausts all symplectic $SU(2)$-actions on compact symplectic 4-manifolds.

A.2 Classification

Suppose thus that W is compact, symplectic, 4-dimensional, and endowed with a symplectic and effective $SU(2)$-action. The latter is hamiltonian, $SU(2)$ being semisimple (*see* II-3.2.6), hence we have a moment map

$$\mu : W \longrightarrow \mathfrak{su}(2)^*$$

which we assume to be *submersive at at least one point* in W.

A.2.1 Orbit types. From II-3.4.7, we know that the principal stabilizer is both discrete and a subgroup of the maximal torus

$$S^1 = \left\{ \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \mid z \in S^1 \right\}.$$

Hence it is a cyclic group $\mathbf{Z}/m$. Principal orbits have dimension 3.

Exercise A.2.2

1. Show that μ is submersive at any point of an exceptional orbit and that the image of any dimension 3 orbit both has dimension 2 and is contained in a coadjoint orbit in $\mathfrak{su}(2)^\star$.
2. Deduce that the stabilizer of any dimension 3 orbit is a $\mathbf{Z}/q$. Using the slice theorem, deduce that $q = m$ and that there is *no exceptional orbit*.

Let us now investigate the *singular* orbits.

Exercise A.2.3 Considering a subgroup S^1 of $SU(2)$, show that, as W is compact, there always exist singular orbits.

As there is no dimension 2 closed subgroup in $SU(2)$, singular orbits must have dimension 0 or 2, and stabilizer $SU(2)$, S^1 or $O(2)$.

Exercise A.2.4 If the stabilizer of a point x contains a circle as a *proper subgroup*, then $\mu(x) = 0$.

Now, we show that $O(2)$ cannot appear as a stabilizer, and more generally that there is no dimension 2 orbit sent to 0 by μ.

Consider a dimension 2 orbit. Its stabilizer is a subgroup of $SU(2)$, has dimension 1 and contains a circle. According to the slice theorem, a neighborhood of such an orbit has the form:

$$SU(2) \times_{SO(2)} \mathbf{R}^2$$

where $SO(2)$ acts linearly on $\mathbf{R}^2$. As we noticed above when looking at examples, for the action to be effective on W, it is necessary that $SO(2)$ acts on $\mathbf{R}^2$ by $v \mapsto t^m v$ with *m odd*.

Suppose now that the orbit under consideration is sent to 0. In particular, $\mathsf{T}_x\mu$ vanishes on the tangent space to the orbit and

$$\mathsf{T}_x(SU(2)\cdot x) \subset \operatorname{Ker} \mathsf{T}_x\mu = \{\mathsf{T}_x(SU(2)\cdot x)\}^\circ$$

for this reason the orbit is *isotropic* (in fact lagrangian: its dimension is half that of W). In particular, according to II-1.3.4, it has a tubular neighborhood of the form $\mathsf{T}^\star S^2$ (up to a covering map). The next exercise might be easier after reading chapter V:

Exercise A.2.5 The bundle $T^*S^2 \to S^2$ *cannot* be written $S^3 \times_{S^1} \mathbf{R}^2$ with S^1 acting on $\mathbf{R}^2$ with an odd weight m (m is actually the Euler class of the bundle over S^2 and that of T^*S^2 is ± 2).

Thus, there is no lagrangian orbit.

Remark. On the contrary, this situation does occur in the $SO(3)$ case. For example if $SO(3)$ acts as a subgroup (the real part) of $SU(3) \subset U(3)$ on $\mathbf{P}^2(\mathbf{C})$, the real part $\mathbf{P}^2(\mathbf{R})$ is a lagrangian orbit [43]. It is this small difference which makes the investigation of $SU(2)$-actions a little easier than that of $SO(3)$-actions.

In short:

Proposition A.2.6 *Let W be a compact symplectic 4-manifold endowed with an effective symplectic action of $SU(2)$. Let $\mu : W \to \mathfrak{su}(2)^*$ be the moment map of this action. Suppose that μ is submersive at least at one point. Then:*

1. *the principal stabilizer is a $\mathbf{Z}/m$ and there are no exceptional orbits,*
2. *if $\mu(x) = 0$ then x is an isolated fixed point,*
3. *there must be singular orbits, which are either isolated points, or symplectic spheres S^2 (with stabilizer S^1).*

The only thing left to check is that the dimension 2 orbits are actually symplectic. The image by μ of such an orbit is the sphere containing $\mu(x)$ (its coadjoint orbit), and $T_x\mu$ is injective when restricted to the orbit. Thus $\mu_{|\text{Orbit}}$ is a diffeomorphism, and we know that it must be symplectic (*see* II-3.4). □

A.2.7 Classification. Remark now:

Lemma A.2.8 $f = \frac{1}{2}\|\mu\|^2 : W \to [0, +\infty[$ *is a quotient map for the $SU(2)$-action on W.*

Proof. We know, thanks to the slice theorem and to the study of stabilizers above that the quotient will be a dimension 1 manifold with boundary, the ends corresponding to the singular orbits. It is thus a compact interval if we assume W to be connected and there are exactly two singular orbits.

The function f descends to the quotient as

$$g : W/SU(2) \longrightarrow [0, +\infty[$$

a smooth function from one interval to another. Computing its derivative:

$$T_x f(Y) = \mu(x) \cdot T_x\mu(Y).$$

If x is a fixed point with $\mu(x) \neq 0$, then

$$Y \in \operatorname{Ker} T_x f \Leftrightarrow T_x\mu(Y) \in \mu(x)^\perp = T_{\mu(x)}S^2$$

where S^2 is the coadjoint orbit of $\mu(x)$. Hence $T_x f$ vanishes exactly when x is on a singular orbit, and g is strictly monotone, which shows that f may be identified with the quotient map. □

Let $[a,b] = f(W)$ and let $c \in]a,b[$. Then $f^{-1}(b)$ is a symplectic sphere S^2, $f^{-1}([c,b])$ is a tubular neighborhood of this sphere, given by the slice theorem and so it is an $E(m)$ ($|m|$, odd, being the order of the principal stabilizer). There are only two possibilities

- Either $a > 0$, then $f^{-1}([a,c])$ is an $E(n)$ as well, necessarily $|m| = |n|$ and actually $m = -n$ in order to be able to glue, so that W is a $\mathbf{P}(E(m) \oplus \mathbf{1})$.
- Or $a = 0$, then $f^{-1}(a)$ is an isolated fixed point near which the $SU(2)$-action is the usual one on $\mathbf{C}^2$. In particular, $|m| = 1$ and W is obtained by gluing a disc to $E(\pm 1)$ so that W is a copy of $\mathbf{P}^2(\mathbf{C})$.

B Appendix: 4-dimensional S^1-manifolds with no invariant symplectic form (examples)

There are a lot of 4-manifolds with S^1-action which do not fit into the framework considered in the present chapter; to be convinced of this, the reader just need have a look at the "list" of S^1-manifolds of dimension 4 given by Fintushel [35].

In this appendix, I shall use the very same methods as in the rest of the chapter (in particular plumbing) to construct some of these examples. The manifolds constructed will be compact, oriented, and endowed with S^1-actions with as manifold of fixed points: some isolated points, possibly a sphere S^2 on the one hand and a connected oriented genus ≥ 1 surface on the other hand. It is certain that no S^1-manifold with these properties can have an invariant symplectic form: having fixed points, the action would be hamiltonian (because of 1.1.1), the hamiltonian would have as extrema an isolated point or a sphere S^2 and a genus ≥ 1 surface, but this is forbidden by 2.3.2.

B.1 Equivariant plumbing on non-simply connected graphs, examples

Nothing obliges us to use only star-shaped graphs to construct 4-manifolds with boundary. Actually, it is obvious that the existence of cycles in the graph will make it more difficult to construct S^1-actions.

B.1.1 Examples with polygonal graphs. We shall only consider here the case where all plumbed surfaces are spheres (the reader will easily generalise these examples following her or his taste). Assume that, over the hemisphere S_{1+}, and with the notations of 5.2.1, the action is given by

$$t \cdot (z,a) = (t^{m_1} z, t^{n_1} a)$$

and that over S_{s-} it is given by

$$t \cdot (z,a) = (t^{m'_s} a, t^{n'_s} z)$$

where

$$\begin{pmatrix} m'_s \\ n'_s \end{pmatrix} = A_s \begin{pmatrix} m_1 \\ n_1 \end{pmatrix}.$$

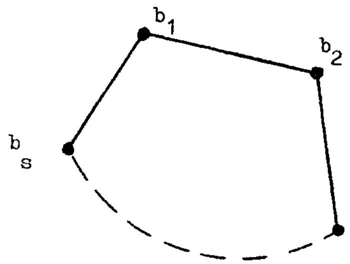

Figure 11

For the actions to agree when we plumb S_{s-} and S_{1+}, it is necessary and sufficient that

$$\begin{pmatrix} n'_s \\ m'_s \end{pmatrix} = \begin{pmatrix} m_1 \\ n_1 \end{pmatrix}$$

that is to say $\begin{pmatrix} m_1 \\ n_1 \end{pmatrix}$ is a non-zero vector fixed by the matrix

$$B_s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_s = \begin{pmatrix} b_s & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_1 & 1 \\ -1 & 0 \end{pmatrix}.$$

In particular, B_s must have 1 as eigenvalue (in other words, this element of $SL(2,\mathbf{Z})$ is *parabolic*).

Reciprocally, let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a parabolic matrix ($a, b, c, d \in \mathbf{Z}$, $ad - bc = 1$, and $a + d = 2$). We can always find integers $s, b_1, \ldots, b_s$ such that $[b_1, \ldots, b_{s-1}] = (-c)/(-d)$ and $[b_1, \ldots, b_s] = a/b$: look first for a development as a continued fraction of $(-c)/(-d)$ and extend it by $b_s = \frac{a+N_{s-2}}{c} = \frac{b+D_{s-2}}{d}$ (no hypothesis is made on the integers b_i). We then have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} N_s & D_s \\ -N_{s-1} & -D_{s-1} \end{pmatrix} = \begin{pmatrix} b_s & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $\begin{pmatrix} m \\ n \end{pmatrix}$ be a non-zero vector, fixed by B, and suppose m and n are relatively prime. The manifold which we get by plumbing along the graph in figure 11 may be endowed with an (effective) S^1-action, determined by m and n on the sphere number 1.

Remark. If $B_s = \pm\mathrm{Id}$, there are enough fixed vectors to define a T^2-action.

Examples.

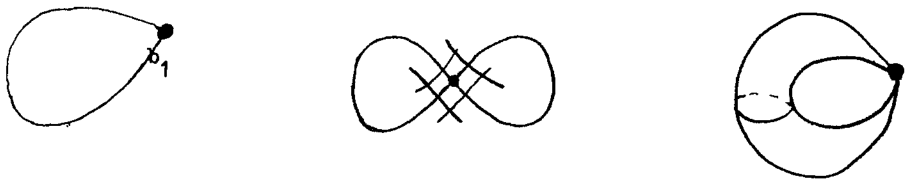

Figure 12

- The case where $s = 1$. The method works for $s = 1$ as soon as the matrix $\begin{pmatrix} b_1 & 1 \\ -1 & 0 \end{pmatrix}$ is parabolic, that is if $b_1 = 2$. The central sphere in the strip has a double point in the plumbed manifold (in figure 12, is shown a real picture of the plumbing and the central sphere with its double point as well).

- The case where $s = 2$. The trace of B_2 is $b_1 b_2 - 2$ and we then have to solve $b_1 b_2 = 4$. Up to a change in the numbering, we find $b_1 = b_2 = \pm 2$ and also $b_1 = \pm 4$ and $b_2 = \pm 1$.

- More generally, remark that $(b_1, \ldots, b_s) = (2, \ldots, 2)$ *always* give a solution. By induction, we actually have

$$\underbrace{[2, \ldots, 2]}_{j \text{ times}} = \frac{j+1}{j}$$

and thus

$$B_s = \begin{pmatrix} N_s & D_s \\ -N_{s-1} & -D_{s-1} \end{pmatrix} = \begin{pmatrix} s+1 & s \\ -s & -(s-1) \end{pmatrix}$$

which is parabolic, the fixed vector being $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. In the same way, one checks (paying attention to signs) that

$$\underbrace{[-2, \ldots, -2]}_{j \text{ times}} = \frac{(-1)^j (j+1)}{(-1)^{j+1} j}$$

and that the unimodular matrix under consideration is thus

$$B_s = (-1)^s \begin{pmatrix} s+1 & -s \\ s & -(s-1) \end{pmatrix}$$

which is parabolic if and only if s is even, the fixed vector being then $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

- The case where $s = 3$. In addition to the solution we just gave, we see for example that all triangles which have two vertices weighted by -1 are solutions, as

$$\begin{pmatrix} -m & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & m-1 \\ 0 & 1 \end{pmatrix}$$

is parabolic, with fixed vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We see that for $m = 1$ the plumbed manifold even has a T^2-action, but this is not very surprising, as the plumbed manifold may then be identified to a neighborhood of the union of the three coordinate lines $x = 0$, $y = 0$, and $z = 0$ in $\mathbf{P}^2(\mathbf{C})$: the tubular neighborhood of any projective line in $\mathbf{P}^2(\mathbf{C})$ is an $E(1)$ (exercise).

- The case where $s = 4$. We find analogously that all quadrilaterals weighted by $(0, a, 0, b)$ are solutions as the matrix

$$\begin{pmatrix} b & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -(a+b) \\ 0 & 1 \end{pmatrix}$$

is parabolic, with fixed vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. If $b = -a$, there is again a T^2-action and notice that the plumbed manifold can be identified with a neighborhood of the union of two fibers and of the two sections in $\mathbf{P}(E(a) \oplus \mathbf{1}) \to \mathbf{P}^1(\mathbf{C})$, thus the T^2-action. These two remarks about torus actions will be made more precise in chapter VI.

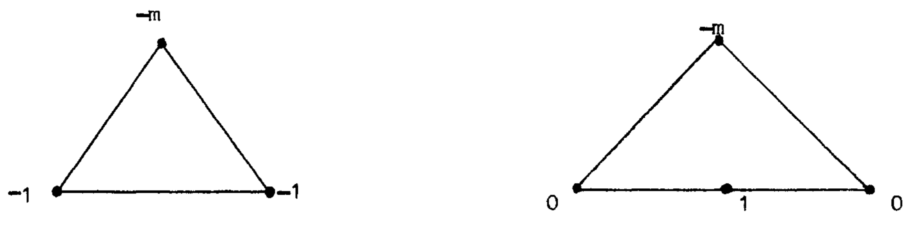

Figure 13

- We may of course blow up points, for example the triangle $(-m, -1, -1)$ gives a quadrilateral $(-m, 0, 1, 0)$ once we blow up the point corresponding to the edge connecting the two -1 vertices (*see* figure 13).

B.1.2 Fixed surfaces. Consider a polygonal weighted graph $(b_1, \ldots, b_s)$ defining a parabolic matrix, and the effective S^1-action on the plumbed manifold associated with the choice of a primitive fixed vector.

Fixed surfaces may only appear as cores of certain plumbed strips. Suppose that such a fixed sphere actually exists. Up to a shift (mod s) in the numbering of the b_i's, we may assume that it is the sphere number 1.

There the action is written $(t^{\pm 1}x, y)$; in particular, in this notation, the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is fixed by B_s, which must be of the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ for some $a \in \mathbf{Z}$ (this is the case for all triangles and quadrilaterals studied above).

B.1.3 The boundary. The boundaries of the S^1-manifolds of dimension 4 we obtained are endowed with a fixed point free S^1-action. In contrast to what could happen at a free end of the graph in 5.2, there cannot exist any exceptional orbit. The boundary is thus a 3-manifold with a *principal* S^1-action. Let us now determine this manifold.

First, *the quotient is a torus* T^2.

Exercise B.1.4 If m and n are relatively prime, the map

$$\begin{array}{ccc} D^2 \times S^1 & \longrightarrow & D^2 \\ (a, z) & \longmapsto & a^n z^{-m} \end{array}$$

is the quotient map for the S^1-action: $t \cdot (a, z) = (t^m a, y^n z)$.

Begin now to plumb along the linear graph $(b_1, \ldots, b_s)$ as in 5.2.5. We saw there that we were gluing $S_{1+} \times S^1$ and $S_{s-} \times S^1$ using the map φ defined by the matrix A_s. If the circle acts on $S_{1+} \times S^1$ by $t \cdot (a, u) = (t^{m_1} a, t^{n_1} u)$ and on $S_{s-} \times S^1$ by $t \cdot (b, v) = (t^{m'_s} b, t^{n'_s} v)$ we know that

$$\begin{pmatrix} m'_s \\ n'_s \end{pmatrix} = A_s \begin{pmatrix} m_1 \\ n_1 \end{pmatrix},$$

thus, forming the quotient as the preceding exercise allows, we see that the manifold we obtain (lens space) is fibered over $D^2 \cup_\psi D^2$ where precisely $\psi(z) = \bar{z}$. The quotient is indeed a sphere S^2.

We still must plumb the bundles numbered 1 and s using small discs in S_{1+} and S_{s-} (figure 14). Passing to the quotient, we shall have to glue the boundaries of the small discs which are the images in the sphere: the quotient is a torus.

We still have to compute the Euler class of the bundle under consideration.

Proposition B.1.5 *Let (e_1, e_2) be an integral basis of $\mathbf{Z}^2$ such that $e_1 = \begin{pmatrix} m \\ n \end{pmatrix}$ is the integral vector fixed by B_s which defines the S^1-action we are dealing with. In this basis, the matrix B_s is written $\begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix}$ and the integer e so defined is the Euler class of the bundle.*

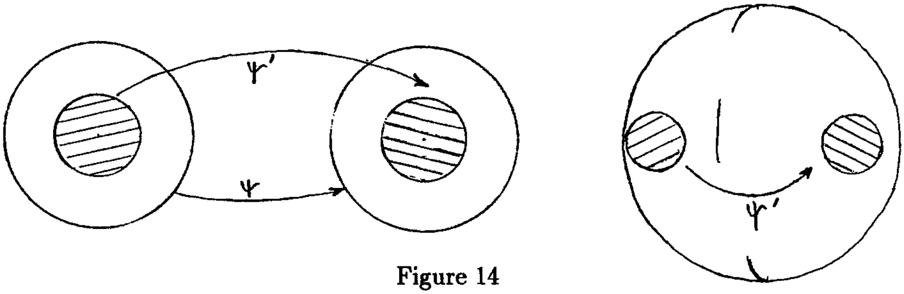

Figure 14

Remark. The absolute value $|e|$ depends only on B_s: the form $\begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix}$ is the Jordan form of B_s. The sign of e, on the other hand, is determined by the choice of the eigenvector e_1, that is the way the S^1-action makes the fiber turn. It is the case for the sign of the Euler class as well.

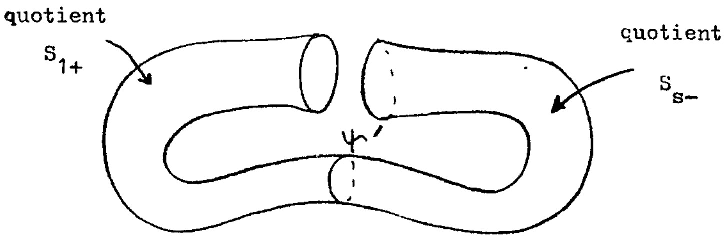

Figure 15

Proof. Consider the tube or annulus A on figure 15. Over the annulus, the bundle is trivialised exactly as on the complement of a small disc of S_{1+}, in other words by

$$\begin{array}{ccc} (D^2 - \{0\}) \times S^1 & \longrightarrow & (D^2 - \{0\}) \times S^1 \\ (a, u) & \longmapsto & (a^n z^{-m}, a^p u^{-q}) \end{array}$$

where we have identified $\mathring{A}$ with $D^2 - \{0\}$, p and q are two integers such that $mp - nq = 1$ (in particular, $e_1 = \begin{pmatrix} m \\ n \end{pmatrix}$ and $e_2 = \begin{pmatrix} p \\ q \end{pmatrix}$ form a basis as in the statement of the proposition) and where on the right hand side, the S^1-action is $t \cdot (b, v) = (b, tv)$.

To obtain the whole bundle we still have to glue, using ψ, or in other words using the matrix $B_s \ldots$ which in these coordinates (b, v) is in Jordan form.

The bundle is hence isomorphic to

$$\underbrace{[0,1] \times S^1}_{A} \times S^1/(0,b,v) \sim (1,b,b^e v).$$

We must now be convinced that this way of constructing the bundle by gluing actually gives the Euler class. This is the content of the following lemma, the proof of which we shall postpone until the next chapter (V-B). □

Lemma B.1.6 *Let $V \to T^2$ be a principal S^1-bundle. Then V is isomorphic to the bundle defined by the gluing data*

$$[0,1] \times S^1 \times S^1/(0,b,v) \sim (0,b,b^e v)$$

where the integer e denotes the Euler class.

B.1.7 Compactification. The boundary of the manifolds we are considering being principal S^1-bundles over the torus T^2, it is easy to complete them to get closed manifolds: glue the total space of the disc bundle $E(-e) \to T^2$, thus getting an oriented compact manifold of dimension 4, endowed with an S^1-action with fixed points. These are the points of the torus T^2 we have just added, isolated fixed points and possibly fixed spheres S^2 coming from the cores of the strips we plumbed: the S^1-manifolds we get have *no invariant symplectic form.*

Call $\mathcal{U}$ the result of plumbing, $\mathcal{V}$ the disc bundle which we glued to it, $\partial\mathcal{V}$ the principal S^1-bundle which is their common boundary and W the resulting closed manifold.

Exercise B.1.8

1. Show that $\pi_1(\mathcal{U}) \cong \mathbf{Z}$, that $\pi_1(\partial\mathcal{V}) \xrightarrow{\psi} \pi_1(\mathcal{V})$ is onto and that the generator of $\operatorname{Ker}\psi$ vanishes in $\pi_1(\mathcal{U})$. Using van Kampen's theorem, deduce that the natural map $\pi_1(\mathcal{U}) \to \pi_1(W)$ is an isomorphism and in particular that the first Betti number $b_1(W)$ equals 1.

2. Show that $\dim H^2(\mathcal{U}) = s$ (the number of bundles we used for plumbing). Using the Mayer-Vietoris exact sequence, deduce that $b_2(W) = s$.

Remark. In the case where $\partial\mathcal{V} \to T^2$ is the trivial bundle (*i.e.* $e = 0$ and the plumbed manifold actually has a T^2-action) there may be other ways to glue the bundle $T^2 \times D^2$. For example, for a plumbing along the triangle $(-1,-1,-1)$ it is possible to glue $T^2 \times D^2$ in such a way that we get $\mathbf{P}^2(\mathbf{C})$ with an S^1-action without fixed point on the piece we added. The results of the previous exercise show just how much the manifold we get here is different (for $\mathbf{P}^2(\mathbf{C})$ we have $b_1 = 0$ and $b_2 = 1$). A systematic construction of this kind of examples will be discussed in chapter VI.

B.1.9 Parabolic Inoue surfaces. We investigate here the case where the polygon is weighted by 2 (all the bundles have Euler class -2). The matrix B_s is, as we have already seen

$$\begin{pmatrix} s+1 & s \\ -s & -(s-1) \end{pmatrix} \sim \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

and so the bundle we glue has Euler class $-s$. The S^1-action has s isolated fixed points and a fixed surface (the torus T^2).

There exists an analytic complex surface, endowed with a $\mathbf{C}^*$-action with these properties. The spheres we used to plumb constitute a cycle of rational curves, and the torus T^2 is an elliptic curve. This surface (*parabolic Inoue surface*) has remarkable properties, but has no nonconstant meromorphic functions (*see* for example [53] and [58]). For instance, it has no symplectic form defining the orientation we consider (*a fortiori* calibrating the complex structure). In fact the intersection pairing is negative definite (the proof of this fact is easy and thus left as an exercise).

B.2 Generalisations

Consider now a weighted connected graph, and let g be the dimension of its H_1.

Assume the weights are such that one can define an S^1-action on the plumbed manifold: for example, any vertex which is the end of three or more edges must correspond to a fixed sphere, and it determines the operation (if any) on the cycles containing this vertex. The matrix associated with each of the cycles as above must have the form $\begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix}$.

If the graph has no free (*i.e.* which is the origin of a unique edge) vertex, the action is principal on the boundary, and the quotient is a genus g surface, which allows us to compactify as above to obtain a lot of S^1-manifolds of dimension 4 with no invariant symplectic form.

Chapter V

Equivariant cohomology and the Duistermaat-Heckman theorems

Another group of famous and spectacular theorems in the theory of hamiltonian torus actions is due to Duistermaat and Heckman[1][34]. The two theorems in their paper assert once again the importance of linear phenomena in the theory.

In the first theorem (here 3.2.2), the authors consider the reduced symplectic forms ω_ξ and ω_η on the quotients of two regular levels $\mu^{-1}(\xi)$ and $\mu^{-1}(\eta)$ which correspond to values ξ and η located in the same component of the set of regular values of the moment map μ. One may then identify the quotients B_ξ and B_η and consider the difference $[\omega_\xi] - [\omega_\eta] \in H^2(B_\xi)$. The theorem asserts that it is a *linear* function of $\xi - \eta$.

The other statement (here 6.1.1) is usually shortened to: a periodic hamiltonian satisfies the exact stationary phase formula.

It was N. Berline and M. Vergne [22] who first explained that this statement is a special case of a "localisation theorem" in equivariant cohomology. This was a very good idea: actually the language of equivariant cohomology fits very well with the investigation of hamiltonian actions, as one might be readily convinced by looking at 3.1.1.

Later came two "survey" papers on the subject, one by Atiyah and Bott [20] in 1984 and one by Ginsburg [37] in 1987. From these come the methods and results I shall discuss in the present chapter.

As I have already said, the language of equivariant cohomology fits so well here that we shall see that the first of the Duistermaat-Heckman theorems becomes practically tautological (this will not detract from its beauty)... Of course we will have to pay for this, namely we will have to define equivariant cohomology, the Borel

[1] I just learn (december 1990) from A. Weinstein that similar results were proved by Karasev [45] in 1981.

construction and classifying spaces... Thus we shall begin by recalling the classical results and constructions of Milnor and Dold, recommending the reader to consult these authors and Husemoller's book [15] for details.

1 Principal and universal bundles

1.1 Principal bundles

A principal G-bundle is a space E endowed with a *free* G-action such that the orbit map $E \to B$ is a locally trivial fibration.

When the slice theorem is true, the last condition is automatically satisfied. We shall only consider compact groups here, but we cannot restrict to the case where E and B are manifolds, mainly because we shall need some infinite dimensional spaces.

Examples.

1. In chapter I, we investigated principal S^1-bundles over S^1 and over surfaces.
2. Of course $S^{2n+1} \to \mathbf{P}^n(\mathbf{C})$ is a principal S^1-bundle as well.
3. Consider now slightly more complicated groups and to this end look at the complex Stiefel manifold $V_k(\mathbf{C}^{n+k})$ of unitary k-frames in $\mathbf{C}^{n+k}$. The unitary group $U(k)$ acts naturally on $V_k(\mathbf{C}^{n+k})$, it is a free action and the quotient is the complex Grassmann manifold $G_k(\mathbf{C}^{n+k})$ of all k-dimensional vector subspaces in $\mathbf{C}^{n+k}$. The projection
$$V_k(\mathbf{C}^{n+k}) \longrightarrow G_k(\mathbf{C}^{n+k})$$
is a principal $U(k)$-bundle (for $k = 1$, it is $S^{2n+1} \to \mathbf{P}^n(\mathbf{C})$).
4. If the group G is discrete, a principal G-bundle is a Galois covering.

The notion of *isomorphism* of principal G-bundles is clear.

I shall very often take the liberty of denoting with the same letter the bundle and its total space.

1.2 Universal bundles

1.2.1 We shall consider *numerable* coverings, that is locally finite coverings $(U_i)_{i\in I}$ such that there exists a partition of unity $(u_i)_{i\in I}$ with:

$$\overline{u_i^{-1}(]0,1])} \subset U_i, \quad \forall i \in I$$

Example. If B is paracompact, it has a numerable covering.

A principal bundle over B is said to be *numerable* if there exists a numerable covering of B which makes it locally trivial.

1.2.2 Induced bundles. If $p : E \to B$ is a principal G-bundle, and $f : B' \to B$ a continuous map,

$$f^*E = \{(e, b') | p(e) = f(b')\}$$

is a principal G-bundle as well, and one shows[2] without too much difficulty:

Theorem 1.2.3 *If $f_t : B' \to B$ is a homotopy, then f_0^*E and f_1^*E are isomorphic.*

□

1.2.4 The universal bundle problem. We try to construct a principal G-bundle which is universal, in other words a bundle $\mathcal{E} \to \mathcal{B}$ such that any principal G-bundle is induced by a map $B \to \mathcal{B}$, and more precisely:

Definition 1.2.5 *A numerable principal G-bundle $\mathcal{E} \to \mathcal{B}$ is universal if*

1. *For any numerable principal G-bundle $E \to B$, one can find a map $f : B \to \mathcal{B}$ such that E is isomorphic to $f^*\mathcal{E}$.*
2. *Two maps $f, g : B \to \mathcal{B}$ induce isomorphic bundles if and only if they are homotopic.*

Milnor [56] gave a very beautiful, and furthermore explicit construction of universal bundles.

1.2.6 The Milnor join. Let

$$EG = G \star G \star \cdots \star G \star \cdots$$

be the "infinite join", or more explicitely:

$$EG = \varinjlim EG(n)$$

where $EG(n)$ is the quotient of

$$G^{n+1} \times \Delta^n = \Big\{(x_0, t_0; x_1, t_1; \ldots; x_n, t_n) | x_i \in G, t_i \in [0,1], \sum t_i = 1\Big\}$$

by

$$(x_0, t_0; \ldots; x_n, t_n) \sim (x_0', t_0'; \ldots; x_n', t_n') \Longleftrightarrow \left\{ \begin{array}{lllr} t_i & = & t_i' & \forall i \\ t_i & = & t_i' \neq 0 & \Rightarrow x_i = x_i' \end{array} \right.$$

We shall write $\langle x_0, t_0; \ldots; x_n, t_n \rangle$ for the equivalence class of the element under consideration.

[2] *see* for example the book by Husemoller [15].

Examples.

1. $G = \mathbf{Z}/2 = \{\pm 1\}$. Then $EG(n)$ is identified with S^n by:

$$\begin{array}{ccc} EG(n) & \longrightarrow & S^n \\ \langle x_0, t_0; \ldots; x_n, t_n \rangle & \longmapsto & (\sqrt{t_0}x_0, \ldots, \sqrt{t_n}x_n) \end{array}$$

2. $G = S^1$. The very same formulas identify $EG(n)$ with S^{2n+1}.

The limit EG is defined by the inclusion maps

$$\begin{array}{ccc} EG(n) & \subset & EG(n+1) \\ \langle x_0, t_0; \ldots; x_n, t_n \rangle & \mapsto & \langle x_0, t_0; \ldots; x_n, t_n; x_{n+1}, 0 \rangle \end{array}$$

Any element in EG will thus be written

$$\langle x_0, t_0; \ldots; x_n, t_n; \ldots \rangle$$

or for brevity $\langle x, t \rangle$.

Endow this space with the least expensive *topology* such that all the maps

$$\begin{array}{ccc} EG & \longrightarrow & [0,1] \\ \langle x_0, t_0; \ldots; x_n, t_n \rangle & \longmapsto & t_i \end{array}$$

and

$$\begin{array}{ccc} t_i^{-1}(]0,1]) & \longrightarrow & G \\ \langle x_0, t_0; \ldots; x_n, t_n \rangle & \longmapsto & x_i \end{array}$$

are continuous.

The group G acts on $EG(n)$ and EG by

$$g \cdot \langle x, t \rangle = \langle gx, t \rangle.$$

It is clear that these actions are free and the quotients will be denoted respectively $BG(n)$ and BG.

Example.

1. We saw, for $G = \mathbf{Z}/2$ that $EG(n) = S^n$ (and $EG = S^\infty$). The group acts by the antipodal map, $BG(n) = \mathbf{P}^n(\mathbf{R})$ and $BG = \mathbf{P}^\infty(\mathbf{R})$.

2. In the same way, for $G = S^1$, $BG(n) = \mathbf{P}^n(\mathbf{C})$, and $BG = \mathbf{P}^\infty(\mathbf{C})$.

We shall now explain (without too much proof) that these spaces are actually *made* to be universal:

One begins by showing that all numerable bundles have a *numerable* partition of unity which makes them locally trivial.

Granted this, consider the t_i's on EG as a universal partition of unity ($\sum t_i = 1$), to show:

Proposition 1.2.7 *For any numerable principal G-bundle, E over B, there exists a map $f : B \to BG$ such that $f^*EG \cong E$.*

Proof. Construct two maps f et g making the diagram

$$\begin{array}{ccc} E & \xrightarrow{g} & EG \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & BG \end{array}$$

commute. That for, consider a partition of unity $(u_n)_{n\geq 0}$ on B such that $E_{|u_n^{-1}(]0,1])}$ is trivial. Let $U_n = u_n^{-1}(]0,1])$ and let

$$\begin{array}{ccc} U_n \times G & \xrightarrow{h_n} & E_{|U_n} \\ \downarrow{\scriptstyle q_n} & & \\ G & & \end{array}$$

be a local trivialisation. Simply define g by:

$$g(z) = \langle q_0 h_0^{-1}(z), u_0(p(z)); \ldots ; q_n h_n^{-1}(z), u_n(p(z)); \ldots \rangle$$

This is better defined than one might think: $h_n^{-1}(z)$ is defined only if $z \in p^{-1}(U_n)$, but, otherwise, $u_n(p(z)) = 0$.

The map g is necessarily an isomorphism and more generally:

Lemma 1.2.8 *Let $E_i \to B_i$ $(i = 1, 2)$ both be principal G-bundles. Any morphism:*

$$\begin{array}{ccc} E_2 & \xrightarrow{g} & E_1 \\ \downarrow & & \downarrow{\scriptstyle p} \\ B_2 & \xrightarrow{f} & B_1 \end{array}$$

induces an isomorphism $E_2 \xrightarrow{g'} f^\star E_1$.

Proof of the lemma.

$$f^\star E_1 = \{(e_1, b_2) | f(b_2) = p(e_1)\}$$

and $g' : E_2 \to f^\star E_1$, defined by $g'(e_2) = (g(e_2), p(e_2))$ is a morphism of principal G-bundles over B_2. It is easy to check that g' is *injective*: $g'(e) = g'(e') \Rightarrow p(e) = p(e') \Rightarrow e' = x \cdot e$ for some $x \in G$, applying g' again, one sees that $x = 1$. Moreover g' is *surjective* as well: let $e \in f^\star E_1$, and let e' be a point in the same fiber as the image of e in E_2. One can find an $x \in G$ such that $g'(xe') = e$. □ □

It is a little more technical, but not much more difficult, to prove that the Milnor join is indeed universal. From this universality, some kind of unicity follows: if $E_1 \to B_1$ and $E_2 \to B_2$ are two universal principal G-bundles, writing explicitely this property for the former gives a morphism:

$$\begin{array}{ccc} E_2 & \longrightarrow & E_1 \\ \downarrow & & \downarrow \\ B_2 & \xrightarrow{f} & B_1 \end{array}$$

and similarly, writing that $E_2 \to B_2$ is universal gives a morphism $g : B_1 \to B_2$. As $f \circ g : B_1 \to B_1$ induces an isomorphism, it is homotopic to the identity, and the same is true for $g \circ f$: thus the homotopy type of the universal space BG is well defined. As we shall use these spaces only in algebraic topology calculations, the unicity of their homotopy type will be sufficient for our purposes.

One proves (*see* [32]) that

Proposition 1.2.9 *If $E \to B$ is any principal G-bundle the total space E of which is contractible, then it is a universal principal G-bundle.* □

In [56], Milnor proved that his spaces are *weakly* contractible (this means that their homotopy groups are zero), but in the mentioned paper, Dold showed that Milnor joins are actually contractible (as it is a little technical, we shall admit this result here).

Examples of applications.

1. If H is a subgroup of G, it acts on EG, and EG/H is a model for BH.

2. If G and H are two groups, the product $G \times H$ acts on $EG \times EH$ in the way you imagine and the quotient is $BG \times BH$ which is a model for $B(G \times H)$.

3. If $f : G \to H$ is a group morphism, it induces a continuous map $BG \to BH$: it is easy to construct on the "Milnor join" models.

Examples.

1. From $\mathbf{Z}/m \subset S^1$, we deduce that the quotient of S^{2n+1} by the equivalence relation:
$$\zeta^m = 1 \Rightarrow (z_0, \ldots, z_n) \sim (\zeta z_0, \ldots, \zeta z_n)$$
is a finite approximation to $B\mathbf{Z}/m$, the latter then being $S^\infty/\mathbf{Z}/m$, the "infinite lens space".

2. In the same way $T^m = S^1 \times \cdots \times S^1$ acts diagonally on $S^{2n+1} \times \cdots \times S^{2n+1}$ which is thus found to be an approximation to ET^m and $BS^1 \times \cdots \times BS^1$ becomes therefore a model for BT^m.

3. Consider the join $EU(k)(n) = \{\langle A_0, t_0; \ldots; A_n, t_n\rangle | A_i \in U(k)\}$ and the map into the Stiefel manifold $V_k(\mathbf{C}^{k(n+1)})$ which, with any $\langle A, t\rangle$ associates the k column vectors in the $k(n+1)$ rows matrix
$$\begin{pmatrix} \sqrt{t_0}A_0 \\ \vdots \\ \sqrt{t_n}A_n \end{pmatrix}$$

Taking limits and passing to quotients, this induces a morphism

$$\begin{array}{ccc} EU(k) & \longrightarrow & V_k(\mathbf{C}^\infty) \\ \downarrow & & \downarrow \\ BU(k) & \longrightarrow & G_k(\mathbf{C}^\infty) \end{array}$$

In order to conclude that the infinite Grassmann manifold $G_k(\mathbf{C}^\infty)$ is a model for $BU(k)$ it is thus enough to be convinced that $V_k(\mathbf{C}^\infty)$ is contractible.

2 The Borel construction and equivariant cohomology

2.1 The Borel construction

If W is a space acted on by the group G, we know very well that the orbit space W/G may be rather complicated. The idea in the *Borel construction* is to try to get a reasonable substitute for this quotient.

Make G act on $EG \times W$ by

$$g \cdot (e, x) = (g \cdot e, g \cdot x).$$

This is a free action, as it is free on the first factor.

Definition 2.1.1 *The quotient $W_G = EG \times_G W$ is called the Borel construction on W.*

Remark. In this way we have only defined the homotopy type of this space.

Consider the two projection maps from the product $EG \times W$:

- $$\begin{array}{ccc} EG \times W & \longrightarrow & EG \\ W_G & \longrightarrow & BG \end{array}$$

  induces a fibration with fiber W.

- The second projection induces a map $\sigma : W_G \to W/G\ldots$ which is not a fibration in general, but for which we have:

Proposition 2.1.2 *If G is a compact Lie group acting smoothly and freely on a manifold W, then*

$$\sigma : W_G \longrightarrow W/G$$

is a fibration with contractible fibers, and in particular is a homotopy equivalence.

Proof. Let us investigate first the fibers of σ in the general case. Let $x \in W$ and let $[x] \in W/G$ be its orbit.

$$\sigma^{-1}([x]) = \{[e,y] \in EG \times_G W | \text{proj}[e,y] = [x]\}$$

Let us find the classes $[e,x]$ for $e \in EG$:

$$[e',x] \sim [e,x] \Leftrightarrow \exists g \in G | e' = g \cdot e, x = g \cdot x$$

that is: $g \in G_x$, and $\sigma^{-1}([x]) = EG/G_x \sim BG_x$.

When the G-action is free, all the fibers are thus spaces EG, in particular they are contractible. The local triviality is left as an exercise (of course!)–it is a consequence of that of $W \to W/G$ (with the hypothesis made, it is the slice theorem). □

It is in this sense that one may say that W_G is the *homotopy quotient* of W: when the genuine quotient is "good", it has the same homotopy type, and W_G is a "good" quotient in general.

On the contrary, if the G-action on W is trivial, then $W_G = BG \times W$; this is the case when W is a point, in which case $W_G = BG$.

2.2 Equivariant cohomology

Definition 2.2.1 *If W is a topological space acted on by a group G, the equivariant cohomology of W, written $H^*_G(W)$, is the cohomology of W_G.*

There will be therefore as many possible theories as there are cohomology theories. Here we shall mainly use de Rham cohomology. That is to say that we shall assume W to be a smooth manifold, with a G-action which is smooth as well. Unfortunately, neither EG nor *a fortiori* W_G are manifolds, which causes some trouble: what could be a differentiable form on W_G? All the groups we shall be interested in are subgroups of some $U(n)$, and this allows us to use the "infinite Stiefel manifold" $V_n(\mathbf{C}^\infty) = \cup_k V_n(\mathbf{C}^{n+k})$ or any other union of manifolds that would fit as a model for EG, a differentiable form then being a family of differentiable forms, that agree *via* the inclusions[3].

Remark. The reader may ask why we are doing such complicated things: a simpler way to define an "equivariant cohomology" would seem to be to consider the cohomology of G-invariant forms on W. Actually, when G is compact, this is nothing else than the usual de Rham cohomology of W, as the next (easy) exercise shows.

Exercise 2.2.2 Show that, if α is an invariant k-form ($\alpha \in \Omega^k(W)^G$), $d\alpha$ is an invariant $(k+1)$-form, in other words that there is a commutative diagram:

$$\begin{array}{ccc} \Omega^k(W)^G & \xrightarrow{d} & \Omega^{k+1}(W)^G \\ \downarrow j & & \downarrow j \\ \Omega^k(W) & \xrightarrow{d} & \Omega^{k+1}(W) \end{array}$$

[3]In general, we shall content ourselves with a finite approximation of EG or W_G.

If G is compact, show that the inclusion j of invariant forms in the space of all forms is a "quasi-isomorphism" (*i.e.* induces an isomorphism at the cohomology level).

This fake "equivariant cohomology" has nothing more to say than the cohomology of W. On the contrary, we have very good reasons to think that the cohomology of W_G says more: for example, if W is a point, then $W_G = BG$ and $H_G^\star(pt) = H^\star(BG)$, which is never zero.

On the other hand, $H_G^\star(W)$, being the cohomology of a space W_G, is, in particular, a ring. It is even more: using the projection map $W_G \xrightarrow{\pi} BG$, it receives the structure of an $H_G^\star(pt)$-module. We shall mainly concentrate on this structure, which has a lot to say about the action: for instance if it is free, $H_G^\star(W) \cong H^\star(W/G)$ is a *torsion* $H_G^\star(pt)$-module.

The first thing to do is to understand what kind of a ring $H_G^\star(pt)$ is. We limit ourselves to the case where G is a torus, and we begin of course by the $G = S^1$ case.

Theorem 2.2.3 *$H^\star(BS^1)$ is a polynomial ring, on a degree 2 generator u.*

Remark. This is true over $\mathbf{Z}$ and therefore over any ring of coefficients, as will be the case for all statements in this chapter which are not given with more precision.

We have to calculate the cohomology of $\mathbf{P}^\infty(\mathbf{C})$. We have more precisely:

Proposition 2.2.4

$$H^\star(\mathbf{P}^n(\mathbf{C}); \mathbf{Z}) \cong \mathbf{Z}[u]/u^{n+1}$$

and the inclusion $j : \mathbf{P}^n(\mathbf{C}) \hookrightarrow \mathbf{P}^{n+1}(\mathbf{C})$ induces

$$\begin{array}{ccc} \mathbf{Z}[u]/u^{n+2} & \xrightarrow{j^\star} & \mathbf{Z}[u]/u^{n+1} \\ u & \longmapsto & u \end{array}$$

from which the theorem is easily deduced, and of which the proof will be given in appendix A. Merely recall here that u is the *Euler class* of the complex line bundle $\mathcal{O}(1)$, the *canonical* bundle over $\mathbf{P}^\infty(\mathbf{C})$, for more details we refer the reader to the mentioned appendix.

Write

$$T^n \cong \underbrace{S^1 \times \cdots \times S^1}_{n}$$

to derive:

Corollary 2.2.5 *Let T be a dimension n torus, then $H^\star(BT)$ is a polynomial ring on n degree 2 variables.*

2.3 Generators for de Rham cohomology

2.3.1 Cohomology of BS^1. Using the previous results, we shall now exhibit "the" generator of $H^2(BS^1; \mathbf{R})$ and take this opportunity to introduce the notion of a *connection*.

Let $S^{2n-1} \subset \mathbf{C}^n$ be the unit sphere $\sum |z_i|^2 = 1$. Make S^1 act diagonally ($t \cdot (z_1, \ldots, z_n) = (tz_1, \ldots, tz_n)$), with fundamental vector field

$$X_{(z_1,\ldots,z_n)} = (iz_1, \ldots, iz_n),$$

or, writting $z_j = q_j + ip_j$,

$$X = \sum_{j=1}^{n} \left(-p_j \frac{\partial}{\partial q_j} + q_j \frac{\partial}{\partial p_j} \right).$$

Let α be the 1-form (on $\mathbf{C}^n$, or on S^{2n-1})

$$\alpha = \frac{1}{2} \sum_{j=1}^{n} (-p_j dq_j + q_j dp_j).$$

By definition, $i_X \alpha \equiv 1$. On the other hand,

$$\begin{aligned} d\alpha &= \tfrac{1}{2}(-\sum dp_j \wedge dq_j + \sum dq_j \wedge dp_j) \\ &= -\omega \end{aligned}$$

where $\omega = \sum dp_j \wedge dq_j$ is the canonical symplectic form on $\mathbf{C}^n$. Remark once again that $i_X \omega = dH$ for $H = \frac{1}{2} \sum |z_j|^2$ and thus, that on S^{2n-1}

$$\begin{aligned} i_X \alpha &\equiv 1 \\ \mathcal{L}_X \alpha &= d i_X \alpha + i_X d\alpha = 0 - dH = 0 \end{aligned}$$

as H is constant on S^{2n-1}.

In other words, α is invariant (second condition) and moreover it is a *connection form* on the S^1-principal bundle $\pi : S^{2n-1} \to \mathbf{P}^{n-1}(\mathbf{C})$ (*see* definition 2.4.4).

Consider now the 2-form $d\alpha$ on S^{2n-1}. It satisfies $i_X d\alpha = 0$ and $\mathcal{L}_X d\alpha = d i_X d\alpha = 0$. We see that we can thus find a 2-form σ on $\mathbf{P}^{n-1}(\mathbf{C})$ such that $\pi^\star \sigma = d\alpha$ (for this reason, $d\alpha$ is called a *basic form*)[4]. Moreover σ is closed because $d\alpha$ is. Actually, it is, up to sign, the reduced symplectic form on projective space, in particular, for $n \geq 2$, it cannot be exact, and thus represents a *nonzero* element in $H^2(\mathbf{P}^{n-1}(\mathbf{C}); \mathbf{R})$.

All these definitions are compatible *via* the inclusion maps $\mathbf{C}^n \subset \mathbf{C}^{n+1}$ and we have thus actually defined a nonzero element σ in $H^2(BS^1; \mathbf{R})$. We still have to compare it with u, which is characterised by the fact that $\langle u, [\mathbf{P}^1(\mathbf{C})] \rangle = 1$.

Exercise 2.3.2 Check that

$$\int_{\mathbf{P}^1(\mathbf{C})} \sigma = -2\pi.$$

[4] σ is the *curvature* form.

Thus $-\sigma/2\pi = u$.

2.3.3 The cohomology of BT. Once a basis $(X_1, \ldots, X_m)$ of $\mathfrak{t} \cong \mathbf{R}^m$ in chosen, we saw that we get an isomorphism

$$\varphi : \underbrace{S^1 \times \cdots \times S^1}_{m} \longrightarrow T$$

a homotopy equivalence

$$\Phi : BS^1 \times \cdots \times BS^1 \longrightarrow BT$$

and a ring isomorphism

$$\Phi^* : H^*BT \longrightarrow H^*BS^1 \otimes \cdots \otimes H^*BS^1 \cong \mathbf{R}[u_1, \ldots, u_m]$$

(using de Rham cohomology).

Make T act on $\mathbf{C}^{mn}$ by

$$(t_1, \ldots, t_m) \cdot (Z_1, \ldots, Z_m) = (t_1 Z_1, \ldots, t_m Z_m)$$

where $Z_i \in \mathbf{C}^n$.

We can thus use $\underbrace{S^{2n-1} \times \cdots \times S^{2n-1}}_{m}$ as a finite approximation to ET. We have m forms $\alpha_1, \ldots, \alpha_m$ with $i_{X_j}\alpha_i = \delta_{i,j}$ and the isomorphism

$$H^2(BT) \longrightarrow \mathfrak{t}^*$$

which sends $(u_1, \ldots, u_m)$ on the basis dual to $(X_1, \ldots, X_m)$.

Exercise 2.3.4 This isomorphism does *not* depend on the choice of the basis $(X_1, \ldots, X_m)$.

Hence we can identify $H^2(BT; \mathbf{R})$ with $\mathfrak{t}^*$ and the polynomial ring $\mathbf{R}[u_1, \ldots, u_m]$ becomes the algebra $S(\mathfrak{t}^*)$ of multilinear symmetric forms[5] on $\mathfrak{t}$.

2.4 Euler classes for fixed point free T-actions

Using equivariant cohomology, we shall define the *Euler class* for an S^1-bundle or a T-bundle, thus generalising the definition of chapter I.

Begin by considering a *principal* S^1-action on a manifold W, and the quotient B. We saw in 2.1.2 that the natural map

$$\sigma^* : H^*(B) \to H^*_{S^1}(W)$$

is an isomorphism.

Definition 2.4.1 *The unique element in $H^2(B; \mathbf{Z})$ whose image by σ^* is $-u$ is called the Euler class of the principal S^1-bundle $W \to B$.*

[5]The reader may find that it is not very elegant to choose a basis, then use it to define an isomorphism and then prove the latter does not depend on the basis. There exists of course an intrinsic method, namely Chern-Weil theory.

Example. The Euler class of the principal S^1-bundle $S^{2n+1} \to \mathbf{P}^n(\mathbf{C})$ is $-u$ (one can either use this example as an exercise or look for a proof in appendix B).

By universality of the principal S^1-bundle $S^{2n+1} \to \mathbf{P}^n(\mathbf{C})$ (n large enough), we deduce the existence of connections on principal S^1-bundles:

Proposition 2.4.2 *Let $\pi : V \to B$ be a principal S^1-bundle. Then there exists an invariant 1-form α on V, satisfying furthermore $i_X\alpha \equiv 1$ (and called connection form). Moreover, $d\alpha = \pi^*\eta$ where η is a 2-form over B, the curvature form, such that $[\eta]$ is the Euler class of the bundle on $H^2(B;\mathbf{R})$.* □

It is even easy to be more precise:

Exercise 2.4.3 Let $\pi : V \to B$ be a principal S^1-bundle, and let η be a 2-form on B such that $[\eta]$ is the Euler class of the bundle. Show that there exists a connection form α on V such that $d\alpha = \pi^*\eta$.

In the case of a torus T of dimension m, we may also consider $(\alpha_1, \dots, \alpha_m)$ as a *connection*; let us state more generally the following definition:

Definition 2.4.4 *Let $E \to B$ be a principal G-bundle. A connection on E is a differentiable 1-form θ on E taking values in $\mathfrak{g}$, and which satisfies the two properties*

1. *$\forall X \in \mathfrak{g}$, we have $i_{\underline{X}}\theta = X$*
2. *θ is invariant, in the sense $\forall g \in G$, $g^*\theta = \mathrm{Ad}_g \cdot \theta$.*

Example. If G is the torus T, consider $(\alpha_1, \dots, \alpha_m)$ (on the universal T-bundle) as a 1-form with values in $\mathbf{R}^m \cong \mathfrak{t}^*$. Of course, it satisfies the above two properties.

We have the existence of a connection on any principal G-bundle as soon as we are able to construct one on a universal principal G-bundle. We shall use only the cases $G = S^1$ or T, for which have already actually proved the existence of connections.

The definition (2.4.1) is extended to all *fixed point free* S^1-actions by the very same method as in I-3.4: choose a common multiple n of all orders of exceptional orbit stabilizers, and consider W' with the $S^1/\mathbf{Z}/n \cong S^1$-action and the projection map $p : W \to W'$.

We may apply the previous definition to W'. Look at the diagram

$$\begin{array}{ccc} H^2(B) & \xrightarrow{\sigma'^*} & H^2_{S^1/\mathbf{Z}/n}(W') \\ & & \downarrow p^* \\ & & H^2_{S^1}(W) \end{array}$$

Let e' be the Euler class of the principal S^1-bundle $W' \to B$: it is the unique element in $H^2(B)$ which satisfies $\sigma'^*e' = -u'$. But $p^*u' = nu$. This assertion may be checked at the classifying space level, where we must show that the n-fold covering map

$$S^1 \xrightarrow{p} S^1$$

induces multiplication by n in $H^2(BS^1)$. But the induced map from BS^1 onto itself is precisely the one which classifies the bundle $\mathcal{O}(n)$ as we shall see in appendix B.

If the ring of coefficients we are using allows division by n, we thus have a unique class $e \in H^2(B)$ (namely e'/n) such that $\sigma^\star e = -u$. We call it the *Euler class* of the S^1-action on W. We still have to prove (having used the same name) that if $\dim W = 3$, it is the same Euler class as in chapter I. This will be done in the mentioned appendix, which will be specially devoted to the study of Euler classes.

In the same way, dealing with a principal (*resp.* without singular orbits) torus T-action ($\dim T = m$) on the manifold W with quotient B, one can define an Euler class $e \in H^2(B)^m$ (*resp.* with coefficients in a ring containing the inverses of the orders of the stabilizers).

If we do not want to be intrinsic, it is the unique element $e = (e_1, \ldots, e_m) \in (H^2(B))^m$ such that

$$\sigma^\star e = (-u_1, \ldots, -u_m) \in H^2_T(W).$$

If we do, then using de Rham cohomology, $e \in H^2(B; \mathbf{R}) \otimes \mathfrak{t}$ is the unique class such that, in the diagram

$$\begin{array}{ccc} H^2(B) & \xrightarrow{\sigma^\star} & H^2_T(W) \\ & & \uparrow \\ & & H^2(BT) \quad \cong \mathfrak{t}^\star \ni \xi \end{array}$$

we have $\sigma^\star \langle e, \xi \rangle = \xi$ (where $\langle e, \xi \rangle$ is the element of $H^2(B)$ we get using duality in $\mathfrak{t}$ in an obvious way).

For example, any regular level of the moment map $\mu : W \to \mathfrak{t}^\star$ of a hamiltonian T-action on the manifold W (assumed to be symplectic!) defines a Euler class e in the cohomology of the symplectic reduced orbifold (*see* II-3.6.6), at least with rational coefficients.

3 Equivariant cohomology and hamiltonian actions

3.1 Relationships between hamiltonian actions and equivariant cohomology

As we have already had the opportunity to mention, equivariant cohomology is a very convenient language in which to speak of hamiltonian group actions. This is clear in the statement of the easily proven:

Proposition 3.1.1 *Let (W,ω) be a symplectic manifold endowed with a symplectic action of the Lie group G. The action is hamiltonian if and only if there exists a closed 2-form $\omega^\natural$ on the Borel construction W_G whose restriction to each fiber of $W_G \to BG$ is ω.*

Proof. Let θ be a connection form[6] on EG. Asume the G-action is *hamiltonian*, let $\mu : W \to \mathfrak{g}^\star$ be its moment map. Then $\theta \otimes \mu$ is a 1-form taking values in $\mathfrak{g} \otimes \mathfrak{g}^\star$, and may be contracted to give a 1-form taking values in $\mathbf{R}$, and denoted $\langle \theta, \mu \rangle$.

Consider now the 2-form $\tilde{\omega} = \omega + d\langle \theta, \mu \rangle$ on $EG \times W$. It is G-invariant by definition and moreover:

$$\begin{aligned} \forall X \in \mathfrak{g} \quad i_{\underline{X}}\tilde{\omega} &= i_{\underline{X}}\omega + i_{\underline{X}} d\langle \theta, \mu \rangle \\ &= i_{\underline{X}}\omega - d i_{\underline{X}}\langle \theta, \mu \rangle \\ &= i_{\underline{X}}\omega - d\langle i_{\underline{X}}\theta, \mu \rangle \\ &= i_{\underline{X}}\omega - d\langle X, \mu \rangle \\ &= i_{\underline{X}}\omega - d\tilde{\mu}_X = 0. \end{aligned}$$

Hence $\tilde{\omega}$ satisfies $\mathcal{L}_{\underline{X}}\tilde{\omega} = 0$ and $i_{\underline{X}}\tilde{\omega} = 0$, moreover it is obvious that it is closed, it thus comes from a closed 2-form $\omega^\natural$ on W_G. □

Exercise 3.1.2 Prove the converse (you need to use the acyclicity of EG: closed forms are exact).

Remark. The construction of the symplectic form in IV-2.4.2 is completely analogous.

3.2 Variation of the reduced symplectic forms

Consider now a torus T acting on a symplectic manifold (W,ω) with moment map $\mu : W \to \mathfrak{t}^\star$. Let ξ be a regular value of μ and $j_\xi : V_\xi \subset W$ the inclusion of the corresponding level. As j_ξ is T-equivariant, there is a map

$$ET \times_T V_\xi \overset{1 \times_T j_\xi}{\hookrightarrow} ET \times_T W$$

and $\omega_\xi^\natural = (1 \times T j_\xi)^\star \omega^\natural$ is the unique closed form on $ET \times_T V_\xi$ which can be lifted to $ET \times V_\xi$ as $j_\xi^\star \omega + \langle d\theta, \xi \rangle$ because we actually have

$$(1) \qquad \begin{aligned} \tilde{\omega} &= \omega + d\langle \theta, \mu \rangle \\ &= \omega + \langle d\theta, \mu \rangle - \langle \theta, d\mu \rangle \end{aligned}$$

[6] I agree that I have only proved the existence of such a form in the case where G is a torus, but it is the only case where we shall need the present result. The reader may read the statement and proof assuming G is a torus, but it is true in a more general set up.

and by the definition of V_ξ:

$$(1 \times j_\xi)^\star \tilde{\omega} = j_\xi^\star \omega + \langle d\theta, \xi\rangle \tag{2}$$

From (2) we deduce immediatly the inverse image of the class $[\omega_\xi^\natural]$ by the morphism $\sigma_\xi^\star : H^2(B_\xi) \longrightarrow H_T^2(V_\xi)$. As $j_\xi^\star\omega$ gives the reduced symplectic form and $d\theta$ the Euler class of the T-bundle $V_\xi \longrightarrow B_\xi$:

$$(\sigma_\xi^\star)^{-1}[\omega_\xi^\natural] = [\omega_\xi] + \langle e, \xi\rangle. \tag{3}$$

Let now $\mathcal{U}$ be a convex open subset of $\mathfrak{t}^\star$, all points of which are regular values of μ. Thus the T-action on $\mu^{-1}(\mathcal{U}) \subset W$ has no singular orbit. Consider then for $\xi \in \mathcal{U}$ the diagram of quotients and inclusions:

$$\begin{array}{ccc} V_\xi & \overset{j_\xi}{\hookrightarrow} & \mu^{-1}(\mathcal{U}) \\ \downarrow{\scriptstyle \sigma_\xi} & & \downarrow{\scriptstyle \sigma} \\ B_\xi & \overset{j_\xi/T}{\hookrightarrow} & \mu^{-1}(\mathcal{U})/T \end{array}$$

and its cohomological analogue:

$$\begin{array}{ccccc} H_T^2(V_\xi) & \overset{j_\xi^\star}{\longleftarrow} & H_T^2(\mu^{-1}(\mathcal{U})) & \ni [\omega^\natural] \\ \uparrow{\scriptstyle \sigma_\xi^\star} & & \uparrow{\scriptstyle \sigma^\star} & \\ H^2(B_\xi) & \overset{(j_\xi/T)^\star}{\longleftarrow} & H^2(\mu^{-1}(\mathcal{U})/T) & \end{array}$$

where all the maps are isomorphisms (j_ξ is a homotopy equivalence being the inclusion of one fiber over the contractible space $\mathcal{U}$) and where we consider $[\omega^\natural]$ as an element of $H_T^2(\mu^{-1}(\mathcal{U}))$ by restriction.

There is thus a unique (*i.e.* independent of $\xi \in \mathcal{U}$) class

$$x = (\sigma^\star)^{-1}[\omega^\natural] \in H^2(\mu^{-1}(\mathcal{U})/T)$$

the image of which by $\sigma_\xi^\star$ is $[\omega_\xi^\natural]$. Hence, from (3) :

Proposition 3.2.1 *Let $\mathcal{U}$ be a convex open subset of $\mathfrak{t}^\star$ consisting of regular values of the moment map $\mu : W \longrightarrow \mathfrak{t}^\star$. There exists a class x in $H^2(\mu^{-1}(\mathcal{U})/T)$ such that*

$$\forall \xi \in \mathcal{U},\ (j_\xi/T)^\star x = [\omega_\xi] + \langle e, \xi\rangle \in H^2(B_\xi)$$

where ω_ξ is the reduced symplectic form at the level ξ and e is the Euler class of any of the T-bundles $V_\xi \longrightarrow B_\xi$. □

Fixing an element ξ_0 in $\mathcal{U}$, we get an equivariant isomorphism $\mu^{-1}(\mathcal{U}) \cong \mathcal{U} \times V_{\xi_0}$ and, passing to quotients, a way to identify each $H^2(B_\xi)$ with $H^2(B_{\xi_0})$, and thus a way to compare the various classes $[\omega_\xi]$. As a corollary this gives the famous

Theorem 3.2.2 (Duistermaat and Heckman [34])

$$[\omega_\xi] - [\omega_{\xi_0}] = \langle -e, \xi - \xi_0\rangle$$

□

Remark. The sign in this statement comes from the orientation conventions in chapter I where we managed to get -1 as Euler class for the Hopf bundle.

3.2.3 Consider for example the case of a circle action on a 4-manifold. The theorem asserts that the function

$$t \longmapsto \operatorname{Vol} B_t$$

is piecewise linear, more precisely that it is linear (affine) over each component of the set of regular values, the slope being the opposite of the Euler class.

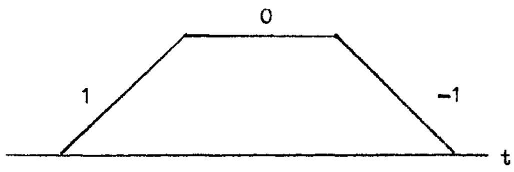

Figure 1

It is perfectly completed by lemma IV-1.2.1 which expresses the variation of the slope at each critical value. For example the discussion in IV-2.3.3 is shortened in a very simple way by the graph of the "volume function" (figure 1).

4 Duistermmat-Heckman with singularities

In this section, we describe a construction, independently due to M. Brion and C. Procesi [27] on the one hand and to V. Guillemin and S. Sternberg [39] on the other, and which generalises the previous remarks.

For the sake of simplicity, we shall assume throughout that all actions are semifree (see nevertheless the remark below).

4.1 The "simple" situation

Let us come back to the situation of the exercises 2.2.2, 3.1.2 and 3.1.3 in chapter IV. We begin with a solution of these exercises. Let H be a periodic hamiltonian on a symplectic manifold W and consider a critical submanifold Z of H with signature $(2, 2p)$. That is to say that, transversally to Z, $W = \mathbf{C} \times \mathbf{C}^p$ with

$$t \cdot (x, y_1, \ldots, y_p) = (tx, \bar{t}y_1, \ldots, \bar{t}y_p)$$

and $H = |x|^2 - |y_1|^2 - \cdots - |y_p|^2$. Transversally to Z we thus have the same situation as in IV-figure 1. Following the gradient flow, we get a map

$$\begin{array}{ccc} \mathbf{C} \times \mathbf{C}^p & \longrightarrow & \mathbf{C}^p \\ (x, y_1, \ldots, y_p) & \longmapsto & (xy_1, \ldots, xy_p) \end{array}$$

which we may restrict to each level of H and quotient. It is easily seen that this map is the quotient map when restricted to any positive level. When $p \geq 2$ it is no longer true at negative levels: the gradient manifold $(y_1, \ldots, y_p)$ is contracted to a point. In any case, we get a diagram

$$\begin{array}{ccccc} V_- & \subset & \mathbf{C} \times \mathbf{C}^p & (x, y) \\ \downarrow & & \downarrow & \downarrow \\ B_- & \to & \mathbf{C}^p & xy \end{array}$$

where

$$\begin{array}{rcl} V_- & = & \left\{(x,y) \mid |x|^2 - |y|^2 = -\varepsilon\right\} \\ & \cong & \mathbf{C} \times S^{2p-1} \\ & & \left(x, \frac{(y_1,\ldots,y_p)}{\sqrt{|x|^2+\varepsilon}}\right) \end{array}$$

and the quotient B_- is the total space of the tautological line bundle $\mathcal{O}(-1)$ over the quotient $\mathbf{P}^{p-1}(\mathbf{C})$ of S^{2p-1}, in other words the blow-up $\widetilde{\mathbf{C}^p}$ of $\mathbf{C}^p$ at 0. The map $B_- \to \mathbf{C}^p$ contracts $[y_1, \ldots, y_p]$ and is precisely the blow up.

We thus remark that the surgery upstairs (from V_- to V_+) corresponds downstairs to the blow up of a point $B_- \to B_+$ (transversally to Z).

Remarks.

1. On reduced symplectic forms: a straightforward computation shows that at the $-\varepsilon$ level the reduced symplectic form on B_- integrates to ε on the generator of the second cohomology group of the exceptionnal divisor. We thus see, using the hamiltonian H, a (real) 1-parameter family of symplectic manifolds which, for negative values of the parameter ε are $\widetilde{\mathbf{C}^p}$'s such that the volume of their exceptional divisor decreases to 0 as $\varepsilon \to 0$. For nonnegative values of the parameter the exceptional divisor has disappeared and the symplectic manifolds are $\mathbf{C}^p$'s.

2. On weights: once we accept working with orbifolds as quotients, there is no essential difference with non semifree actions... in particular with actions of the form $(tx, \bar{t}^m y_1, \ldots, \bar{t}^m y_p)$ which will appear in the following, even after having started with semifree actions.

4.2 General case of a fixed submanifold of signature $(2p, 2q)$

Transversally to Z we have now $W = \mathbf{C}^{p+q}$, $Z = 0$, and

$$t \cdot (y_1, \ldots, y_p, z_1, \ldots, z_q) = (ty_1, \ldots, ty_p, \bar{t}z_1, \ldots, \bar{t}z_q)$$

and the trick is to blow up W along Z in order to be in the previous simple situation. Thus look at

$$\widetilde{\mathbf{C}^{p+q}} = \{([\eta, \zeta], y, z) \text{ with incidence relation}\}$$

with the S^1-action extended linearly as usually:

$$t \cdot ([\eta, \zeta], y, z) = ([t\eta, \bar{t}\zeta], ty, \bar{t}z)\,.$$

The "point" $Z = 0$ becomes two fixed submanifolds included in the exceptional divisor $y = z = 0$: a $\mathbf{P}^{q-1}(\mathbf{C})$ ($\eta = 0$) and a $\mathbf{P}^{p-1}(\mathbf{C})$ ($\zeta = 0$). Note that all points of the exceptional divisor now have a $\mathbf{Z}/2$ as stabilizer. Let $\widetilde{H}$ be a hamiltonian for the S^1-action (with respect to some invariant symplectic form on the blown up manifold). Look now at the normal bundles and indices of critical submanifolds:

$$\mathbf{P}^{q-1}(\mathbf{C}) \subset_{p\mathcal{O}(-1)} \mathbf{P}^{p+q-1}(\mathbf{C}) \subset_{\mathcal{O}(-1)} \widetilde{\mathbf{C}^{p+q}}$$

the normal bundle thus being $p\mathcal{O}(-1)\oplus\mathcal{O}(-1) \to \mathbf{P}^{q-1}(\mathbf{C})$ where the first summand corresponds to the coordinate η and the second one to an incidence coordinate λ (such that $z = \lambda\zeta$). The S^1-action on those is given by $t\cdot(\eta,\lambda) = (t\eta, \bar{t}\lambda)$. Similarly for $\mathbf{P}^{p-1}(\mathbf{C}) \subset \mathbf{P}^{p+q-1}(\mathbf{C}) \subset \widetilde{\mathbf{C}^{p+q}}$ where we get $(\bar{t}\zeta, t\mu)$.

We can conclude as in [39] that we now have two copies of the preceeding simple situation. But let us continue and look at the following picture (figure 2). Call V'

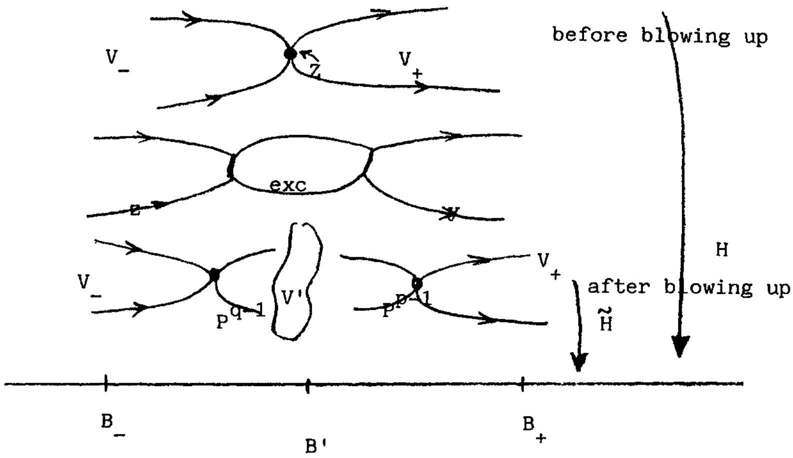

Figure 2

the regular level $\widetilde{H}^{-1}(0)$ and B' the quotient. Following the gradient or minus the gradient we get a diagram

$$\begin{array}{ccc} B' & \longrightarrow & B_+ \\ \downarrow & & \downarrow \\ B_- & \longrightarrow & B_0 \end{array}$$

as in [27] where B' was defined as a fiber product, a definition which could seem rather artificial from the topological viewpoint (if not from the algebraic one).

Remark. Recall that V_0 was a critical level, thus B_0 is not smooth in general.

Let us now describe more precisely the topology of V' and B' (and the map $B' \to B_0$). The regular level $V' = \widetilde{H}^{-1}(0)$ meets the exceptional divisor $\mathbf{P}^{p+q-1}(\mathbf{C})$

along $|\eta|^2 = |\zeta|^2$. This submanifold of $\mathbf{P}^{p+q-1}(\mathbf{C})$ is the quotient of $S^{2p-1} \times S^{2q-1} \subset S^{2p+2q-1}$ by the diagonal S^1-action. Being the common boundary of tubular neighborhoods of $\mathbf{P}^{p-1}(\mathbf{C})$ and $\mathbf{P}^{q-1}(\mathbf{C})$ in $\mathbf{P}^{p+q-1}(\mathbf{C})$, it is a sphere bundle over $\mathbf{P}^{q-1}(\mathbf{C})$ $(S(p\mathcal{O}(-1)) \to \mathbf{P}^{q-1}(\mathbf{C}))$ as well as over $\mathbf{P}^{p-1}(\mathbf{C})$ $(S(q\mathcal{O}(-1)) \to \mathbf{P}^{p-1}(\mathbf{C}))$. Look

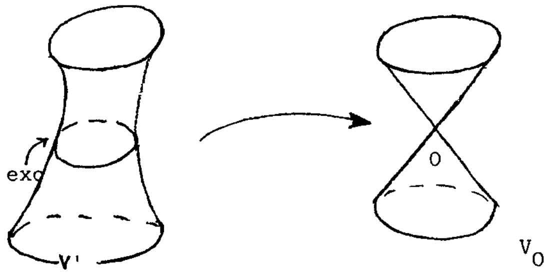

Figure 3

now at quotients: to construct B', we removed the singular point and replaced it with $V' \cap \mathrm{exc}/S^1$. This is the same as the quotient of $S(p\mathcal{O}(1)) \to \mathbf{P}^{q-1}(\mathbf{C})$ by $[\eta, \zeta] \sim [t\eta, \bar{t}\zeta] \sim [t^2\eta, \zeta]$ and thus is nothing other than the projectivised bundle $\mathbf{P}(p\mathcal{O}(1)) \to \mathbf{P}^{q-1}(\mathbf{C})$. Of course the projectivised bundle is trivial and the quotient is a $\mathbf{P}^{p-1}(\mathbf{C}) \times \mathbf{P}^{q-1}(\mathbf{C})$.

Remark. If q (or p) equals 1, we get the blowing up of a point (or a smooth submanifold) of B_0 which in this case is smooth as we have already remarked. In general the map $B' \to B_0$ is a blow-up in the algebraic sense.

$$\begin{array}{ccccc} B' & \xrightarrow{\sigma_-} & B_+ & \supset \mathbf{P}^{p-1} \\ \downarrow{\scriptstyle \sigma_+} & \searrow{\scriptstyle \sigma} & \downarrow & \\ B_- & \longrightarrow & B_0 & \\ \cup & & & \\ \mathbf{P}^{q-1} & & & \end{array} \tag{4}$$

4.3 Application: The Duistermaat-Heckman problem at critical values

Why are we looking for a diagram like (4)? In the Duistermaat-Heckman theorem 3.2.2 all the regular quotients of $H^{-1}(\pm\varepsilon)$ were identified with the same manifold $B_\pm$ and this allowed us to compare the classes $[\sigma_\varepsilon] \in H^2(B_+)$ (*resp.* $[\sigma_{-\varepsilon}] \in H^2(B_-)$) and to prove that in each case, staying in the same component of the set of regular values, $[\sigma_\varepsilon]$ was an affine function of ε.

The diagram (4) will be used to compare $[\sigma_\varepsilon]$ and $[\sigma_{-\varepsilon}]$.

Recall first the 4-dimensional case (*see* IV-2.2) in which B_+, B_0, B_- are the same surface B and thus it is very easy to compare σ_ε and $\sigma_{-\varepsilon}$ just by looking at the function

$$t \longmapsto \int_B \sigma_t.$$

Over each component of the regular set, the graph is a straight line (with integral slope in the semifree case we are considering). We have already studied the surgery when crossing the critical level V_0: the slope decreases by 1. In this situation, the change in the topology of the quotient is by... blowing up a point in $\mathbf{C}$ (*i.e.* no change at all). This apparently trivial action is nontrivial at the level of reduced symplectic forms where the -1 corresponds to a blow-up.

In the general case of an S^1-action: look at the diagram H^2(diagram (4)) in which we see

$$H^2(B_-) \hookrightarrow H^2(B') \hookleftarrow H^2(B_+)$$

which enables us to compare the classes of the reduced symplectic forms in $H^2(B')$.

Consider the curve $t \mapsto [\sigma_t] - [\sigma'_t]$ defined, for all $t > 0$, as follows:

- $[\sigma_t]$ is the class of the reduced symplectic form at level t considered as an element in $H^2(B')$ *via* the inclusion.
- $[\sigma_t]$ for $t < 0$, being an affine function of t, is easy to extend as an affine function for all t, $[\sigma'_t]$ denotes its value for the positive values of t (once again the inclusion is omitted).

Theorem 4.3.1 (**[27]**) *The curve $t \mapsto [\sigma_t] - [\sigma'_t]$ is a half-line, directed by the class of the exceptional divisor of the map $B' \to B_0$.*

Proof. The exceptional divisor of the map $B' \to B_0$ is the sum of the two exceptional divisors of $B' \to B_\pm$. Since everything is affine it is thus sufficient to prove the theorem in the "simple" case where it is an obvious consequence of the remark on reduced symplectic forms in 4.1. □

Another remark in the Guillemin and Sternberg paper is that everything we did, we could have done in an equivariant way for any group action which commutes with the S^1-action. Consider for instance a hamiltonian torus action with moment map $\mu : W \to \mathfrak{t}^*$ and try to compare the reduced symplectic forms when crossing a codimension 1 wall between two polytopes of regular values of μ. It is sufficient to be able to cross it orthogonally and thus to consider a $S^1 \times T^{n-1}$ situation which can be studied as before.

5 Localisation at fixed points

5.1 The support of a $H^\star BT$-module

Once again, $H^\star$ denotes de Rham cohomology. Identify

$$H^\star BT \cong S(\mathfrak{t}^\star) \cong \mathbf{R}[u_1, \ldots, u_n]$$

(the last isomorphism depending on the choise of a basis in $\mathfrak{t}$) in particular, consider elements of $H^\star BT$ as functions on $\mathfrak{t}$ or as polynomials in $(u_1, \ldots, u_n)$.

Let M be a $H^\star BT$-module. Its *support* is

$$\operatorname{Supp} M = \bigcap_{\{f \mid f \cdot M = 0\}} V_f \subset \mathfrak{t}$$

where, for $f \in H^\star BT \cong S(\mathfrak{t}^\star)$, $V_f = \{X \in \mathfrak{t} \mid f(X) = 0\}$.

Examples.

1. If M is free, then $f \cdot M = 0 \Rightarrow f = 0 \Rightarrow V_f = \mathfrak{t} \Rightarrow \operatorname{Supp} M = \mathfrak{t}$: modules having a proper support does have torsion.

2. If $T = S^1$, the ring $H^\star BT$ is a principal ideal domain (polynomials in one variable). If f is a generator of the ideal annihilating M, then $\operatorname{Supp} M$ consists of the zeros of f. In this example, one can imagine why it might be more convenient to use complex coefficients; it suffices for that to replace $H^\star BT$ by $H^\star BT \otimes \mathbf{C} \cong H^\star(BT; \mathbf{C})$ and $\mathfrak{t}$ by $\mathfrak{t} \otimes \mathbf{C}$ thus, when M does have torsion, there is a (proper) support. We shall see that in the hamiltonian case there is no problem and we can "stay real": the supports we shall have to consider will be unions of vector subspaces.

3. The module $\{0\}$ has empty support.

Lemma 5.1.1 *If $M' \xrightarrow{a} M \xrightarrow{b} M''$ is an exact sequence of $H^\star BT$-modules, then*

$$\operatorname{Supp} M \subset \operatorname{Supp} M' \bigcup \operatorname{Supp} M''.$$

Proof. Put $S = \operatorname{Supp} M$, $S' = \operatorname{Supp} M'$ and $S'' = \operatorname{Supp} M''$. Let $x \notin S' \cup S''$. Then, as $x \notin S'$, there exists a polynomial f such that $f(x) \neq 0$ and $f \cdot M' = 0$. For the same reasons, there exists a polynomial g annihilating M'' but not x. But

$$b(g \cdot M) = g \cdot b(M) \subset g \cdot M'' = 0$$

thus $g \cdot M \subset a(M')$ and $f \cdot (g \cdot M) \subset a(f \cdot M') = 0$ hence $(fg) \cdot M = 0 \ldots\ f(x)g(x) \neq 0$ and so $x \notin S$. □

Similarly:

Exercise 5.1.2 If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of H^*BT-modules, then

$$\operatorname{Supp} M = \operatorname{Supp} M' \cup \operatorname{Supp} M''.$$

and also:

Lemma 5.1.3 *If M et M' are H^*BT-algebras*[7] *and if $a : M' \to M$ is a morphism of algebras, then* $\operatorname{Supp} M \subset \operatorname{Supp} M'$.

Proof. If $x \notin \operatorname{Supp} M'$, there is a polynomial f annihilating M' but not x. The unit $1_{M'}$ is killed by f, $a(f \cdot 1_{M'}) = f \cdot 1_M = 0$ so f kills M and $x \notin \operatorname{Supp} M$. □

5.2 Supports of $H_T^*(\mathcal{U})$, examples

Let us now begin to investigate the H^*BT-module structure of the equivariant cohomology of T-manifolds.

Lemma 5.2.1 *Let H be a closed subgroup of T. Then*

$$\operatorname{Supp} H_T^*(T/H) = \mathfrak{h} \subset \mathfrak{t}.$$

Proof. Remark first that $ET \times_T T/H \sim ET/H \sim BH$, whence we deduce that $H_T^*(T/H)$ is simply H^*BH with the H^*BT-module structure given by the restriction

$$H^*BT \longrightarrow H^*BH.$$

On the other hand, H is a closed subgroup of the torus T, and so, up to the fact that it might not be connected, it is a torus. More precisely, it is the product of a torus H_0 by some finite group that de Rham cohomology will not even notice and hence we suppose that H is a torus.

The structure of H^*BH as an H^*BT-module is rather easy to understand. The inclusion of the Lie algebra

$$i : \mathfrak{h} \longrightarrow \mathfrak{t}$$

induces the restriction of polynomials

$$S({}^t i) : S(\mathfrak{t}^*) \longrightarrow S(\mathfrak{h}^*).$$

The torsion elements of H^*BH are thus all the polynomials on $\mathfrak{t}$ whose restriction to $\mathfrak{h}$ vanishes. Thus

$$\operatorname{Supp} H_T^*(T/H) = \bigcap_{f_{|\mathfrak{h}}=0} V_f = \mathfrak{h}$$

□

Corollary 5.2.2 *Let X be a T-manifold. Assume there exists an equivariant map $X \to T/H$. Then* $\operatorname{Supp} H_T^*(X) \subset \mathfrak{h}$.

[7] with units, of course.

Proof.

$$f^\star : H_T^\star(T/H) \longrightarrow H_T^\star(X)$$

is a ring morphism. Applying lemma 5.1.3 we find that

$$\operatorname{Supp} H_T^\star(T/H) \supset H_T^\star(X)$$

□

The corollary may be applied in particular when X is an equivariant tubular neighborhood of any orbit of the smooth T-action on a manifold W (thanks to the slice theorem) and gives:

Corollary 5.2.3 *If X is an equivariant tubular neighborhood of a type (H) orbit in the T-manifold W, then* $\operatorname{Supp} H_T^\star(X) \subset \mathfrak{h}$. □

and in the same way

Corollary 5.2.4 *If T acts freely on W, then* $\operatorname{Supp} H_T^\star(W) = 0$. □

Considering now the union of all orbits in W whose stabilizer is a proper subgroup of T *i.e.* the complement of the fixed point set F, we are going to prove:

Proposition 5.2.5 *Let F be the fixed point set of a T-action on a compact manifold W. Then*

$$\operatorname{Supp} H_T^\star(W - F) \subset \bigcup_H \mathfrak{h}$$

where H describes the (finite) set of all the (proper) stabilizers of points in W.

Remark. In particular, this support is a proper subspace of $\mathfrak{t}$, contained in a finite union of vector subspaces, thus $H_T^\star(W - F)$ is a torsion $H^\star BT$-module.

Proof. The space $W - F$ has the same (equivariant) homotopy type as the complement of a small equivariant neighborhood of F. The complement may be covered by open subsets $\mathcal{U}_1, \dots, \mathcal{U}_h$ which are equivariant tubular neighborhoods of orbits (with proper subgroups of T as stabilizers).

We may apply corollary 5.2.3 to each $\mathcal{U}_i$, and this will allow us to prove by induction on h :

$$\operatorname{Supp} H_T^\star(\mathcal{U}_1 \cup \dots \cup \mathcal{U}_h) \subset \mathfrak{h}_1 \cup \dots \cup \mathfrak{h}_h.$$

Start the induction by applying 5.2.3 to $\mathcal{U}_1$. Now suppose the result to hold up to order h, put $\mathcal{V}_h = \mathcal{U}_1 \cup \dots \cup \mathcal{U}_h$ and thus $\mathcal{V}_{h+1} = \mathcal{V}_h \cup \mathcal{U}_{h+1}$.

We have $\operatorname{Supp} H_T^\star(\mathcal{V}_h) \subset \bigcup_{i=1}^h \mathfrak{h}_i$ by the induction hypothesis, and hence we can use the Mayer-Vietoris exact sequence:

$$\begin{array}{ccccc} H_T^\star(\mathcal{V}_h \cap \mathcal{U}_{h+1}) & \longrightarrow & H_T^\star(\mathcal{V}_{h+1}) & \longrightarrow & H_T^\star(\mathcal{V}_h) \oplus H_T^\star(\mathcal{U}_{h+1}) \\ M' & \longrightarrow & M & \longrightarrow & M'' \end{array}$$

in which

- $\mathcal{V}_h \cap \mathcal{U}_{h+1} \subset \mathcal{U}_{h+1}$ is endowed with an equivariant map onto the central orbit of $\mathcal{U}_{h+1}$ in such a way that $\operatorname{Supp} M' \subset \mathfrak{h}_{h+1}$.
- $\operatorname{Supp} M'' = \operatorname{Supp} H_T^*(\mathcal{V}_h) \cup \operatorname{Supp} H_T^*(\mathcal{U}_{h+1}) = \bigcup_{i=1}^h \mathfrak{h}_i \cup \mathfrak{h}_{h+1}$ by exercise 5.1.2.

This gives the result for $\operatorname{Supp} \mathcal{V}_{h+1}$ and for $\operatorname{Supp} M$ it follows from lemma 5.1.1 and the proposition. □

5.3 The localisation theorem

Our goal in this investigation is to prove rather precise versions of the fact that, forgetting torsion, the H^*BT-module $H_T^*(W)$ looks very much like the *free* H^*BT-module $H_T^*(F)$.

Theorem 5.3.1 *Let $i : F \hookrightarrow W$ be the inclusion of fixed points. Then both the kernel and cokernel of*

$$i^* : H_T^*(W) \longrightarrow H_T^*(F)$$

have support included in $\bigcup_{H \text{ stabilizer} \neq T} \mathfrak{h}$.

Proof. Let $\mathcal{U}$ be an equivariant tubular neighborhood of the fixed point set F. We know that

$$\operatorname{Supp} H_T^*(W - \mathcal{U}) \subset \bigcup_{H \neq T} \mathfrak{h}$$

and that the same is true of $\operatorname{Supp} H_T^*(\partial(W - \mathcal{U}))$. Using the long (equivariant) cohomology exact sequence of the pair $(W - \mathcal{U}, \partial(W - \mathcal{U}))$, the same is true for the support of $H_T^*(W - \mathcal{U}, \partial(W - \mathcal{U}))$.

Let $\mathcal{V}$ be another equivariant tubular neighborhood, a little larger than $\mathcal{U}$, such that $\mathcal{V} - \mathcal{U} \sim \partial(W - \mathcal{U}) = \partial\mathcal{U}$. Thus:

$$H_T^*(W, F) \cong H_T^*(W, \mathcal{V}) \overset{\text{excision}}{\cong} H_T^*(W - \mathcal{U}, \mathcal{V} - \mathcal{U}) \cong H_T^*(W - \mathcal{U}, \partial(W - \mathcal{U}))$$

and in particular, $\operatorname{Supp} H_T^*(W, F) \subset \bigcup \mathfrak{h}$.

The exact sequence of the pair (W, F)

$$H_T^*(W, F) \longrightarrow H_T^*(W) \xrightarrow{i^*} H_T^*(F) \longrightarrow H_T^*(W, F)$$

alows us to conclude. □

The rank of the H^*BT-module $H_T^*(W)$ is thus the same as the rank of the free module $H_T^*(F)$. It is possible to give an even more precise statement, using the notion of *localisation.*

If $f \in H^*BT$ is a nonzero element, consider the *localised* subring $(H^*BT)_f$ of the ring of fractions H^*BT, consisting of all the fractions which have a power of f as denominator. In the same way, for any module M, consider

$$M_f = M \bigotimes_{H^*BT} (H^*BT)_f$$

the *localised* $(H^*BT)_f$-module obtained by extension of scalars.

Let f be a polynomial which vanishes on all the $\mathfrak{h}$'s. Consider the induced morphism

$$i^\star : (H_T^\star(W))_f \longrightarrow (H_T^\star(F))_f$$

It is an isomorphism because of

Lemma 5.3.2 *Let $M' \xrightarrow{a} M \xrightarrow{b} M''$ be an exact sequence of modules. Then the sequence*

$$M'_f \xrightarrow{a'} M_f \xrightarrow{b'} M''_f$$

is exact.

In other words, $(H^\star BT)_f$ is flat.

Proof. $b' \circ a' = 0$ as $b \circ a$, hence $\operatorname{Im} a' \subset \operatorname{Ker} b'$. Reciprocally, let $x \in \operatorname{Ker} b'$. By the definition of M_f, we can find an integer m such that $f^m x \in M$ and as we have $b'(x) = 0$, $b(f^m x) = f^m b'(x) = 0$ so that $f^m x \in \operatorname{Ker} b = \operatorname{Im} a$. We can thus write $f^m x = a(y)$ and

$$x = \frac{a(y)}{f^m} = a'\left(\frac{y}{f^m}\right) \in \operatorname{Im} a'.$$

□

According to the lemma, the kernel and cokernel of the (localised) morphism $i^\star$ are the localisations of the kernel (N) and cokernel (C). On the other hand, we know that $\operatorname{Supp} N \subset V_f$ and $\operatorname{Supp} C \subset V_f$ by the definition of f. Thus N_f and C_f are zero and $i^\star$ is actually an isomorphism.

Because of formula (8) (*see* appendix A) relating $i^\star$ and the Gysin morphism $i_\star$:

$$i^\star i_\star 1 = e_T(\nu),$$

we suspect that $i_\star$ is more or less the inverse isomorphism, up to the inversion of some Euler classes. This is what we shall make precise now.

Let us begin with two properties of $i_\star$ analogous to 5.3.1 for $i^\star$:

Proposition 5.3.3 *The kernel and cokernel of*

$$i_\star : H_T^\star(F) \longrightarrow H_T^\star(W)$$

both have support in the union $\bigcup \mathfrak{h}$ of the Lie algebras of the stabilizers $\neq T$ of points of W.

Proof. Consider the cohomology exact sequence of the pair $(W, W - F)$

$$\begin{array}{ccccccc} H_T^\star(W-F) & \longrightarrow & H_T^\star(W, W-F) & \longrightarrow & H_T^\star(W) & \longrightarrow & H_T^\star(W-F) \\ & & \downarrow \cong & \nearrow i_\star & & & \\ & & H_T^\star(F) & & & & \end{array}$$

and deduce the result from the previous lemmas. □

Let us now invert Euler classes. Any polynomial f vanishing on the support is suitable to define a localisation. We are going to construct a minimal such f.

Let Z be a component of the fixed point set F of the T-action on W. Consider its equivariant Euler class $e_T(\nu_Z) \in H_T^\star(Z)$ (*see* A.2.2). As the T-action on Z is trivial, $H_T^\star(Z) \cong H^\star(Z) \otimes H^\star BT$ and we may decompose

$$e_T(\nu_Z) = \sum_{i,j} a_i^j \otimes f_{2m-i}^j \in H^\star(Z) \otimes H^\star BT$$

where the lower index denotes the degree of the cohomology class under consideration, and $2m$ is the codimension of the submanifold Z in W.

Of course, all degree > 0 elements in $H^\star(Z)$ are nilpotent, and thus:

Lemma 5.3.4 *With these notations, $e_T(\nu_Z)$ is invertible in $H_T^\star(Z)$ if and only if $\sum_j a_0^j \otimes f_{2m}^j$ is invertible.* □

Let $z \in Z$ be a fixed point. The sum $\sum a_0^j \otimes f_{2m}^j$ may be interpreted as the restriction of $e_T(\nu_Z)$ in $H^0(z) \otimes H^{2m}BT \cong H_T^\star(z)$. Let us compute this restriction.

Write

$$\mathsf{T}_z W = \mathsf{T}_z Z \oplus L_1 \oplus \ldots \oplus L_m$$

where T acts on L_i by the morphism $\alpha_i : T \to S^1$, and denote also by $\alpha_i : \mathfrak{t} \to \mathbf{R}$ the linear form which is the derivative of α_i at 1. Then the restriction of $e_T(\nu_Z)$ is (up to sign) the product of the linear forms α_i (considered as a polynomial on $\mathfrak{t}$)[8].

Remark. It is clear that the kernel of α_i is one of the $\mathfrak{h}$'s in the support of $H_T^\star(W - F)$. Indeed, $H = \operatorname{Ker} \alpha_i \subset T$ fixes vectors in the subspace L_i, but the T-action on L_i is nothing other than the derivative at z of that on W.

Let us put $f_Z = \prod \alpha_i$ (it is $e_T(\nu_z)$ for some z in Z) and $f = \prod_{Z \subset F} f_Z$. Then all the classes $e_T(\nu_Z)$ are invertible in $(H_T^\star(F))_f$ and the map

$$Q = \sum_{Z \subset F} \frac{i_Z^\star}{e_T(\nu_Z)}$$

(where each i_Z is the inclusion of the component Z) defines a homomorphism

$$Q : (H_T^\star(W))_f \longrightarrow (H_T^\star(F))_f$$

inverse to $i_\star$, in other words:

Theorem 5.3.5 (localisation at fixed points) *If $x \in H_T^\star(W)$, in a suitable localisation,*

$$x = \sum_{Z \subset F} \frac{i_\star^Z i_Z^\star x}{e_T(\nu_Z)}.$$

□

[8]This is the definition of equivariant Euler classes *see* A.2.2.

5.3.6 Actually, in the hamiltonian case, using a well chosen projection of the moment map and the ideas of [36], one shows that the H^*BT-module H_T^*W is the free module generated by H^*W. The restriction to the fiber $H_T^*W \to H^*W$ is surjective and the restriction to the fixed points $i^* : H_T^*W \to H_T^*F$ is injective.

6 The Duistermaat-Heckman formula

6.1 The Duistermaat-Heckman formula

Theorem 6.1.1 (**[34]**) *Let (W,ω) be a compact symplectic manifold of dimension $2n$ and let H be a periodic hamiltonian on W with only isolated fixed points. Then*

$$\int_W e^{-Hu}\frac{\omega^{\wedge n}}{n!} = \sum_{Z\in F}\frac{e^{-uH(Z)}}{e_{S^1}(\nu_Z)}.$$

Remarks.

- It is not absolutely necessary to assume that the fixed points are isolated. There is a more general statement in [34].
- Similarly, it is not necessary to consider only circle actions, analogous results hold with higher dimensional tori.
- The formula must be read as an equality of formal series in the variables u and u^{-1}. We shall come back to this remark later on.
- If the formal variable u is replaced by it where t is a real parameter, this formula is an *exact* (*i.e.* without rest) "stationary phase formula".
- Reading again it instead of u, and up to Fourier transform, the formula expresses the fact that the image measure on $\mathbf{R}$ (or on $\mathfrak{t}^*$ if we are dealing with a higher dimensional torus) of the symplectic volume $\omega^{\wedge n}/n!$ under the moment map has a piecewise polynomial density (with respect to Lebesgue measure).

Proof. As H is a periodic hamiltonian, we know that the symplectic form ω may be "extended" to W_{S^1} (3.1.1) as a closed 2-form $\omega^\natural$. Consider the latter and all its powers. In order to deal with all of them at the same time, we use formally $\alpha = \exp[\omega^\natural]$ as if this was an element of $H_{S^1}^*(W)$. Apply first the localisation theorem 5.3.5:

$$\alpha = \sum_{Z\subset F}\frac{i_*^Z i_Z^*\alpha}{e_{S^1}(\nu_Z)}. \tag{5}$$

Then "integrate in the fibers" (*see* A.6.1) of the fibration $\pi : W_{S^1} \to BS^1$, in other words apply π_*. The right hand side of (5) becomes (as $\pi_* i_* = 1$, *see* A.6.4 applied to the section $BT = ET \times_T Z \hookrightarrow ET \times_T W = W_T$):

$$\sum_{Z\in F}\frac{i_Z^*\alpha}{e_{S^1}(\nu_Z)}.$$

We therefore calculate $i_Z^*\alpha = \exp i_Z^*\omega^\sharp$. The "form" $\omega^\sharp$ is the projection of $\omega + d(\theta H)$, so $i_Z^*\omega^\sharp$ is the projection of $i_Z^*(\omega + d\theta H - \theta \wedge dH)$. As Z is a point, this is simply the projection of $d\theta H(Z)$ that is $-uH(Z)$.

The right hand side of (5) is thus

$$\sum_{Z\in F} \frac{e^{-uH(Z)}}{e_{S^1}(\nu_Z)}.$$

Look now at the left hand side: $\pi_*\alpha = \pi_* \exp \omega^\sharp$. We thus have to consider $\exp(\omega + d\theta H - \theta \wedge dH)$ on $ES^1 \times W$, to integrate over W and to project. During the integration process over W, all the terms in the exponential of degree different from $2n$ vanish. There remains only $\omega^{\wedge n}/n!$ and $(dH)^{2n}/(2n)!$, the latter form being exact, it also integrates to zero. The form $d\theta$ still projects onto $-u$. The left hand side in (5) is thus

$$\int_W e^{-Hu}\frac{\omega^{\wedge n}}{n!}$$

as we annouced. □

6.2 Examples of applications

In this paragraph, we make formula 6.1.1 talk. The left hand side is a formal series in u, but not *a priori* the right hand side: if S^1 acts near the fixed point Z by

$$t \cdot (z_1, \ldots, z_n) = (t^{a_1} z_1, \ldots, t^{a_n} z_n)$$

then $\pm a_1 \ldots a_n u^n = e_{S^1}(\nu_Z)$ appears in the denominator. The Duistermaat-Heckman formula implies therefore that certain elements must vanish.

Here are some examples of these cancellations. Assume for simplicity that H is the hamiltonian of a semifree action (*i.e.* $a_i = \pm 1$). Let $\lambda(Z)$ be the number of minus signs (the index of Z as a critical point of H is $2\lambda(Z)$). The formula is then:

$$\int_W e^{-Hu}\frac{\omega^{\wedge n}}{n!} = \frac{1}{u^n}\sum_{Z\in F}(-1)^{\lambda(Z)}e^{-uH(Z)}.$$

Put "$u = 0$" in this formula.

1. The case where $n = 1$.

$$\int_W \omega = \left[\frac{1}{u}\left(\sum_{Z\in Min} e^{-uH(Z)} - \sum_{Z\in Max} e^{-uH(Z)}\right)\right]_{u=0}$$

in other words

$$\operatorname{Vol} W = H(max) - H(min)$$

as we know (III-3.2.1) that there are only one local minimum and one local maximum (*see* figure 4).

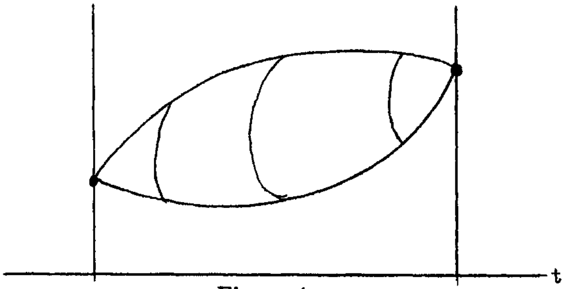

Figure 4

2. The case where $n = 2$. Let ρ_i be the number of index $2i$ critical points and let F_i be the set of such points.

$$\int_W \frac{\omega^{\wedge 2}}{2} = \left\{ \frac{1}{u^2} \left[\rho_0 - \rho_1 + \rho_2 - u \left(\sum_{Z \in F_0} H(Z) - \sum_{Z \in F_1} H(Z) + \sum_{Z \in F_2} H(Z) \right) \right.\right.$$
$$\left.\left. + \frac{u^2}{2} \left(\sum_{Z \in F_0} H(Z)^2 - \sum_{Z \in F_1} H(Z)^2 + \sum_{Z \in F_2} H(Z)^2 \right) \right] \right\}_{u=0}$$

The degree 0, 1 and 2 terms in the brackets give respectively

- $\rho_0 - \rho_1 + \rho_2 = 0$ but we know that $\rho_0 = \rho_1 = 1$ and thus there are exactly two index 2 points. Call them a and b.
- $H(min) - (H(a) + H(b)) + H(max) = 0$, in other words the two intervals $[min, max]$ and $[a, b]$ have the same middle. We already noticed these two properties in IV-2.3.3 and in the discussion following 3.2.2 (*see* figure 1).
- The third term computes the volume:

$$\operatorname{Vol} W = \frac{1}{2} \left[H(min)^2 - (H(a)^2 + H(b)^2) + H(max)^2 \right]$$

  It is the surface of the trapezium in figure 1! in other words this is the integral of the volume function $t \mapsto \operatorname{Vol} B_t$. Actually in this case, using the methods of chapter IV, it is rather easily seen that $W = S^2 \times S^2$, and that the S^1-action extends to a T^2-action... of course the trapezium is the image of the moment map.

3. The case where $n = 3$. With the same notations, and the same hypotheses, the same method gives for example

$$\rho_0 - \rho_1 + \rho_2 - \rho_3 = 0$$

This, added to the fact that $\rho_0 = \rho_3 = 1$ gives $\rho_1 = \rho_2$. More generally, notice we allways have $\rho_i = \rho_{n-i}$ in dimension $2n$ when the fixed points are isolated: this is Poincaré duality, as H is a "perfect" Morse function and so $\rho_k = \dim H^{2k}(W)$.

Remark. In dimension 6 ($n = 3$) and with the hypotheses above, it can be shown that $\rho_1 = \rho_2 = 3\ldots$ using essentially the same method, a localisation formula near fixed points analogous to à 5.3.5, but in *equivariant K-theory.* In some sense it is even easier: the group $K_T(W)$ is defined (starting with T-vector bundles over W) much more simply than $H_T^*(W)$ for which we had to use classifying spaces and the Borel construction.

The functor K_T has properties analogous to that of H_T^*, which allows us to prove a localisation theorem. From this theorem, Hattori remarked in [41] that it was possible to deduce, among other things, with our hypotheses (periodic hamiltonian, isolated critical points, semifree action) that $\rho_i = \binom{n}{i}$. For $n = 2$, we have already explained this several times; for $n = 3$ remark that the method, even if analogous, is nevertheless more powerful, as we are not able to deduce $\rho_1 = \rho_2 = 3$ from 5.3.5 ou 6.1.1.

It is also with the help of the localisation formula in equivariant K-theory, for hamiltonian torus actions on toric varieties (*see* chapter VI) that M. Brion gave very elegant proofs of some results about convex polyhedra (*see* [26]).

Exercise 6.2.1 Let (W, ω) be a compact symplectic manifold of dimension $2n$ endowed with a hamiltonian action of a torus T^n (of half the dimension). Let P be the polyhedron which is the image of W under the moment map μ.

1. Let AB be an edge in P. Show that it is the image of a symplectic sphere $S \subset W$.

2. Write $\overrightarrow{AB} = \lambda\vec{u}$ where $\vec{u} \in \mathbf{Z}^n$ is indivisible and $\lambda \in \mathbf{R}$. What can be said about the stabilizer of the points in S? What is the volume $\int_S \omega$ of the sphere S?

A Appendix: some algebraic topology

We shall give here some hints of proofs and references for the main notions of algebraic topology used in this chapter. There are a lot of good books. We already mentioned the one by Husemoller [15] when we used it at the beginning of the chapter. I like very much and used a lot the one by Milnor [16] especially for this appendix.

A.1 The Thom class of an oriented vector bundle

Let $E \to B$ be an oriented (for example, complex) vector bundle of (real) rank n. Denote E_0 the complement of the zero section in E, F a fiber and F_0 the intersection $F \cap E_0$. F is the vector space $\mathbf{R}^n$ and we have

$$H^n(F, F_0; \mathbf{Z}) = H^n(\mathbf{R}^n, \mathbf{R}^n - \{0\}; \mathbf{Z}) \cong H^{n-1}(\mathbf{R}^n - \{0\}; \mathbf{Z}) \cong H^{n-1}(S^{n-1}; \mathbf{Z}) \cong \mathbf{Z}$$

so that the chosen orientation (which specifies the last isomorphism) gives, for each fiber F an element U_F which generates the infinite cyclic group $H^n(F, F_0; \mathbf{Z})$. The

Thom isomorphim theorem asserts the existence of a class on E the restriction of which to each fiber is U_F :

Theorem A.1.1 (Thom isomorphism) *There exists a unique cohomology class $U \in H^n(E, E_0; \mathbf{Z})$ whose restriction to $H^n(F, F_0; \mathbf{Z})$ is U_F (for any fiber F). Moreover the map:*

$$\begin{array}{ccc} H^\star(E) & \longrightarrow & H^\star(E, E_0; \mathbf{Z}) \\ y & \longmapsto & y \smile U \end{array}$$

is an isomorphism.

For the proof, first look at the case of the trivial bundle, which is rather easy, then cover the base space by open subsets which trivialise the bundle, and (at least if the base space B is compact) deduce the result by induction on the number of open subsets, with the help of Mayer-Vietoris and the five lemma (*see* [16] for details). □

The class U is called the *Thom class* of the oriented bundle E, it is clear that a change in orientation will change the sign of U. It is also clear that U is *natural:* if $f : B' \to B$ is a continuous map, $U(f^\star E) = f^\star U(E)$.

The isomorphism $y \mapsto y \smile U$ is the *Thom isomorphism.*

A.2 The Euler class of an oriented bundle, equivariant Euler class

Definition A.2.1 *The unique class $e \in H^n(B; \mathbf{Z})$ such that $\pi^\star e = j^\star U$ where U is the Thom class and the maps are the natural ones*

$$H^n(E, E_0; \mathbf{Z}) \xrightarrow{j^\star} H^n(E; \mathbf{Z}) \xleftarrow{\pi^\star} H^n(B; \mathbf{Z})$$

is called the Euler class of the oriented bundle $\pi : E \to B$.

Remark. Changing the orientation of E changes the sign of e. On the other hand, e is natural as is the Thom class we used to define it.

A.2.2 The equivariant Euler class. Recall that $E \to B$ is a *G-vector bundle* if it is a vector bundle, endowed with a G-action which is linear in the fibers and compatible with some G-action on the base space. This is what we need to form the vector bundle $EG \times_G E \to EG \times_G B$.

As such, it has an Euler class (if E is oriented) which we shall denote

$$e_G(E) \in H^\star_G(B)$$

as we intend to call it the *equivariant Euler class* of the bundle $E \to B$.

A G-vector bundle may be trivial as a vector bundle without being trivial as a G-vector bundle, as the following example shows.

Example. If B is a point, a G-vector bundle over B is nothing other than a linear representation of the group G. Such a representation has an Euler class in $H^\star BG$. For instance if $G = S^1$ and if $E = \mathbf{C}$ with the action $t \cdot v = t^m v$ then $ES^1 \times_{S^1} E$ is the bundle $\mathcal{O}(-m)$ over BS^1 (*see* appendix B).

A.3 The Gysin exact sequence

Proposition A.3.1 (Gysin exact sequence) *There exists an exact sequence*

$$\longrightarrow H^i(B)\xrightarrow{\smile e}H^{i+n}(B)\xrightarrow{\pi_0^*}H^{i+n}(E_0)\longrightarrow H^{i+1}(B)\longrightarrow$$

Proof. Use the long cohomology exact sequence of the pair (E, E_0) and the previous isomorphisms.

$$\begin{array}{ccccccccc}
\longrightarrow & H^k(E,E_0) & \xrightarrow{j^*} & H^k(E) & \longrightarrow & H^k(E_0) & \xrightarrow{\delta} & H^{k+1}(E,E_0) & \longrightarrow \\
 & \uparrow\smile U & & \uparrow\pi^* & \nearrow \pi_0^* & & & \uparrow\smile U & \\
 & H^{k-n}(E) & & H^k(B) & & & & H^{k-n+1}(E) & \\
 & \uparrow\pi^* & \nearrow\smile e & & & & & \uparrow\pi^* & \\
 & H^{k-n}(B) & & & & & & H^{k-n+1}(B) &
\end{array}$$

□

A.4 The cohomology of projective space

To prove proposition 2.2.4, we use the Gysin exact sequence for the "tautological" line bundle:

$$E = \mathcal{O}(-1) = \left\{(l,v) | l \subset \mathbf{C}^{n+1}, v \in l\right\} \subset \mathbf{P}^n(\mathbf{C}) \times \mathbf{C}^{n+1}$$

E_0 is the set of all pairs (l, v) where $v \neq 0$, it is easily identified with $\mathbf{C}^{n+1} - \{0\}$ and its homotopy type is that of the sphere S^{2n+1}. Let v be the Euler class of the bundle E (u in the statement 2.2.4 is just $-v$). We may write the Gysin exact sequence:

$$\longrightarrow H^{k+1}(S^{2n+1}) \longrightarrow H^k(\mathbf{P}^n(\mathbf{C}))\xrightarrow{\smile v}H^{k+2}(\mathbf{P}^n(\mathbf{C}))\xrightarrow{\pi_0}H^{k+2}(S^{2n+1}) \longrightarrow$$

from which we immediately deduce that all the $H^{2k+1}(\mathbf{P}^n(\mathbf{C}))$ vanish, and that

$$\smile v : H^{2k}(\mathbf{P}^n(\mathbf{C})) \longrightarrow H^{2k+2}(\mathbf{P}^n(\mathbf{C}))$$

is an isomorphism for $0 \leq 2k \leq 2n - 2$. □

A.5 The Gysin homomorphism (case of an embedding)

Let $Z \overset{i}{\hookrightarrow} W$ be a codimension m submanifold, with normal bundle ν (assumed to be oriented). We define a *Gysin* homomorphism

$$i_* : H^k(Z) \longrightarrow H^{k+m}(W)$$

(which goes the wrong way and which does not preserve the graduation) by the composition:

$$H^k(Z)\xrightarrow{\Phi}H^{k+m}(\nu,\nu_0) \overset{\text{excision}}{\cong} H^{k+m}(W, W - Z)\xrightarrow{\text{restriction}}H^{k+m}(W)$$

where of course, Φ denotes the Thom isomorphism for the bundle ν.

By the very definition of the Euler class, we have the fundamental property

$$i^\star i_\star 1 = e(\nu) \tag{6}$$

and similarly

$$i^\star i_\star x = x \smile e(\nu). \tag{7}$$

Actually, there is a commutative diagram:

$$\begin{array}{ccccc} 1 & \longmapsto & U & & \\ H^0(Z) & \longrightarrow & H^m(\nu,\nu_0) & \longrightarrow & H^m(W) \\ & & \downarrow j^\star & & \downarrow i^\star \\ & & H^m(\nu) & \xleftarrow{\pi^\star} & H^m(Z) \quad \ni e(\nu) \end{array}$$

which allows us to prove (6). Similarly for (7), since $\pi^\star$ and $j^\star$ preserve multiplicative structures and since the Thom isomorphism is $x \mapsto x \smile U$. □

A.5.1 The equivariant case. Suppose G acts on W and Z, the embedding i being equivariant. Up to finite approximations, we may assume that

$$EG \times_G Z \overset{i}{\hookrightarrow} EG \times_G W$$

is the embedding of a submanifold. Therefore we have a Gysin homomorphism $i_\star : H^\star_G(Z) \to H^\star_G(W)$. Computing $i^\star i_\star 1$ gives the Euler class of the normal bundle of $EG \times_G Z$ that is, the equivariant Euler class of ν:

$$i^\star i_\star x = x \smile e_G(\nu). \tag{8}$$

A.6 The Gysin homomorphism: integration in the fibers

A.6.1 Let $\pi : E \to B$ be a locally trivial bundle with compact fibre F. Assume that B, E and F are manifolds and moreover that F is oriented and has dimension m.

Let ω be a k-form on E. We can associate to it a $(k-m)$-form on B, denoted $\pi_\star\omega$ and defined thus: if $k < m$, $\pi_\star\omega = 0$, otherwise

$$(\pi_\star\omega)_b\,(X_1,\ldots,X_{k-m}) = \int_{\pi^{-1}(b)} \alpha$$

where α is the m-form on $\pi^{-1}(b) \cong F$ such that

$$\alpha_x\,(V_1,\ldots,V_m) = \omega_x\left(\tilde{X}_1(x),\ldots,\tilde{X}_{k-m}(x),V_1,\ldots,V_m\right)$$

... in which formula $\tilde{X}_i(x)$ is any vector in T_xE such that $\mathsf{T}_x\pi(\tilde{X}_i(x)) = X_i \in \mathsf{T}_bE$ (it is clear that the result of this process is well defined).

We thus get a map

$$\pi_\star : \Omega^k(E) \longrightarrow \Omega^{k-m}(B)$$

Exercise A.6.2 The diagram

$$\begin{array}{ccc} \Omega^k(E) & \xrightarrow{\pi_\star} & \Omega^{k-m}(B) \\ \downarrow d & & \downarrow d \\ \Omega^{k+1}(E) & \xrightarrow{\pi_\star} & \Omega^{k-m+1}(B) \end{array}$$

commutes.

We thus get a morphism, still denoted $\pi_\star : H^k(E) \to H^{k-m}(B)$, of which it is not hard to make a relative version which will appear in the following example (of course, in this paragraph, we are dealing with de Rham cohomology).

Example. $\pi : S^m \to pt$ may be considered as a fibration with fiber S^m. The homomorphism

$$\begin{array}{cccc} \pi_\star : & H^m(S^m) & \longrightarrow & H^0(pt) \\ & \omega & \longmapsto & \int_{S^m} \omega \end{array}$$

sends the generator to 1. From the definition of the Thom class U, we deduce that if E is an oriented rank n vector bundle over B, then $\pi_\star : H^n(E, E_0) \to H^0(B)$ sends U to 1.

A.6.3 The reader may be frightened by the present terminology and ask (with some reason) what relationship exists between the $_\star$ considered here (case of fibrations) and the $_\star$ of A.4 (case of embeddings). She or he may keep quiet $\pi_\star$ and $i_\star$ are two aspects of a single *Gysin homomorphism* $f_\star$ defined for any *proper* map $f : V \to W$ from one manifold into another. One possible way to define $f_\star$ is precisely to decompose f as an embedding $i : V \to W \times S^n$ (n large) and a fibration $\pi : W \times S^n \to W$ and to put $f_\star = \pi_\star \circ i_\star$ (it is not very hard to prove that it does not depend on the choices).

A.6.4 Consider for instance the case of the inclusion of a *section* $i : B \hookrightarrow E$ of the bundle. We then have, for any $y \in H^\star(B)$, $\pi_\star i_\star y = y$. In fact, $i_\star$ is the composition of the horizontal maps in

$$\begin{array}{ccccc} y & \longmapsto & y \smile U & & \\ H^m(B) & \longrightarrow & H^m(\nu, \nu_0) & \longrightarrow & H^{m+n}(E) \\ & & & \searrow & \downarrow \pi_\star \\ & & & & H^n(B) \end{array}$$

but we saw in the previous example that the Thom class is sent to 1.

B Appendix: various notions of Euler classes

We have already met two notions of Euler classes for fixed point free S^1-actions: in chapter I for 3-manifolds, where we got a *number*, and in 2.4 where a degree 2 cohomology class was constructed. On the other hand, we have also used the Euler class of a (real) oriented vector bundle. In this appendix, we shall investigate the relations between these various notions.

B.1 The case of S^1-bundles

Because all relevant definitions can be ultimately expressed in terms of principal bundles or actions, it is enough to study this latter case. We shall thus prove:

Theorem B.1.1 *Let V be an oriented compact 3-manifold endowed with an S^1-action. Let B be the oriented surface which is the quotient. If $e \in H^2(B)$ is the Euler class of the S^1-bundle $V \to B$ (in the sense of chapter V), then $\langle e, [B] \rangle \in \mathbf{Z}$ is the Euler number in the sense of chapter I.*

Proof.

1. *It suffices to prove this in the case where B is a sphere*: indeed, write $B = \overline{B - D_0} \cup \overline{D_0}$ and the bundle is trivialised over each piece. Define a map $f : B \to S^2$ by collapsing the complement of a collar $\overline{B - D_0}$ onto a point (figure 5).

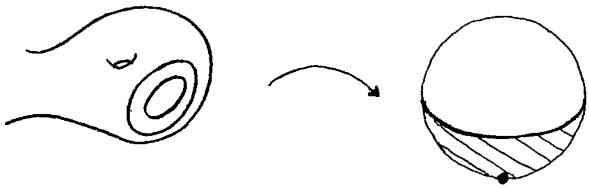

Figure 5

The map $f^* : H^2(S^2) \to H^2(B)$ is an isomorphism. Moreover, by the definition of f, the bundle $V \to B$ is exactly the pull-back by f of the bundle over the sphere which is trivial over each hemisphere and described by the same gluing data as $V \to B$ along the equatorial circle.

We thus assume that B is a sphere S^2.

2. *It suffices to consider the case where $V \to S^2$ is the Hopf bundle:* in fact, if $g : S^2 \to S^2$ is a *degree* m map, it multiplies Euler classes (in the sense V) by m, but in the sense I as well: we may assume it induces $z \mapsto z^m$ on the equatorial circle ∂D_0

$$\begin{array}{ccc} V' & \longrightarrow & V \\ \downarrow & & \downarrow \\ S^2 & \xrightarrow{g} & S^2 \end{array}$$

With obvious notations $a' \mapsto ma$ and similarly $\partial\sigma' \mapsto m\partial\sigma$, but $b' \mapsto b$ as it is an orbit. We deduce that $e = me'$ (in the sense of chapter I).

3. For the Hopf bundle $S^3 \to \mathbf{P}^1(\mathbf{C})$, the statement is clear, as we have $e = -1$ in the sense of I and in the sense of V as well by definition of u.

□

Remark. We have just checked by explicit computation, and used the fact that the Hopf bundle $S^3 \to \mathbf{P}^1(\mathbf{C})$ is universal among the principal S^1-bundles over surfaces.

B.2 Complex line bundles

To any principal S^1-bundle $V \to B$ we associate[9] a rank 1 complex vector bundle $E \to B$ using the usual linear S^1-action over $\mathbf{C}$:

$$E = V \times_{S^1} \mathbf{C}$$

where S^1 acts on $V \times \mathbf{C}$ by $t \cdot (v, z) = (tv, \bar{t}z)$.

Examples.

- Take $V = S^{2n+1}$ with the action defining the Hopf fibration. The map

$$\begin{array}{ccc} S^{2n+1} \times \mathbf{C} & \longrightarrow & \mathbf{P}^n(\mathbf{C}) \\ (z_0, \ldots, z_n; z) & \longmapsto & [z_0, \ldots, z_n] \end{array}$$

  becomes, taking quotients, the vector bundle

$$S^{2n+1} \times_{S^1} \mathbf{C} \longrightarrow \mathbf{P}^n(\mathbf{C})$$

  Using the injection

$$\begin{array}{ccc} S^{2n+1} \times_{S^1} \mathbf{C} & \hookrightarrow & \mathbf{P}^n(\mathbf{C}) \times \mathbf{C}^{n+1} \\ [z_0, \ldots, z_n; z] & \mapsto & ([z_0, \ldots, z_n], z_0 z, \ldots, z_n z) \end{array}$$

  we may identify the vector bundle obtained with the tautological bundle $\mathcal{O}(-1)$.

- For any $m \in \mathbf{Z}$ we also know how to define a bundle called $\mathcal{O}(m)$ over $\mathbf{P}^n(\mathbf{C})$:

$$\mathcal{O}(m) = S^{2n+1} \times \mathbf{C}/(z_0, \ldots, z_n; z) \sim (tz_0, \ldots, tz_n; t^m z).$$

  Thus the bundle $E(m)$ we considered over $\mathbf{P}^1(\mathbf{C})$ in IV-A is $\mathcal{O}(m)$. Consider ζ a primitive m-th root of 1, and the lens space $L^{2n+1}(m)$ which is the quotient of S^{2n+1} by $v \sim \zeta v$. Making $S^1 (= S^1/\mathbf{Z}/m)$ act on $L^{2n+1}(m)$, we still get a principal bundle over $\mathbf{P}^n(\mathbf{C})$.

Exercise B.2.1 Show that in the last example the associated complex line bundle is $\mathcal{O}(-m)$.

[9] This is actually what is called an *associated bundle.*

Conversely, given a complex line bundle E over a manifold B, the choice of a hermitian metric on E allows us to consider its *circle* bundle $S(E)$, which is then a principal S^1-bundle over B (thanks to the linear S^1-action by rotations in the fibers). Of course we may recover E from $S(E)$ as above.

We now have to compare the Euler class of E as an oriented vector bundle and the Euler class of $S(E)$ as a principal S^1-bundle. But we actually defined the class $u \in H^2(\mathbf{P}^n(\mathbf{C}); \mathbf{Z})$ in such a way that $e(\mathcal{O}(-1)) = -u$ (*see* A.4). By universality, we deduce that they coincide.

Exercise B.2.2 Comparing $\mathcal{O}(1)$ and $\mathcal{O}(-1)$, check that $u = e(\mathcal{O}(1))$.

B.2.3 As a conclusion to this appendix, let us now check the assertion about multiplication by n which we used to define the Euler class in the nonprincipal case and in 2.4:

The map $z \mapsto z^n$ induces multiplication by n on $H^2(BS^1)$.

Coming back to the previous examples, we see that we have to prove that the Euler class of $\mathcal{O}(n)$ is nu. On the other hand, as this only concerns H^2, the assertion needs only be checked on $\mathbf{P}^1(\mathbf{C})$. Consider thus the map

$$\begin{array}{rccc} \varphi: & \mathbf{P}^1(\mathbf{C}) & \longrightarrow & \mathbf{P}^1(\mathbf{C}) \\ & [x,y] & \longmapsto & [x^n, y^n] \end{array}$$

It has degree n, which means that it induces multiplication by n in the group H^2. Moreover it is (almost) obvious that it satisfies $\varphi^\star\mathcal{O}(1) \cong \mathcal{O}(n)$. By naturality we thus have

$$e(\mathcal{O}(n)) = \varphi^\star e(\mathcal{O}(1)) = ne(\mathcal{O}(1)) = nu.$$

□

Chapter VI

Toric manifolds

This chapter is a kind of appendix to the rest of these notes: its goal is to present a very beautiful family of symplectic manifolds endowed with hamiltonian torus actions.

These are the *toric varieties*[1] introduced by Demazure and investigated since then by numerous authors[2]. These are algebraic varieties which can be defined over any field (of course we shall restrict to the field of complex numbers). The prototype is the closure of any orbit of a (complex) torus acting in a linear way on a projective space which we already met in III-4.4.1, but we shall present here an alternative description.

The manifolds under consideration have properties which coincide *a priori* with those which we are interested in:

- They are endowed with the action of a "big" torus (of maximal dimension: half that of the manifold) and with symplectic structures for which this action is hamiltonian.
- They are constructed from something very close to a convex polyhedron, namely a *fan* (this is something allowing us to recover the combinatorics of the faces of a convex polyhedron, but *not* the size of the faces, *see* 2.1). Moreover, given an invariant line bundle, one can actually associate with these manifolds a convex polyhedron with integral vertices.
- In the dimension 4 (complex 2) case they are described by graphs which look very much like some of the ones which appeared in chapter IV... which is not surprising since we exhibited there *all* the 4-manifolds endowed with a hamiltonian action of the circle, *a fortiori* these endowed with hamiltonian T^2-actions.

After the publication of the convexity theorem (III-4.2.1), various authors hurried to remark that, in the case where the cohomology class of the symplectic form is

[1]It is to insist on the fact that we are mainly interested in the complex and smooth case that we used the word "manifold" in the title of the chapter.

[2]There is a rather complete, but already a little out of date bibliography in [30] and a more recent one in [58].

the Euler class of a complex line bundle, the polyhedron image by the moment map was actually the one that was classically associated with the line bundle under consideration (*see* [44] and the proof by Khovanski and Arnold of III-4.4.1 in [19] we already used).

Here, we shall describe a complex toric variety[3] as the "quotient" of some $\mathbf{C}^N$ by a complex subtorus of $(\mathbf{C}^*)^N$. It is not necessary here to make precise the sense of the word "quotient", it will be the honest topological quotient of a big open subset of $\mathbf{C}^N$. According to a rather general idea of F. Kirwan, this quotient is identified to the symplectic reduction of some regular level of a moment map on $\mathbf{C}^N$. We thus meet a description due to T. Delzant, from which we deduce, as in [31] that with each (integral or not) convex polyhedron satisfying the necessary conditions implied by III-4.2.5, we are able to associate a compact symplectic manifold of dimension double that of the polyhedron, the latter being the image of the moment map.

This construction gives, in a very natural way, many of the classical results on the topology of these manifolds (simple connectivity, cohomology, invariant line bundles and symplectic forms).

To stay in the spirit of chapter IV of these notes, we shall investigate more precisely the case of dimension 4, where it is almost obvious, using the results of the submentioned chapter, that given the image of the moment map, we are able to recover the whole diffeomorphism type of the manifold: in other words, a compact symplectic manifold of dimension 4 is endowed with a hamiltonian T^2-action if and only if it is diffeomorphic to a complex toric surface. This is a special case of a result in [31] where the analogous statement for hamiltonian T^n-actions in dimension $2n$ is proven.

It goes thus without saying that I claim no originality in this chapter[4]: there are ideas from [46] and [31], which I have chosen to describe here mainly because I wanted to relate the topological aspects of the theory I enjoyed learning in the very beautiful paper by Danilov [30]. I enjoyed reading [28] as well, so I used it, especially in the description of fibrations and of surfaces.

1 The action of $T^N_{\mathbf{C}}$ and its subgroups on $\mathbf{C}^N$

Call T^N the real torus and $T^N_{\mathbf{C}}$ the complex torus:

$$\begin{aligned} T^N &= \left\{(t_1,\ldots,t_N)\in\mathbf{C}^N \,|\, |t_i|=1\right\} \\ T^N_{\mathbf{C}} &= \left\{(t_1,\ldots,t_N)\in\mathbf{C}^N \,|\, t_i\neq 0\right\} \quad = (\mathbf{C}^*)^N \end{aligned}$$

and make these two groups act on $\mathbf{C}^N$:

$$(t_1,\ldots,t_N)\cdot(z_1,\ldots,z_N)=(t_1z_1,\ldots,t_Nz_N)$$

noticing[5] that, restricted to $(\mathbf{C}^*)^N$ the action is free and transitive!

[3] Not in the generality needed by algebraic geometers.

[4] Nor in the others, after all.

[5] Silly as it may seem, this remark is the basis of this chapter.

1.1 Nontrivial stabilizers

Call $(e_1, \ldots, e_N)$ the canonical basis in any of the spaces $\mathbf{Z}^N \subset \mathbf{Q}^N \subset \mathbf{R}^N \subset \mathbf{C}^N$ and, for $I \subset \{1, \ldots, N\}$, e_I the "coordinate" subspace generated by $(e_i)_{i \in I}$:

$$e_I = \{z | j \notin I \Rightarrow z_j = 0\}$$

T_I the corresponding complex torus[6]:

$$T_I = \{t | j \notin I \Rightarrow t_j = 1\}$$

e'_I the open cone

$$e'_I = \{z | j \notin I \Leftrightarrow z_j = 0\}$$

and $\overline{I}$ the complement of I.

Of course, $z \in e_I$ if and only if its stabilizer contains $T_{\overline{I}}$ and $z \in e'_I$ if and only if its stabilizer *is* $T_{\overline{I}}$.

1.2 Subtori

Consider a linear map

$$\pi : \mathbf{Z}^N \longrightarrow \mathbf{Z}^n$$

and use the same letter (when no confusion is to be feared) π for the maps $\pi \otimes \mathbf{Q}$, $\pi \otimes \mathbf{R}$ and $\pi \otimes \mathbf{C}$.

Assume that $\pi \otimes \mathbf{Q}$ is surjective and call $\mathbf{K} \subset \mathbf{Z}^N$ the kernel of π. Similarly $K \subset T^N$ will be the kernel of

$$\mathbf{R}^N/\mathbf{Z}^N \xrightarrow{\pi} \mathbf{R}^n/\mathbf{Z}^n$$

the definition of $K_{\mathbf{C}}$ being left to the reader... all these groups are now acting on $\mathbf{C}^N$ as subgroups of $T^N_{\mathbf{C}}$. Call $k = N - n$ the dimension of K.

Proposition 1.2.1 *The points in the singular orbits of $K_{\mathbf{C}}$ in $\mathbf{C}^N$ are the points in the e_I such that* $\mathbf{K} \otimes \mathbf{C} \cap e_{\overline{I}} \neq 0$.

Proof. We know that z lies in a nonprincipal orbit of $K_{\mathbf{C}}$ if and only if there exists a (proper) subset I of indices in $\{1, \ldots, N\}$ such that $z \in e_I$ and $K_{\mathbf{C}} \cap T_{\overline{I}} \neq 1$. Linearising the latter condition, we find that $\mathbf{K} \otimes \mathbf{C} \cap e_{\overline{I}} \neq 0$... but this gives no information for exceptional orbits (case where $K_{\mathbf{C}} \cap T_{\overline{I}}$ is a finite group). □

Example. Put $N = n = 2$ (and $k = 0$), and let $\pi : \mathbf{Z}^2 \to \mathbf{Z}^2$ be the map with matrix $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ in the canonical basis, in such a way that π itself is injective ($\mathbf{K} = 0$) but its avatar $T^2 \to T^2$ has nevertheless a nontrivial kernel $K \cong \{(\varepsilon, \varepsilon) | \varepsilon = \pm 1\} \subset T^2$. In this case, $\mathbf{K}_{\mathbf{C}}$ acts on $\mathbf{C}^2$ without singular orbit, but the point $(0,0)$ is an exceptional one.

[6] Trying to economise on notation which is already very heavy we shall denote in the same way both the complex and real torus. No confusion is to be feared.

Given the (integral) map π, we thus deduce:

Proposition 1.2.2 *Let*

$$\mathcal{U}_\pi = \mathbf{C}^N - \bigcup_{\mathbf{K} \cap e_{\bar{I}} \neq 0} e_I$$

This is the biggest open subset in $\mathbf{C}^N$ *on which* $K_{\mathbf{C}}$ *acts without singular orbits.*

Remark. $\dim \mathbf{K} = N - n = k$, and if $\# \bar{I} > n$, $\mathbf{K} \otimes \mathbf{C} \cap e_{\bar{I}} \neq 0$. In $\mathcal{U}_\pi$ there is therefore no coordinate subspace e_I such that $\# I < k$.

Example. Let $\pi : \mathbf{Z}^{n+1} \to \mathbf{Z}^n$ be the map which sends the n first vectors $e_1, \ldots, e_n$ of the canonical basis on the vectors having the same names, representing the canonical basis of $\mathbf{Z}^n$ and sending e_{n+1} on $-(e_1 + \cdots + e_n)$. $\mathbf{K}$ is the line generated by $e_1 + \ldots + e_n + e_{n+1}$ and K the diagonal torus $(t, \ldots, t)$. Here $\mathcal{U}_\pi$ is $\mathbf{C}^{n+1} - 0$.

1.3 Real and imaginary parts

If $T_{\mathbf{C}}$ is a complex torus and T the real torus which is its compact component, then $\mathfrak{t}_{\mathbf{C}} = \mathfrak{t} \oplus i\mathfrak{t}$, and we shall simply write the element g of $T_{\mathbf{C}}$ as $g = k \exp iX$ where $k \in T$ and $X \in \mathfrak{t}$. For example, if $T_{\mathbf{C}} = \mathbf{C}^\star$, then $T = S^1$, $\mathfrak{t} = i\mathbf{R}$ and any element $z \in \mathbf{C}^\star$ may be written $z = ue^x$ where $x \in \mathbf{R}$ and $u \in S^1$.

2 Fans and toric varieties

2.1 Fans

The idea one must have in mind is that a fan is what is left from a convex polyhedron when the "sizes" of its faces are forgotten.

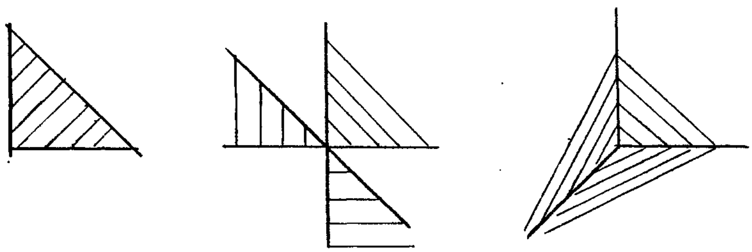

Figure 1

Consider a convex polyhedron P in a vector space which we shall call $E^\star$ because we actually want to work in the dual[7] E.

[7]All the convex polyhedra we already met were indeed living in "dual spaces".

With any face Γ we may associate its *tangent cone*: fix a point m in the (relative) interior of Γ (notation: $m \in \mathring{\Gamma}$) and define

$$\sigma_\Gamma = \bigcup_{r \geq 0} r \cdot (P - m).$$

(*see* the picture in the middle of figure 1). The associated fan is the family $\Sigma(P) = \{\check{\sigma}_\Gamma\}$ of the *convex dual cones*:

$$\check{\sigma} = \{X \in E | \varphi(X) \geq 0 \ \forall \varphi \in \sigma\}$$

Definition 2.1.1 *A fan in $\mathbf{R}^n$ is a family Σ of convex polyhedral cones in $\mathbf{R}^n$ having vertex 0, generated by integral vectors and such that*

1. *any face of a cone in Σ is a cone in Σ*
2. *the intersection of two cones in Σ is a face in each of them.*

Remark. One may replace $\mathbf{R}^n$ by $\mathbf{Q}^n$ in this definition: the important things are the vector space structure, allowing us to define cones, and (last but not least) the lattice $\mathbf{Z}^n$.

Exercise 2.1.2 The family of the convex dual cones associated with a convex polyhedron is indeed a fan.

Example. Figure 1 shows (on the right) the fan associated with the simplex on the left (in the middle are the tangent convex cones). The dimension 1 cones are generated by the vectors $x_1 = e_1$, $x_2 = e_2$, and $x_3 = -(e_1 + e_2)$. The fan contains the dimension 2 convex cones generated by (x_1, x_2), (x_2, x_3) and (x_3, x_1) as well.

The union $\Sigma^{(k)}$ of all dimension k cones is the *k-skeleton*. The 1-skeleton $\Sigma^{(1)}$ may (and actually will) be considered as the set of the N (primitive) vectors in $\mathbf{Z}^n$ which generate the half-lines which are the cones in $\Sigma^{(1)}$.

In the example in figure 1, notice that the family of the dual convex cones, that is the fan defined by the polyhedron, covers the whole space. Remark also that this was not the case for the family of the tangent convex cones. We shall prove that this property is related with the compacity of the polyhedron, which will thus be read on the fan in a very convenient way. It is one of the reasons why people usually prefer to work with the fan instead using the family of tangent convex cones.

Call *support* of the fan Σ the union $|\Sigma|$ of all the cones. We say that a fan is *complete* if its support is the whole space $\mathbf{R}^n$.

Proposition 2.1.3 *The convex polyhedron P is compact if and only if the fan $\Sigma(P)$ is complete.*

Proof. The convex polyhedron P is compact if and only if its projection on any line is compact, and in particular if and only if for any $X \in E$ the map

$$\begin{array}{ccc} E^\star & \longrightarrow & \mathbf{R} \\ \varphi & \longmapsto & \varphi(X) \end{array}$$

sends P onto a compact interval.

Considering $-X$ as well, we thus see that P is compact if and only if

$$\forall X \in E,\ \exists a \in \mathbf{R} \text{ such that } \forall \varphi \in P,\ \varphi(X) \geq a.$$

As P is a convex polyedron, a may be considered as the value at X of a linear form m on the (relative) interior of some face of P. Thus P is compact if and only if

$$\forall X \in E,\ \exists \Gamma \text{ face of } P \text{ and } m \in \mathring{\Gamma} \text{ such that } \forall \varphi \in P,\ \varphi(X) - m(X) \geq 0.$$

The last condition is equivalent to the existence for any X of a face Γ such that X is in the dual convex cone: the fan covers the whole space. □

2.2 Closing a fan, open subsets in $\mathbf{C}^N$, toric varieties

Given the fan Σ, choose a numbering of the primitive vectors generating $\Sigma^{(1)}$

$$\Sigma^{(1)} = (x_1, \ldots, x_N)$$

and consider the unique linear map

$$\pi : \mathbf{Z}^N \longrightarrow \mathbf{Z}^n$$

which sends the vectors $(e_1, \ldots, e_N)$ of the canonical basis respectively onto the vectors $(x_1, \ldots, x_N)$ and assume that $\pi \otimes \mathbf{Q}$ is surjective (this will be the case for instance if Σ contains a dimension n cone).

Denote $\langle x_I$ the cone generated by the vectors x_i $(i \in I)$:

$$\langle x_I = \{\alpha_1 x_{i_1} + \cdots + \alpha_r x_{i_r} | \alpha_j \geq 0,\ I = \{i_1, \ldots, i_r\}\}\,.$$

To the projection π, we know how to associate an open subset $\mathcal{U}_\pi$ in $\mathbf{C}^N$. Just as π, the latter depends only on the 1-skeleton of Σ. We now want to construct a sub-open subset $\mathcal{U}_\Sigma$ of $\mathcal{U}_\pi$ which will really depend on Σ. To this end, we make a restriction on the fans Σ we shall use:

$$(1) \qquad \langle x_I \in \Sigma \Rightarrow e_I \cap \mathbf{K} \otimes \mathbf{C} = \{0\}$$

(there might perfectly well be cones $\langle x_I$ on some vectors in the 1-skeleton which are not in Σ but satisfy the condition in (1)).

We can now define

$$\mathcal{U}_\Sigma = \mathbf{C}^N - \bigcup_{\{I | \langle x_I \notin \Sigma\}} e_I.$$

Condition (1) implies that $\mathcal{U}_\Sigma \subset \mathcal{U}_\pi$, in particular that the complex subtorus $K_\mathbf{C}$ asociated with π acts on $\mathcal{U}_\Sigma$ without singular orbits.

Exercise 2.2.1 Show that the $K_{\mathbf{C}}$-orbits in $\mathcal{U}_\Sigma$ are closed and deduce that X_Σ is Hausdorff.

The quotient is then a space (actually a complex analytic space) X_Σ whose singularities are not too complicated: at worst it is an orbifold and it has a finite branched covering which is smooth. Of course, without condition (1) singularities are more complicated.

Definition 2.2.2 *The space X_Σ is the toric variety associated with Σ.*

Remarks on condition (1).

1. By definition, $\#I \leq 1 \Rightarrow \langle x_I \in \Sigma$ and thus $\langle x_{\bar{I}} \notin \Sigma \Rightarrow \dim e_I \leq N-2$: in order to obtain $\mathcal{U}_\Sigma$, we removed from $\mathbf{C}^N$ only some codimension ≥ 2 subspaces.

2. If $\Sigma' \subset \Sigma$ and if Σ satisfies (1), then Σ' satifies it as well.

3. As we assumed that $\pi \otimes \mathbf{Q}$ was surjective, then

$$e_I \cap \mathbf{K} \otimes \mathbf{C} = \{0\} \Rightarrow \#I \leq n$$

and therefore condition (1) implies that all the dimension n cones in Σ are *simplicial,* that is generated by n vectors.

4. Condition (1) is implied by the condition

(2) Each cone in Σ is generated by a part of some $\mathbf{Z}$-basis.

In fact let $\langle x_I$ be a dimension h cone. Up to renumbering in $\Sigma^{(1)}$, we may assume that $I = \{1, \ldots, h\}$. Then

$$\begin{aligned} e_I \cap \mathbf{K} \otimes \mathbf{C} &= \left\{ \textstyle\sum_{i=1}^h \alpha_i e_i | \pi(\sum \alpha_i e_i) = 0 \right\} \\ &= \left\{ \textstyle\sum_{i=1}^h \alpha_i e_i | \sum \alpha_i x_i = 0 \right\} \\ &= 0. \end{aligned}$$

Hence $\langle x_I \in \Sigma \Rightarrow e_I \cap \mathbf{K} \otimes \mathbf{C} = 0$. Condition (2) is actually equivalent to the smoothness of X_Σ as the following proposition shows.

Proposition 2.2.3 *X_Σ is smooth if and only if all dimension n cones in Σ are generated by a $\mathbf{Z}$-basis of $\mathbf{Z}^n$.*

Such a fan will be said *smooth*: hence a fan is smooth if and only if it satifies condition (2).

Proof. X_Σ is smooth if and only if $K_\mathbf{C}$ has no finite stabilizer in $\mathcal{U}_\Sigma$, in other words if and only if

$$\langle x_I \in \Sigma \Rightarrow K \cap T_I = \{1\}$$

but, as we assumed $\pi \otimes \mathbf{Q}$ to be surjective, this is equivalent to

$$\langle x_I \in \Sigma \,\mathrm{et}\, \#I = n \Rightarrow K \cap T_I = \{1\}$$

as well. But the torus K is the kernel of $\pi : \mathbf{R}^N/\mathbf{Z}^N \to \mathbf{R}^n/\mathbf{Z}^n$. In other words, if K contains a nontrivial element, this means that there exists a nonintegral vector in $\mathbf{Q}^N$ the image of which lies in $\mathbf{Z}^n$. Renumbering the x_i's if necessary, assume that $I = \{1, \ldots, n\}$. Then $K \cap T_I$ is a nontrivial finite group if and only if there exist relatively prime integers $a_1, \ldots, a_n$ and $m \geq 2$ such that

$$\pi\left(\sum_{i=1}^{n} \frac{a_i}{m} e_i\right) \in \mathbf{Z}^n.$$

This says that we have found a vector $\sum a_i x_i$ de $\mathbf{Z}^n$ the "coordinates" of which are relatively prime and which is divisible: thus it is not written in a $\mathbf{Z}$-basis. □

Exercise 2.2.4 Using III-4.2.5, show that, in order that a dimension n convex polyhedron be the image under the moment map (of an effective T^n-action) of a $2n$-manifold, it is necessary that the associated fan is smooth.

A convenient way to present $\mathcal{U}_\Sigma$ from the point of view of the $K_\mathbf{C}$-bundle $\mathcal{U}_\Sigma \to X_\Sigma$ might be the following:

Lemma 2.2.5 *Let* $\mathcal{U}_I = \left\{z \in \mathbf{C}^N \mid z_j = 0 \Rightarrow j \in I\right\}$. *It is the product of* $\#\bar{I}$ *copies of* $\mathbf{C}^\star$ *and of* $\#I$ *copies of* $\mathbf{C}$ *and we have*

$$\mathcal{U}_\Sigma = \bigcup_{\langle x_I \in \Sigma} \mathcal{U}_I$$

Proof. We may write $\mathcal{U}_I = e_I \times e'_{\bar{I}}$. For any $z \in \mathbf{C}^N$, call $I(z) \subset \{1, \ldots, N\}$ the set of the indices of those coordinates which are zero (thus $z \in \mathcal{U}_{I(z)}$). We have

$$z \in e_I \Leftrightarrow \bar{I} \subset I(z)$$

and

$$\begin{aligned} z \in \mathcal{U}_\Sigma &\Leftrightarrow (z \in e_I \Rightarrow \langle x_{\bar{I}} \in \Sigma) \\ &\Leftrightarrow (I \subset I(z) \Rightarrow \langle x_I \in \Sigma) \\ &\Leftrightarrow \langle x_{I(z)} \in \Sigma \end{aligned}$$

so $z \in \mathcal{U}_\Sigma \Leftrightarrow \langle x_{I(z)} \in \Sigma \Leftrightarrow \mathcal{U}_{I(z)} \subset \mathcal{U}_\Sigma$. □

In the case of a smooth fan, and thus of a *principal* $K_\mathbf{C}$-bundle $\mathcal{U}_\Sigma \to X_\Sigma$, the open subsets $\mathcal{U}_I$ of $\mathcal{U}_\Sigma$ give local trivialisations of the bundle.

Proposition 2.2.6 *With each cone* $\langle x_I$ *of the smooth fan* Σ *is associated an open subset* X_I *of the toric variety* X_Σ, *over which the principal* $K_\mathbf{C}$*-bundle* $\mathcal{U}_\Sigma \to X_\Sigma$ *is trivialised.*

Proof. Choose a basis of $\mathbf{K}$. Let A be the matrix (with k columns and N rows) giving this basis in the canonical basis of $\mathbf{C}^N$. The torus $K_{\mathbf{C}}$ acts on $\mathbf{C}^N$ by

$$(u_1,\dots,u_k)\cdot(z_1,\dots,z_N)=(u^{m_1}z_1,\dots,u^{m_n}z_N)$$

where u^{m_i} is the monomial in the u_j's given by the multiexponent $m_i\in\mathbf{Z}^k$ which is the i-th line of A.

Consider this action on one of the open subsets $\mathcal{U}_I$ we are interested in. Remark first that the $\#I=n$ case is actually sufficient; as the fan is complete, any cone $\langle x_J\in\Sigma$ with $\#J<n$ is a face of a cone $\langle x_I\in\Sigma$ with $\#I=n$ and then $\mathcal{U}_J\subset\mathcal{U}_I$. Furthermore, we merely have to change the numbering of the vectors in the 1-skeleton of Σ to be able to assume that $I=\{1,\dots,n\}$. Then $\mathcal{U}_I=\{(z_1,\dots,z_N)\mid z_{n+1}\neq 0,\dots,z_N\neq 0\}$. The fact that $K_{\mathbf{C}}$ then acts freely on $\mathcal{U}_I$ corresponds to the fact that the k last rows[8] of the matrix A give an invertible (over the integers) matrix. In other words, we can change the parmetrisation of $K_{\mathbf{C}}$ in such a way that the action is written:

$$(v_1,\dots,v_k)\cdot(z_1,\dots,z_N)=(v^{p_1}z_1,\dots,v^{p_n}z_n,v_1z_{n+1},\dots,v_kz_N).$$

Denote (*cf.* homogeneous coordinates in the projective space) $[z_1,\dots,z_N]$ the element in X_Σ which represents the class of $(z_1,\dots,z_N)\in\mathcal{U}_\Sigma$ modulo $K_{\mathbf{C}}$. We can define a section $\sigma_I: X_I\to\mathcal{U}_I$ to the projection $\mathcal{U}_I\to X_I$ by

$$\sigma_I[z_1,\dots,z_N]=\left(\frac{z_1}{z^{p_1}},\dots,\frac{z_n}{z^{p_n}},1,\dots,1\right)$$

where z^{p_i} denotes the monomial in $z_{n+1},\dots,z_{n+k}$ associated with $p_i\in\mathbf{Z}^k$. A trivialisation of the principal $K_{\mathbf{C}}$-bundle over X_I is easily deduced:

$$\begin{array}{ccc} K_{\mathbf{C}}\times X_I & \longrightarrow & \mathcal{U}_I\\ (v,[z]) & \longmapsto & v\cdot\sigma_I[z]\end{array}$$

□

The $T^N_{\mathbf{C}}$-action descends to quotients as a $T^N_{\mathbf{C}}$ and even a $T^N_{\mathbf{C}}/K_{\mathbf{C}}$-action on X_Σ: the toric variety X_Σ is endowed with the action of a complex torus we shall denote $Q_{\mathbf{C}}$, the real torus it contains being denoted Q. A quick calculation gives $\dim_{\mathbf{C}}X_\Sigma=\dim_{\mathbf{C}}Q_{\mathbf{C}}=n$.

Each cone $\langle x_I$ in Σ thus defines an open subset X_I in X_Σ. For example, the cone $\{0\}=\langle x_\emptyset$ defines $\mathcal{U}_\emptyset=(\mathbf{C}^*)^N=T^N_{\mathbf{C}}$ and $X_\emptyset=T^N_{\mathbf{C}}/K_{\mathbf{C}}=Q_{\mathbf{C}}$, which thus appears as an open dense subset in X_Σ.

Examples.

1. Take for $(x_1,\dots,x_n)$ the canonical basis of $\mathbf{Z}^n$ and let Σ be the set of all the cones (of the first octant). Then $N=n$, $\pi=\mathrm{Id}$, $\mathbf{K}=0$, $K=1$ and $\mathcal{U}_\pi=\mathcal{U}_\Sigma=\mathcal{U}_\Sigma/K=\mathbf{C}^n$: the toric variety associated with the first octant fan is $\mathbf{C}^n$.

[8] Recall that $N=n+k$.

Figure 2

2. With the same 1-skeleton as in the previous example, define now Σ as the set of all the cones in the first octant except the n-dimensional one. Nothing has changed except $\mathcal{U}_\Sigma$ from which we must take off $e_\emptyset = 0$. Hence $X_\Sigma = \mathcal{U}_\Sigma = \mathbf{C}^n - 0$.

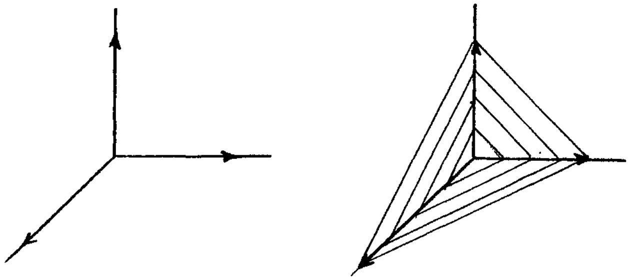

Figure 3

3. Consider the fan on figure 3, that is the vectors $x_i = e_i$ $(1 \leq i \leq n)$ and $x_{n+1} = -(x_1 + \cdots + x_n)$ and the fan in $\mathbf{R}^n$ consisting of all dimension $\leq n$ cones on these vectors (at least if $n = 2$ we already saw that it is the fan associated with the standard simplex). Here $\mathcal{U}_\Sigma = \mathbf{C}^{n+1} - 0$ and $\mathbf{K}$ is the diagonal $\mathbf{Z}$ *i.e.* the set $(m, \ldots, m) \in \mathbf{Z}^{n+1}$ and therefore K is the diagonal $\mathbf{C}^\star$ and X_Σ is the complex projective space $\mathbf{P}^n(\mathbf{C})$ (once again).

4. As in figure 4, consider the fan in $\mathbf{R}^2$ the 1-skeleton of which is $(e_1, e_2, -e_1, -e_2)$ and the 2-skeleton of which consists of all cones $\langle x_{i,i+1}$ (considering the numbering mod 4). Looking for the I's such that $\langle x_I \notin \Sigma$ we find that

$$\mathcal{U}_\Sigma = \mathbf{C}^4 - \left(\{z_1 = z_3 = 0\} \bigcup \{z_2 = z_4 = 0\}\right)$$

and thus we actually get that $\mathcal{U}_\Sigma = (\mathbf{C}^2 - 0) \times (\mathbf{C}^2 - 0)$. The group K is $\mathbf{C}^\star \times \mathbf{C}^\star$ acting diagonally. The toric variety is the product $\mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$.

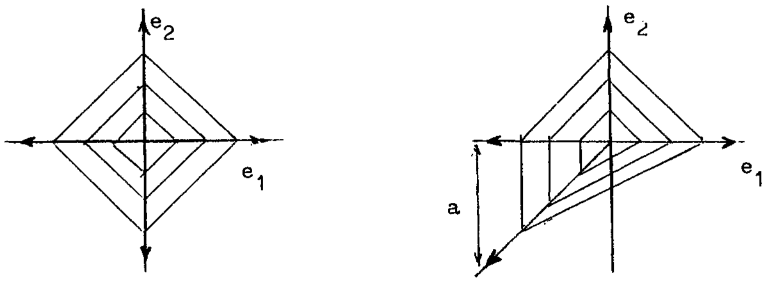

Figure 4

5. $\mathcal{U}_\Sigma$ depends only on the combinatorics of the fan. For instance, all the fans in $\mathbf{R}^2$ the 1-skeleton of which consists of four vectors x_1, x_2, x_3, x_4 and whose 2-skeleton contains all the cones $\langle x_{i,i+1}$ give the same $\mathcal{U}_\Sigma = (\mathbf{C}^2 - 0) \times (\mathbf{C}^2 - 0)$. In fact the torus $K_\mathbf{C}$ changes. Consider the case where $x_1 = e_1$, $x_2 = e_2$, $x_3 = -e_1$, $x_4 = -e_1 - ae_2$ for some integer a (figure 4). The torus $K_\mathbf{C}$ acts this time by
$$(u, v) \cdot (z_1, z_2, z_3, z_4) = (u^a v z_1, u z_2, v z_3, u z_4).$$
The toric variety X_Σ we get this way is again a Hirzebruch surface (already met in chapter IV) as we shall prove in paragraph 5.

6. Here is an example of a nonsmooth fan (figure 5). Working in $\mathbf{Z}^2$, the 1-skeleton is $x_1 = e_1$, $x_2 = e_1 + 2e_2$. As the determinant of (x_1, x_2) equals 2, this is not a $\mathbf{Z}$-basis.

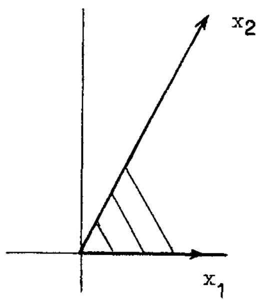

Figure 5

Add now the cone generated by x_1 and x_2. Then $\mathcal{U}_\Sigma = \mathcal{U}_\pi = \mathbf{C}^2$, and $\mathbf{K} = 0$. The group K is the kernel of
$$\pi = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} : T^2 \longrightarrow T^2,$$
which we already considered in 1.2, where we saw that K is generated by $(-1, -1) \in T^2$. The variety X_Σ is the quotient $\mathbf{C}^2/(x, y) \sim (-x, -y)$, it has a

singularity at 0. Here is another description of this complex surface: the map

$$\begin{array}{ccc} \mathbf{C}^2 & \longrightarrow & \mathbf{C}^3 \\ (x,y) & \longmapsto & (x^2,y^2,xy) \end{array}$$

descends to quotient as an injection

$$X_\Sigma \longrightarrow \mathbf{C}^3$$

the image of which is the surface satisfying the equation $Z^2 = XY$.

The open set $\mathcal{U}_\Sigma$ itself is a toric variety of dimension N, associated with the fan $\tilde{\Sigma}$ obtained when "closing" Σ: $\tilde{\Sigma}^{(1)} = (e_1, \ldots, e_N)$ and $\langle e_I \in \tilde{\Sigma} \Leftrightarrow \langle x_I \in \Sigma$. In figure 3, the reader can imagine the fan pictured on the right as being (in relief) in $\mathbf{R}^3$.

By definition, $\tilde{\Sigma}$ contains only cones of the first octant, moreover, as soon as $N > n$ (which will allways be the case if Σ is complete), it contains no dimension N cone. In other words we then have $\mathcal{U}_\Sigma \subset \mathbf{C}^N - 0$.

3 Fans, symplectic reduction, convex polyhedra

In the penultimate example we considered above, we saw a new avatar of $\mathbf{P}^n(\mathbf{C})$ written as the quotient of $\mathbf{C}^{n+1} - 0$ by the action of $\mathbf{C}^\star$ with the help of a fan. For this particular toric variety, we make the following observations:

1. $\mathbf{C}^{n+1}$ can be replaced by S^{2n+1} noticing that $S^{2n+1} \times]0, +\infty[\cong \mathbf{C}^{n+1} - 0$.

2. The previous decomposition is compatible with that of $\mathbf{C}^\star$ as the product $S^1 \times]0, +\infty[$, and thus $\mathbf{P}^n(\mathbf{C})$ is the quotient of S^{2n+1} by S^1.

3. Moreover, all this can be done in the hamiltonian framework: the sphere is a level manifold of a moment map and the projective space is its symplectic reduction.

4. Of the action of the torus T^{n+1} on $\mathbf{C}^{n+1}$, there remains a (hamiltonian) T^n-action on the projective space, for which the image of the moment map is the standard simplex in $\mathbf{R}^n$... polyhedron the associated fan of which is precisely the one we used to construct[9] $\mathbf{P}^n(\mathbf{C})$ as a toric variety!

Of course, the aim of this section is to generalise the above remarks to all toric varieties.

3.1 The moment map for the K-action

The action of the real torus T^N on $\mathbf{C}^N$ is hamiltonian with moment map

$$\mu(z_1, \ldots, z_N) = \frac{1}{2}(|z_1|^2, \ldots, |z_N|^2)$$

[9] *see* the above examples and figures 1 and 3.

taking values in $\mathfrak{t}^\star = (\mathbf{R}^N)^\star$. In the same way as $\mathfrak{t}$ denotes the Lie algebra of the torus T, $\mathfrak{k}$ denotes that[10] of the subtorus K. Dual to the inclusion $K \subset T$, we shall use the projection $p : \mathfrak{t}^\star \to \mathfrak{k}^\star$.

The composed map

$$\mu' : \mathcal{U}_\Sigma \subset \mathcal{U}_\pi \subset \mathbf{C}^N \xrightarrow{\mu} \mathfrak{t}^\star \xrightarrow{p} \mathfrak{k}^\star$$

is the moment map for the K-action on $\mathcal{U}_\Sigma$.

The image of μ is the first octant in $\mathfrak{t}^\star$, all the interior points are regular values. As p is linear and *surjective*, all the points in the image of the interior are regular values for μ' (and are actually values of μ': to obtain $\mathcal{U}_\Sigma$ we only removed some coordinates subspaces, hence all points in the interior of the first octant are in $\mu(\mathcal{U}_\Sigma)$).

Actually, as the K-action on $\mathcal{U}_\Sigma$ has no singular orbit, $\mu'(\mathcal{U}_\Sigma)$ consists of regular values of μ', it is an open subset in $\mathfrak{k}^\star$: the one which plays the analogous role to $]0, +\infty[$ in the projective space example.

The following proposition is a special case of a result of F. Kirwan ([46, theorem 7.4]). The present proof is not very different from her proof of the general case.

Proposition 3.1.1 *Let $\eta \in \mu'(\mathcal{U}_\Sigma)$. Then the inclusion $\mu'^{-1}(\eta) \subset \mathcal{U}_\Sigma$ induces a homeomorphism $\mu'^{-1}(\eta)/K \to \mathcal{U}_\Sigma/K_{\mathbf{C}} = X_\Sigma$.*

Thus, with each η in the open convex cone $\mu'(\mathcal{U}_\Sigma) \subset \mathfrak{k}^\star$, we associate, by symplectic reduction of $\mathbf{C}^N$, a symplectic form ω_η on the manifold X_Σ.

Proof of the proposition. Choose a metric on $\mathfrak{k}^\star$ and consider the function $f : \mathbf{C}^N \to \mathbf{R}$ which to z associates the real number $\|\mu'(z) - \eta\|^2$. It is nonnegative and minimal on $\mu'^{-1}(\eta)$. Its derivative is given by $T_z f(X) = 2(\mu'(z) - \eta) \cdot T_z\mu'(X)$.

In particular, as η was chosen among the regular values of μ', the only critical points of f are those in $\mu'^{-1}(\eta)$. Hence for any $z \in \mathcal{U}_\Sigma$, the gradient trajectory descending from z will reach a point in $\mu'^{-1}(\eta)$.

Let us calculate this gradient:

$$\begin{aligned}
(\operatorname{grad}_z f, X) &= 2\left((\mu'(z) - \eta), T_z\mu'(X)\right) \\
&= 2\left({}^tT_z\mu'(\mu'(z) - \eta), X\right) \\
&= 2\left(i_{\underline{\mu'(z)-\eta}_z}\omega, X\right) \\
&= 2\omega_z(\underline{\mu'(z) - \eta}_z, X) \\
&= -2i(\underline{\mu'(z) - \eta}_z, X)
\end{aligned}$$

[10]It is actually a gothic k.

hence $\text{grad}_z f = -2i\underline{\mu'(z) - \eta}_z$, where of course we have used the metric to identify $\mu'(z) - \eta$ with an element of $\mathfrak{k}$ and underlinuing denotes as usual the fundamental associated vector field. Thus the gradient flow consists of exponentials of elements of $i\mathfrak{k}$: this is the "imaginary part" of the torus $K_{\mathbf{C}}$ (the analogue of $]0, +\infty[$ in $\mathbf{C}^\star$).

In particular, any gradient trajectory is included in a $K_{\mathbf{C}}$-orbit. From what we said before, we derive that each $K_{\mathbf{C}}$-orbit meets $\mu'^{-1}(\eta)$. Let us now show that they meet along a unique K-orbit, that is to say that if $z \in \mu'^{-1}(\eta)$, then $K_{\mathbf{C}} \cdot z \cap \mu'^{-1}(\eta) = K \cdot z$.

Assuming that $\mu'(g \cdot z) = \eta$ for some $g \in K_{\mathbf{C}}$ we shall show that we then have $g \cdot z = k \cdot z$ for some $k \in K$. As $\mu'^{-1}(\eta)$ is K-invariant, we may assume that $g = \exp iX$ for some $X \in \mathfrak{k}$. We thus want to check that $g \cdot z = z$.

Consider the function $h : \mathbf{R} \to \mathbf{R}$ defined by $h(t) = \mu'((\exp itX) \cdot z) \cdot X$. We have $h(0) = h(1) = \eta \cdot X$ and therefore h' must vanish between 0 and 1: $\exists t_0 \in [0, 1]$ such that $h'(t_0) = 0$. But

$$\begin{aligned} h'(t) &= T_y\mu'(i\underline{X}_y) \cdot X \\ &= \omega_y(i\underline{X}_y, \underline{X}_y) = \pm \left\| \underline{X}_y \right\|^2 \end{aligned}$$

putting $y = \exp itX$ to save notations. For $y_0 = \exp it_0X$, we thus have $\underline{X}_{y_0} = 0$ and therefore $\exp itX$ fixes y_0 and z. □

3.2 What is left from the $T^N_{\mathbf{C}}$-action

Fix any η as above in such a way that X_Σ is endowed with the symplectic form ω_η. The action of the real torus Q is hamiltonian and the image of the moment map

$$\nu : X_\Sigma \longrightarrow \mathfrak{q}^\star$$

is the intersection P_η of $\mu(\mathcal{U}_\Sigma)$ with the subspace parallel to $\mathfrak{q}^\star$ whose image[11] by $p : \mathfrak{t}^\star \to \mathfrak{k}^\star$ is η.

Using the fan Σ of $\mathbf{R}^n$, we constructed a manifold X_Σ. Choosing an η gave us a convex polyhedron in $(\mathbf{R}^n)^\star = \mathfrak{q}^\star$. Of course, we shall now prove:

Proposition 3.2.1 *Σ is the fan associated to any of the polyhedra P_η.*

Proof. Consider the exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{K} \otimes \mathbf{R} & \longrightarrow & \mathbf{R}^N & \longrightarrow & \mathbf{R}^n & \longrightarrow & 0 \\ 0 & \longrightarrow & \mathfrak{k} & \longrightarrow & \mathfrak{t} & \longrightarrow & \mathfrak{q} & \longrightarrow & 0 \end{array}$$

and, dually

$$0 \longrightarrow \mathfrak{q}^\star \longrightarrow \mathfrak{t}^\star \xrightarrow{p} \mathfrak{k}^\star \longrightarrow 0$$

[11]The reader has probably noticed that $\mathfrak{q}^\star = \operatorname{Ker} p$.

with $\eta \in \mathfrak{k}^\star$. Choose $\xi_0 \in \mathfrak{t}^\star$ such that $p(\xi_0) = \eta$, in order to consider $p^{-1}(\eta)$ as the subspace parallel to $\mathfrak{q}^\star$ through ξ_0.

As

$$\operatorname{Im}\mu = \{\varphi : \mathfrak{t} \longrightarrow \mathbf{R} | \varphi(e_i) \geq 0 \, \forall i\}$$

we find that

$$P_\eta = p^{-1}(\eta) \cap \operatorname{Im}\mu = \left\{ f \in \mathfrak{t}^\star | \varphi(e_i) \geq 0 \, \forall i, \, \varphi_{|\mathfrak{k}} = \xi_{0|\mathfrak{k}} \right\}.$$

To say that the restriction of $\varphi - \xi_0$ to $\mathfrak{k}$ vanishes is equivalent to saying that the $\langle \varphi - \xi_0, e_i \rangle$ depend only on the $x_i = \pi(e_i)$. In other words:

$$P_\eta = \{\psi \in \mathfrak{q}^\star | \psi(x_i) \geq \xi_0(x_i)\}$$

and in particular the $x_i \in \mathfrak{q}$ are the directions of the dimension $n-1$ faces of P_η as they were the generators of the 1-skeleton of Σ. The cones of P_η are the intersections of $p^{-1}(\eta)$ with the coordinates cones $\langle e_I$ in $\mu(\mathcal{U}_\Sigma)$, in other words with the e_I such that $\langle x_{\overline{I}} \in \Sigma$. □

3.2.2 Complex curves and fixed points. If $\langle x_I$ is a dimension $n-1$ cone, that is to say, if $\# I = n-1$, then $e_{\overline{I}}$ is a subspace of $\mathbf{C}^N$ of dimension $k+1$, $e_{\overline{I}} \cap \mathcal{U}_\Sigma$ is an open subset (as $\langle x_I \in \Sigma$) on which the complex torus $K_\mathbf{C}$ still acts. We thus have the inclusion

$$e_{\overline{I}} \cap \mathcal{U}_\Sigma / K_\mathbf{C} \subset \mathcal{U}_\Sigma / K_\mathbf{C}$$

of a complex curve in X_Σ. The image of this curve under ν is the edge $p^{-1}(\eta) \cap \langle e_{\overline{I}} \subset \mathfrak{t}^\star$ of P_η, the compacity of the curve will follow as we shall see in 4.1. As it is endowed with an action of the torus $Q_\mathbf{C}$, it is a $\mathbf{P}^1(\mathbf{C})$. This is the sphere which is sent onto the edge of P_η under consideration as in exercise V-6.2.1. Its stabilizer in $Q_\mathbf{C}$ corresponds to the orthogonal (in $\mathfrak{q}$) of the edge we are dealing with in $\mathfrak{q}^\star$.

Similarly, the dimension n cones of Σ correspond to the fixed points of the $Q_\mathbf{C}$-action on X_Σ.

4 Properties of the toric manifolds X_Σ

We shall describe in terms of Σ some of the properties of the manifolds we have constructed.

4.1 Compacity of X_Σ

We expressed $X_\Sigma = \mu'^{-1}(\eta)/K$ where K was a compact group. For X_Σ to be compact, it is thus necessary (and sufficient) that $\mu'^{-1}(\eta)$ is.

On the other hand, $\mu'^{-1}(\eta) = \mu^{-1}(p^{-1}(\eta))$ and it is clear that μ is proper. We need thus only know when $p^{-1}(\eta) \cap \operatorname{Im}(\mu)$ is compact, but it is precisely the polyhedron under consideration. We have therefore proved:

Proposition 4.1.1 *X_Σ is compact if and only if one of the polyhedra P_η is compact.* □

Taking 2.1.3 into account, we thus also have:

Proposition 4.1.2 *X_Σ is compact if and only if the fan Σ is complete.* □

4.2 Topology of X_Σ and classes of invariant symplectic forms

If Σ is smooth and complete, we have thus constructed a (smooth and) compact *manifold* X_Σ, of (real) dimension $2n$, endowed with an action of a torus Q of dimension n and a family of invariant symplectic forms giving moment maps and convex polyhedra.

Conversely, notice that any dimension n convex polyhedron P the fan of which is smooth and complete (that is to say that P is compact and that its dimension n cones are generated by $\mathbf{Z}$-bases), allows us to construct such a manifold with, moreover, an invariant symplectic form: write P as the intersection of the ≥ 0 cone in $\mathbf{R}^N$ (N is the number of codimension 1 faces in P) with a suitable affine subspace, giving a value of η using which we can make the symplectic reduction as above.

We shall now prove that all the de Rham cohomology classes containing invariant symplectic forms may be obtained this way. Once again it is very easy. We first have

Proposition 4.2.1 *The toric variety X_Σ is simply connected and the principal $K_{\mathbf{C}}$-bundle $\mathcal{U}_\Sigma \to X_\Sigma$ defines a map*

$$H^2(BK;\mathbf{Z}) \longrightarrow H^2(X_\Sigma;\mathbf{Z})$$

which is an isomorphism.

In other words, the Euler class of this principal $K_{\mathbf{C}}$-bundle (which lives in $H^2(X_\Sigma;\mathbf{Z})^k$ where $k = \dim K$, *see* V-2.4) is a $\mathbf{Z}$-basis of $H^2(X_\Sigma;\mathbf{Z})$.

Proof. In order to construct $\mathcal{U}_\Sigma$, we removed from $\mathbf{C}^n$ some vector subspaces, but no hyperplane. Thus, $\mathcal{U}_\Sigma$ is 2-connected, and $\mathcal{U}_\Sigma \to X_\Sigma$ may be used as a principal K-bundle, universal up to dimension 2 (*see* [15]). □

Corollary 4.2.2 ([31]) *The map from the convex open cone $\mu'(\mathcal{U}_\Sigma)$ of $\mathfrak{k}^\star$ to the vector space $H^2(X_\Sigma;\mathbf{R})$, which associates with η the cohomology class of ω_η, may be extended as an affine isomorphism*

$$\mathfrak{k}^\star \longrightarrow H^2(X_\Sigma;\mathbf{R})$$

Proof. The convex open cone $\mu'(\mathcal{U}_\Sigma)$ generates $\mathfrak{k}^\star$ by definition. Moreover, the map $\eta \mapsto [\omega_\eta]$ is affine according to the Duistermaat-Heckman theorem (*see* V-3.2.2), thus there is no difficulty to extend it as an affine map

$$\mathfrak{k}^\star \longrightarrow H^2(X_\Sigma;\mathbf{R}).$$

It suffices now to apply the previous proposition together with V-3.2.2 to see that the "slope" of this map is given by a $\mathbf{Z}$-basis of $H^2(X_\Sigma;\mathbf{Z})$. It therefore cannot avoid being a diffeomorphism. □

4.3 Integral polyhedra and invariant line bundles

Classically, algebraic geometers are able to associate a convex polyhedron with the toric variety X_Σ, endowed with an invariant (complex) line bundle[12] The polyhedron is integral, namely its vertices lie in the integral lattice of $\mathfrak{q}^*$.

We did just something similar, associating a convex (but not necessarily integral) polyhedron with each invariant symplectic form, in particular with any element of a convex open cone in $H^2(X_\Sigma; \mathbf{R})$.

Let us now investigate the relationship between these two constructions, showing that the "real" one is an interpolation of the integral one. To this end, we begin by understanding the invariant complex line bundles.

Let $g : \mathbf{R}^n \to \mathbf{R}$ be a map satisfying the two properties:

1. It is the restriction of a linear map on each of the cones in Σ.
2. It takes integral values on $|\Sigma| \cap \mathbf{Z}^n$.

Remark. This is equivalent to requiring that g is the projection of a *linear* map, still denoted $g : \mathbf{Z}^N \to \mathbf{Z}$. We shall consider g sometimes as one and sometimes as the other.

With such a map, we associate a fan Σ_g in $\mathbf{R}^{n+1} = \mathbf{R}^n \times \mathbf{R}$ which is more or less the graph of g on Σ:

1. Put $x'_i = (x_i, g(x_i))$ for $1 \leq i \leq N$ and $x'_{N+1} = (0, 1)$, this defining $\Sigma_g^{(1)}$ from $\Sigma^{(1)}$.
2. More generally, we accept a cone $\langle x'_I$ as a member of Σ_g when (and only when)

$$\begin{cases} N+1 \notin I & \text{and} \quad \langle x_I \in \Sigma \\ I = J \cup \{N+1\} & \text{and} \quad \langle x_J \in \Sigma \end{cases}$$

Proposition 4.3.1 *The toric variety X_{Σ_g} is smooth if and only if X_Σ is. It is endowed with a natural map*

$$X_{\Sigma_g} \to X_\Sigma$$

which makes it a complex line bundle over X_Σ.

Proof. The former assertion is obvious, taking into account the characterisation of smooth toric varieties by their fans in 2.2.3.

To prove the latter, consider carefully the open subset $\mathcal{U}_{\Sigma_g}$ of $\mathbf{C}^{N+1}$:

$$\mathcal{U}_{\Sigma_g} = \mathbf{C}^{N+1} - \bigcup_{\langle x'_I \notin \Sigma_g} e_I$$

[12] All the notions of invariance we use for toric varieties are relative to the action of the *big torus* $Q_{\mathbf{C}}$.

to notice that

$$\mathcal{U}_{\Sigma_g} = \mathcal{U}_\Sigma \times \mathbf{C}$$

as actually

$$\mathcal{U}_{\Sigma_g} = \mathbf{C}^{N+1} - \bigcup_{\langle x'_I \notin \Sigma_g} f_I$$

where we called $f = (e_1, \ldots, e_N, f_{N+1})$ the canonical basis. Now

$$\langle x'_I \in \Sigma_g \Leftrightarrow \begin{cases} I \subset \{1, \ldots, N\} & \text{et} \quad \langle x_I \in \Sigma \;\Leftrightarrow\; f_I = e_I \times \mathbf{C} \\ I = J \cup \{N+1\} & \text{et} \quad \langle x_J \in \Sigma \;\Leftrightarrow\; f_I = e_J \times 0 \end{cases}$$

and therefore

$$\begin{aligned} \mathcal{U}_{\Sigma_g} &= \mathbf{C}^{N+1} - \textstyle\bigcup_{\langle x_I \in \Sigma} (e_I \times 0 \cup e_I \times \mathbf{C}) \\ &= \mathbf{C}^{N+1} - \textstyle\bigcup_{\langle x_I \in \Sigma} e_I \times \mathbf{C} \\ &= \mathcal{U}_\Sigma \times \mathbf{C}. \end{aligned}$$

Let us now construct the complex torus K' associated with the situation of Σ_g:

$$0 \longrightarrow \mathbf{K}' \longrightarrow \mathbf{Z}^N \times \mathbf{Z} \xrightarrow{\pi'} \mathbf{Z}^n \times \mathbf{Z}$$

where $\pi'(a, b) = (\pi(a), g(a) + b)$, that is $\mathbf{K}' = \{(a, b) \mid \pi(a) = 0 \,\text{et}\, b = -g(a)\}$ and the map

$$\begin{array}{ccc} \mathbf{K} & \longrightarrow & \mathbf{K}' \\ a & \longmapsto & (a, -g(a)) \end{array}$$

is an isomorphism. The complex torus $K'_{\mathbf{C}} \subset T_{\mathbf{C}}^{N+1} = T_{\mathbf{C}}^N \times \mathbf{C}^\star$ is the isomorphic image of $K_{\mathbf{C}}$ by the restriction of

$$\begin{array}{ccc} T_{\mathbf{C}}^N & \longrightarrow & T_{\mathbf{C}}^N \times \mathbf{C}^\star \\ t & \longmapsto & (t, \tilde{g}(t)^{-1}) \end{array}$$

where $\tilde{g}$ is the multiplicative avatar of the linear map g. In other words, the $K'_{\mathbf{C}}$-action on $\mathcal{U}_{\Sigma_g}$ may be identified with that of $K_{\mathbf{C}}$ on $\mathcal{U}_\Sigma \times \mathbf{C}$ by

$$t \cdot (x, y) = (t \cdot x, g(t)^{-1} y).$$

Taking quotients of both sides gives the announced line bundle.

$$\begin{array}{ccc} \mathcal{U}_{\Sigma_g} = \mathcal{U}_\Sigma \times \mathbf{C} & \longrightarrow & \mathcal{U}_\Sigma \\ \big\downarrow K'_{\mathbf{C}} & & \big\downarrow K_{\mathbf{C}} \\ X_{\Sigma_g} & \longrightarrow & X_\Sigma \end{array}$$

□

Example. In the case where Σ is the fan describing $\mathbf{P}^n(\mathbf{C})$, if we define $g : \mathbf{Z}^{n+1} \to \mathbf{Z}$ by $g(e_i) = 0$ for $1 \le i \le n$ and $g(e_{n+1}) = m$, we get the bundle $\mathcal{O}(-m)$ (*see* V-B).

For $n = 1$, figure 6 shows the bundle $\mathcal{O}(-1)$ over $\mathbf{P}^1(\mathbf{C})$.

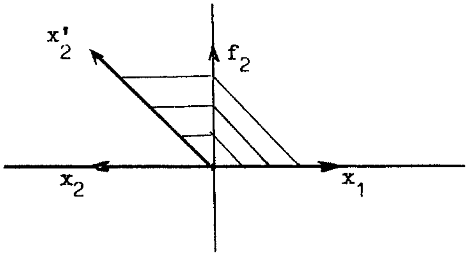

Figure 6

Remark. If g is already a linear map on$\mathbf{Z}^n$, the bundle obtained is trivial. Indeed, notice first that the bundle associated with $g = 0$ is (the torus $K'_{\mathbf{C}}$ does not act on the factor $\mathbf{C}$ in this case). Then, for $g \in (\mathbf{Z}^n)^\star$, define $\tilde{g} : \mathbf{Z}^n \times \mathbf{Z} \to \mathbf{Z}^n \times \mathbf{Z}$ by $(a, b) \mapsto (a, b + g(a))$ and remark that the map $\tilde{g}$ defines an isomorphism from Σ_0 onto Σ_g and thus an isomorphism from X_{Σ_0} onto X_{Σ_g} over X_Σ.

We may thus consider that we have defined a map $g \mapsto X_{\Sigma_g}$ from the set of all linear maps $\mathbf{Z}^N \to \mathbf{Z}$ (modulo those $\mathbf{Z}^n \to \mathbf{Z}$) into the set of isomorphism classes of Q-invariant complex line bundles over X_Σ.

The former is $(\mathbf{Z}^N)^\star/(\mathbf{Z}^n)^\star\ldots$ in other words $\mathbf{K}^\star$. As to the latter, and although we did not require any analyticity, we might call it $\mathrm{Pic}_Q(X_\Sigma)$.

The map in question is a morphism of groups; if g_1, $g_2 \in (\mathbf{Z}^N)^\star$, the group $K_{\mathbf{C}}$ acts on $\mathcal{U}_\Sigma \times \mathbf{C}$

- using g_1 by $t \cdot_1 (x, y) = (t \cdot x, g_1(t)^{-1} y)$,
- using g_2 by $t \cdot_2 (x, y) = (t \cdot x, g_2(t)^{-1} y)$,
- and using $g_1 + g_2$ by $t \cdot (x, y) = (t \cdot x, (g_1(t) g_2(t))^{-1} y)$, which corespond to the tensor product of line bundles (this is where the group structure on $\mathrm{Pic}_Q(X_\Sigma)$ comes from).

Composing with the Euler class, we get a group morphism

$$\mathbf{K}^\star \longrightarrow H^2(X_\Sigma; \mathbf{Z}).$$

Proposition 4.3.2 *The morphism $\mathbf{K}^\star \to H^2(X_\Sigma; \mathbf{Z})$ is an isomorphism and may be extended as the isomorphism $\mathfrak{k}^\star \to H^2(X_\Sigma; \mathbf{R})$ of* 4.2.2 *(given by symplectic reduction).*

Remark. Because X_Σ is simply connected, we have $H^2(X_\Sigma; \mathbf{Z}) \subset H^2(X_\Sigma; \mathbf{Z}) \otimes \mathbf{R} \cong H^2(X_\Sigma; \mathbf{R})$, and we consider of course $\mathbf{K}^\star$ as the integral lattice in $\mathfrak{k}^\star$ (this gives the meaning of the word "extend" in the statement).

Proof. We already have a $\mathbf{Z}$-basis of $H^2(X_\Sigma;\mathbf{Z})$ (*see* 4.2.1). Once a basis of $\mathbf{K}$ is chosen, we may write the torus $K_\mathbf{C}$ as a product of k factors $\mathbf{C}^\star$ and it is easily shown that $\mathbf{K}^\star \to H^2(X_\Sigma;\mathbf{Z})$ is an isomorphism, making the nice

Exercise 4.3.3 Let $(f_1,\ldots,f_k)$ be a basis of $\mathbf{K}$ and $(g_1,\ldots,g_k)$ be the dual basis in $\mathbf{K}^\star$. For any $i \in [1,k]$, define a fan Σ_i in $\mathbf{R}^{n+1}$ by decomposing the projection

$$\pi : \mathbf{Z}^N \xrightarrow{\pi_i} \mathbf{Z}^{n+1} \longrightarrow \mathbf{Z}^n$$

in such a way that $\operatorname{Ker}\pi_i = \mathbf{K}/(\mathbf{Z}\cdot f_i)$ and that Σ_i has the same combinatorics as Σ and $\tilde{\Sigma}$, thus $\mathcal{U}_{\Sigma_i} = \mathcal{U}_\Sigma \subset \mathbf{C}^N$.

Show that the projection $\mathbf{Z}^{n+1} \to \mathbf{Z}^n$ induces a principal $\mathbf{C}^\star$-bundle $X_{\Sigma_i} \to X_\Sigma$. Comparing $\mathcal{U}_{\Sigma_i}$ and $\mathcal{U}_{\Sigma_{g_i}} \subset \mathbf{C}^{N+1}$, show that X_{Σ_i} is the complement of the zero section in $X_{\Sigma_{g_i}}$.

Thus the element g_i in the basis of $\mathbf{K}^\star$ we consider is sent to the Euler class of the i-th principal $\mathbf{C}^\star$-bundle forming the principal $K_\mathbf{C}$-bundle $\mathcal{U}_\Sigma \to X_\Sigma$ and the homomorphism in question is actually an isomorphism over $\mathbf{Z}$.

Let us come now to our main subject of interest, namely the interpolation statement. The most important tool in the proof is our old friend, the 1-form $\alpha = \frac{1}{2}\sum(p_i dq_i - q_i dp_i)$ on $\mathbf{C}^N$. In addition to being a primitive of the symplectic form of $\mathbf{C}^N$, it satisfies:

If $g \in \mathfrak{t}^\star$ and $X \in \mathfrak{t}$, then, on $\mu^{-1}(g)$, we have $i_{\underline{X}}\alpha = g(X)$

from which we immediately deduce that if $\eta \in \mathfrak{k}^\star$ and $Y \in \mathfrak{k}$, then, on $\mu'^{-1}(\eta)$, $i_{\underline{Y}}\alpha = \eta(Y)$.

Let g be an integral element in $\mathfrak{t}^\star$ and $\eta = g_{|\mathfrak{k}} \in \mathfrak{k}^\star$ be its projection, such that $\eta \in \mu'(\mathcal{U}_\Sigma)$. Call σ $(= dz/z)$ the volume form on S^1. Consider the principal S^1-bundle which is the circle bundle of $X_{\Sigma_g} \to X_\Sigma$:

$$\begin{array}{ccc} \mu'^{-1}(\eta)\times S^1 & \longrightarrow & \mu'^{-1}(\eta) \\ \downarrow & & \downarrow \\ X_{\Sigma_g} & \longrightarrow & X_\Sigma \end{array}$$

(the vertical arrows are the quotients by the dimension k compact tori K', K).

On $\mu'^{-1}(\eta)\times S^1$, $\alpha\oplus\sigma$ is a connection 1-form for the trivial bundle represented by the upper horizontal arrow. The torus K' acts on $\mu'^{-1}(\eta)\times S^1$ with fundamental vector fields $(\underline{Y}, -\eta(Y))$; in particular,

$$\forall Y \in \mathfrak{k},\ i_{\underline{Y}}(\alpha\oplus\sigma) = i_{\underline{Y}}\alpha - \eta(Y) = 0$$

and therefore $\alpha\oplus\sigma$ descends to the quotient and gives a 1-form of connection on X_{Σ_g}. Its *curvature form*, the cohomology class of which is the Euler class of $X_{\Sigma_g} \to X_\Sigma$, is the reduced symplectic form ω_η. □

Corollary 4.3.4 *Assume* Σ *is smooth and complete. Let* u_i *be the Euler class of the complex line bundle over* X_Σ *which is associated with* $e_i^\star \in (\mathbf{Z}^N)^\star$. *The group* $H^2(X_\Sigma;\mathbf{Z})$ *is the commutative group generated by* $(u_1,\ldots,u_N)$, *with the relations*

$$\left(\sum_{i=1}^{N} g(e_i)u_i = 0\right)_{g\in(\mathbf{Z}^n)^\star}.$$

Proof. We explicitely describe $\mathbf{K}^\star$ and the above isomorphism:

$$0 \longrightarrow (\mathbf{Z}^n)^\star \longrightarrow \left(\mathbf{Z}^N\right)^\star \longrightarrow H^2(X_\Sigma;\mathbf{Z}) \longrightarrow 0.$$

□

4.4 The cohomology of X_Σ

Let us begin with an exercise:

Exercise 4.4.1 Using a well chosen Morse function on X_Σ, show that the ring $H^\star(X_\Sigma;\mathbf{Z})$ is generated by its degree 2 elements (*see* [30]). If X_Σ is simplicial, but not smooth, this is still true with *rational* cohomology.

The ring $\mathbf{Z}[u_1,\ldots,u_N]$ (with the same notations as in corollary 4.3.4) may be interpreted as the ring $H^\star(BT;\mathbf{Z})$. The linear forms $\sum_{i=1}^{N} g(e_i)u_i$ (for $g \in (\mathbf{Z}^n)^\star$) in question in this statement thus describe a system of generators for the ideal I, kernel of the restriction $H^\star BT \to H^\star BK$.

From this exercise and the previous statements we deduce:

Corollary 4.4.2 *If* Σ *is smooth and complete, the composition*

$$H^\star(BT;\mathbf{Z}) \longrightarrow H^\star(BK;\mathbf{Z}) \longrightarrow H^\star(X_\Sigma;\mathbf{Z})$$

is a surjective ring homomorphism. □

Thus when the variety X_Σ is smooth and compact, $H^\star(X_\Sigma;\mathbf{Z})$ is a ring of polynomials. More precisely:

Theorem 4.4.3 $H^\star(X_\Sigma;\mathbf{Z}) \cong \mathbf{Z}[u_1,\ldots,u_N]/(I+J)$, *where* I *is the ideal generated by the linear relations* $(\sum g(e_i)u_i = 0)_{g\in(\mathbf{Z}^n)^\star}$ *and* J *that generated by the monomials* u^I *such that* $\langle x_I \notin \Sigma$.

There is a proof of this theorem in [30]. In 5.2, we shall give a complete proof in the case of surfaces ($n = 2$). In the next exercise, we give some hints for a proof having the two following advantages:

1. It shows the respective roles of the two ideals I and J.
2. It uses the construction $\mathcal{U}_\Sigma \to X_\Sigma$ and computes the equivariant cohomology of X_Σ with respect to the action of its big torus.

Assuming Σ smooth and complete, the cohomology is with integral coefficients.

Exercise 4.4.4

1. Show that $ET \times_Q X_\Sigma$ and $ET \times_T \mathcal{U}_\Sigma$ have the same homotopy type. Derive a ring isomorphism $H_T^\star \mathcal{U}_\Sigma \cong H_Q^\star X_\Sigma$, the structure of a $H^\star BT$-module on $H_Q^\star X_\Sigma$ and a commutative diagram
$$\begin{array}{ccc} H^\star BT & \longrightarrow & H_Q^\star X_\Sigma \\ \downarrow & & \downarrow \\ H^\star BK & \longrightarrow & H^\star X_\Sigma \end{array}$$

2. Applying carefully the ideas of V-5.2, show that $H^\star \mathcal{U}_\Sigma$ is a torsion $H^\star BT$-module and that
$$\operatorname{Supp} H_T^\star \mathcal{U}_\Sigma = \bigcup_{\langle x_I \in \Sigma} e_I$$
where $(e_1, \ldots, e_N)$ is considered as the canonical basis of $\mathbf{R}^N = \mathfrak{t}$. In particular, this support is a union of dimension n vector subspaces.

3. Let J be the ideal of $H^\star BT$ generated by all the monomials u^J such that $\langle x_J \notin \Sigma$. Check that the torsion $H^\star BT$-module $H^\star BT/J$ has the same support as $H_T^\star \mathcal{U}_\Sigma$.

4. We said in V-5.3.6 that $H_Q^\star X_\Sigma$ is the free $H^\star BQ$-module generated by $H^\star X_\Sigma$. On the other hand, it is easy to check, using the spectral sequence of the fibration $ET \times_T \mathcal{U}_\Sigma \to BT$, that $H^\star BT \to H_T^\star \mathcal{U}_\Sigma$ is a surjective homomorphism. Using these two results, show that $H^\star BT \to H_Q^\star X_\Sigma$ is actually the quotient $H^\star BT \to H^\star BT/J$ and deduce the theorem.

5 Complex toric surfaces

5.1 Graphs associated with dimension 2 fans

In dimension 2, a fan is well defined once we know its 1-skeleton, and thus is given by a family of N primitive vectors $(x_1, \ldots, x_N)$ of $\mathbf{Z}^2$ (indices are considered as if in $\mathbf{Z}/N$) such that all the (x_{i-1}, x_i) are $\mathbf{Z}$-bases with the same orientation. The dimension 2 cones are then the $\langle x_{i-1,i}$.

Writing the direct basis (x_{i+1}, x_i) in terms of the direct basis (x_i, x_{i-1}), notice that
$$x_{i+1} = -x_{i-1} + m_i x_i$$
for some integer m_i. To give a fan is thus equivalent to give the integers $(m_1, \ldots, m_N)$. The graph consisting of the polygon with N vertices weighted by $m_1, \ldots, m_N$ in the same order gives the fan back. In order that such a graph defines a fan, the integers in question must satisfy two conditions

- On the one hand, the equality
$$\begin{pmatrix} m_1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} m_2 & 1 \\ -1 & 0 \end{pmatrix}\cdots\begin{pmatrix} m_N & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
in order that the fan closes up, that is that (x_N, x_1) is a direct $\mathbf{Z}$-basis.

- On the other hand, during this operation, we are not allowed to turn around twice (or more).

In the graph, any vertex now represents some $\mathbf{P}^1(\mathbf{C})$ and any edge corresponds to a fixed point of the action of the *big torus* $Q_{\mathbf{C}}$, intersection of two such $\mathbf{P}^1(\mathbf{C})$.

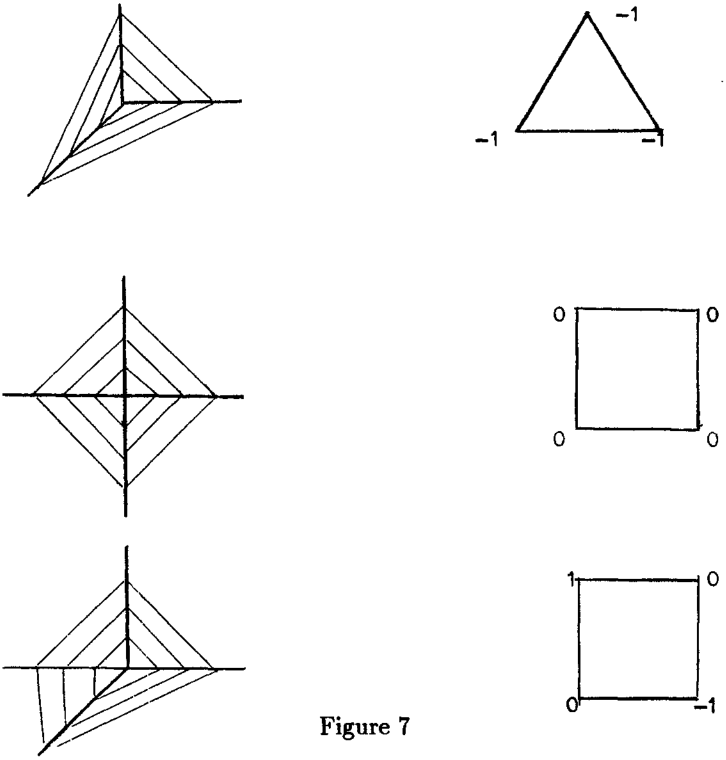

Figure 7

Figure 7 shows the weighted graphs associated respectively to $\mathbf{P}^2(\mathbf{C})$, $\mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$ and to a surface which is nothing other than the blow-up of $\mathbf{P}^2(\mathbf{C})$ at one point as the reader will see when solving the next exercise.

Exercise 5.1.1 Let Σ be a dimension 2 fan such that

$$\Sigma^{(1)} = (x_1, \ldots, x_N),\ \Sigma^{(2)} = ((x_{i,i+1})_{i \bmod N}$$

. Show that $z \notin \mathcal{U}_\Sigma \subset \mathbf{C}^N$ if and only if z has two nonconsecutive (mod N) zero coordinates. Let Σ' be a complete fan the 1-skeleton of which is $(x_1, \ldots, x_N, x_{N+1} = x_1 + x_N)$. Show that the map

$$\begin{array}{ccc} \mathbf{C}^{N+1} & \xrightarrow{\tilde{\sigma}} & \mathbf{C}^N \\ (z_1, \ldots, z_{N+1}) & \longmapsto & (z_1 z_{N+1}, z_2, \ldots, z_{N-1}, z_N z_{N+1}) \end{array}$$

sends $\mathcal{U}_{\Sigma'}$ onto $\mathcal{U}_\Sigma$. If $K_{\mathbf{C}}$ acts on $\mathbf{C}^N$ by

$$t \cdot (z_1, \dots, z_N) = (t^{b_1} z_1, \dots, t^{b_N} z_N)$$

show that $K'_{\mathbf{C}} = K_{\mathbf{C}} \times \mathbf{C}^\star$ acts on $\mathbf{C}^{N+1}$ by

$$(t, u) \cdot (z_1, \dots, z_{N+1}) = (t^{b_1} \overline{u} z_1, t^{b_2} z_2, \dots, t^{b_N} \overline{u} z_N, u z_{N+1})$$

and deduce that $\widetilde{\sigma}$ induces a map

$$\sigma : X_{\Sigma'} \longrightarrow X_\Sigma$$

which is the blow-up of the fixed point of $Q_{\mathbf{C}}$ associated with the cone $\langle x_{N,1}$ in Σ.

Remark. It is possible to show (*see* [58]) that all the smooth compact complex toric surfaces are obtained by a finite sequence of blow-ups, starting with $\mathbf{P}^2(\mathbf{C})$ or with a Hirzebruch surface.

5.2 Interpretation of the m_i

Let Σ_i be the fan consisting of all the cones in Σ generated by x_{i-1}, x_i and x_{i+1}. According to the study of the line bundles in 4.3, we know that the manifold X_{Σ_i} is the total space of a complex line bundle over $\mathbf{P}^1(\mathbf{C})$, the one which has Euler class $-m_i$.

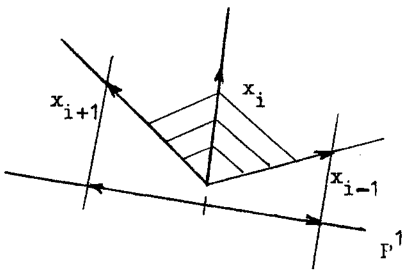

Figure 8

The inclusion $e_{\overline{i}} \cap \mathcal{U}_{\Sigma_i}/K_{\mathbf{C}} \subset \mathcal{U}_{\Sigma_i}/K_{\mathbf{C}} \subset X_\Sigma$ thus presents X_{Σ_i} as a tubular neighborhood of the curve in X_Σ which is associated with the cone $\langle x_i$. This complex curve carries a homology class dual to u_i (notation of 4.3.4).

For example, the fan on figure 4 represents the toric surface obtained by gluing the total spaces of the line bundles $\mathcal{O}(a)$ and $\mathcal{O}(-a)$ over $\mathbf{P}^1(\mathbf{C})$. It is the Hirzebruch surface $\mathbf{P}(\mathcal{O}(a) \oplus \mathbf{1})$.

So the formal analogy between the plumbing diagrams in IV-B and the graph associated with a fan here is not accidental:

Proposition 5.2.1 *The manifold plumbed along the weighted graph Γ is an open subset of the toric surface associated with the same graph.* □

We noticed in IV-B that the polygonal graphs we consider here also represent plumbed manifolds which can be endowed with actions of dimension 2 tori. It is nevertheless recommended to be careful with identifying the *closed* manifolds obtained: they are all obtained by gluing a $T^2 \times \mathbf{R}^2$ to the same manifold with boundary in both cases, but *not* with the same gluing map. The manifold in chapter IV allways has a nontrivial fundamental group, its second Betti number is N and it has no invariant symplectic form, on the contrary the toric manifold we consider here is simply connected, its second Betti number is $N-2$ and it has plenty of invariant symplectic forms.

Investigating the multiplicative structure of $H^\star(X_\Sigma;\mathbf{Z})$, taking corollary 4.3.4 and the above remarks into account, we easily obtain a proof of 4.4.3 in the case of surfaces, namely:

Proposition 5.2.2 $H^\star(X_\Sigma;\mathbf{Z}) \cong \mathbf{Z}[u_1,\ldots,u_N]/(I+J)$, *where I is the ideal generated by the linear relations $(\sum g(e_i)u_i = 0)_{g\in(\mathbf{Z}^2)^\star}$ and J the one generated by the monomials u^I such that $\langle x_I \notin \Sigma$.*

Remark. J is generated here by the degree > 4 monomials and the $u_i u_j$ where i and j are not consecutive mod N.

Proof. Taking into account the structure of $H^2(X_\Sigma;\mathbf{Z})$ and the obvious incidence relations between the curves dual to the u_i, we have a surjective morphism

$$\mathbf{Z}[u_1,\ldots,u_N]/(I+J) \longrightarrow H^\star(X_\Sigma;\mathbf{Z}).$$

In order to be sure that this is an isomorphism, it suffices to calculate the degree 4 parts of both rings. On the right, we know that $H^4(X_\Sigma;\mathbf{Z}) \cong \mathbf{Z}$ and that any of the products $u_i u_{i+1}$ may be chosen as a generator (the curves associated with u_i and u_{i+1} meet at $+1$ point). Let us thus show that, in the ring on the left, we have $u_{i-1}u_i = u_i u_{i+1}$ for any i. Fix $i \in [1,N]$, then (x_{i-1},x_i) is a basis of $\mathbf{Z}^2$ and the projection $\mathbf{Z}^N \to \mathbf{Z}^2$ has matrix

$$\begin{pmatrix} a_1 & \ldots & a_{i-2} & 1 & 0 & -1 & a_{i+2} & \ldots & a_N \\ b_1 & \ldots & b_{i-2} & 0 & 1 & m_i & b_{i+2} & \ldots & b_N \end{pmatrix}$$

(where $x_j = a_j x_{i-1} + b_j x_i$) and therefore

$$\begin{cases} f_1 = a_1 e_1^\star + \ldots + a_{i-2}e_{i-2}^\star + e_{i-1}^\star - e_{i+1}^\star + a_{i+2}e_{i+2}^\star + \ldots + a_N e_N^\star \\ f_2 = b_1 e_1^\star + \ldots + b_{i-2}e_{i-2}^\star + e_i^\star + m_i e_{i+1}^\star + b_{i+2}e_{i+2}^\star + \ldots + b_N e_N^\star \end{cases}$$

form a basis of $(\mathbf{Z}^2)^\star \subset (\mathbf{Z}^N)^\star$. The ideal I is thus generated by the two elements

$$\begin{cases} v_1 &= a_1u_1 + \ldots + a_{i-2}u_{i-2} + u_{i-1} - u_{i+1} + a_{i+2}u_{i+2} + \ldots + a_Nu_N \\ v_2 &= b_1u_1 + \ldots + b_{i-2}u_{i-2} + u_i + m_iu_{i+1} + b_{i+2}u_{i+2} + \ldots + b_Nu_N \end{cases}$$

in particular $u_iv_1 = 0$ that is $u_{i-1}u_i - u_iu_{i+1} = 0$, and this is what we wanted to prove. Moreover $u_iv_2 = 0$ gives us $u_i^2 = -m_iu_iu_{i+1}$, this means that $-m_i$ is actually the self-intersection number of the curve corresponding to the cone $\langle x_i$, as we must find, as it is the Euler class[13] of the normal bundle of this curve. □

5.3 4-manifolds with hamiltonian T^2-actions

Proposition 5.3.1 ([31]) *A compact symplectic 4-manifold W may be endowed with a hamiltonian T^2-action if and only if its T^2-equivariant diffeomorphism type is that of a complex toric surface.*

More generally in [31] dimension $2n$ symplectic manifolds endowed with hamiltonian T^n-actions are considered. The proof of this special case is very easy and in the "low dimensions" spirit of chapter IV.

Proof. As always call μ the moment map and consider the image polyhedron $\mu(W)$. Choose a rational direction of projection which is orthogonal to some edge (to make things simpler). In other words, choose a circle $S^1 \subset T^2$ with a fixed sphere. The associated hamiltonian H is the function $\langle \mu, X \rangle$, where X generates the Lie algebra of the circle under consideration. Replacing X be $-X$ if need be, we may assume that the fixed sphere is the minimum of H.

The complement of an equivariant tubular neighborhood of the maximum of H is then the manifold plumbed along a star-shaped graph with two branches. We may easily reconstruct a T^2-action on this plumbing: the circle S^1 we are dealing with is a direct summand in T^2 and we require that some supplement fixes the sphere B_{min}, this being enough to define the action everywhere. We still must compactify, which is done by addition of either a vertex if the maximum is reached on a sphere or an edge otherwise. The graph thus obtained is actually the one associated with a fan as it comes from the convex polyhedron which is the image of the moment map μ. □

[13] Another definition of the Euler class!

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