LITERATURE CITED

1. I. M. Gel'fand and I. Ya. Dorfman, "Hamiltonian operators and algebraic structures connected with them," Funkts. Anal. Prilozhen., 13, No. 4, 13-30 (1979).
2. A. M. Astashov, "Normal forms of Hamiltonian operators in field theory," Dokl. Akad. Nauk SSSR, 270, No. 5, 1033-1037 (1983).
3. F. Magri, "A simple model of the integrable Hamiltonian equation," J. Math. Phys., 19, No. 5, 1156-1162 (1978).
4. O. I. Mokhov, "Third-order local Poisson brackets," Usp. Mat. Nauk, 40, No. 5, 257-258 (1985).
5. B. A. Dubrovin and S. P. Novikov, "Hamiltonian formalism of one-dimensional systems of hydrodynamical type and the Bogolyubov—Uizem averaging method," Dokl. Akad. Nauk SSSR, 270, No. 4, 781-785 (1983).
6. B. A. Kupershmidt and G. Wilson, "Modifying Lax equations and the second Hamiltonian structure," Invent. Math., 62, No. 3, 403-436 (1981).
7. A. S. Focas and B. Fuchssteiner, "On the structure of symplectic operators and hereditary symmetries," Lett. Nuovo Cimento, 28, No. 8, 299-303 (1980).
8. N. Kh. Ibragimov, Transformation Groups in Mathematical Physics [in Russian], Nauka, Moscow (1983).
9. A. M. Vinogradov, "Hamiltonian structures in field theory," Dokl. Akad. Nauk SSSR, 241, No. 1, 18-21 (1978).
10. C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, "Korteweg—de Vries equation and generalizations. VI. Methods for exact solutions," Commun. Pure Appl. Math., 27, 97-133 (1974).
11. P. D. Lax, "Almost-periodic solutions of the KdV equation," SIAM Rev., 18, No. 3, 351-375 (1976).
12. I. M. Gel'fand and I. Ya. Dorfman, "Integrable KdV-G. Dim equations," in: Contemporary Problems of Mathematical Physics and Computational Mathematics [in Russian], Nauka, Moscow (1982), pp. 102-112.

COBORDISMS OF LAGRANGIAN IMMERSIONS IN THE SPACE OF THE COTANGENT BUNDLE OF A MANIFOLD

M. Audin

UDC 512.7

In a previous paper [2] we showed how to calculate the cobordism groups defined by V. I. Arnol'd of Legendre immersions in the space of 1-jets $J^1(\mathbf{R}^n, \mathbf{R})$. Here we show how to reduce the calculation of the groups of Lagrange cobordisms in the total spaces of the cotangent bundles of manifolds to this. In particular, we show that the oriented Legendre bordism group [of Legendre immersions in $J^1(\mathbf{R}^n, \mathbf{R})$] is a natural direct summand in the oriented Lagrange bordism group (of Lagrange immersions in $T^*\mathbf{R}^n$), and the supplementary summands are defined by continuous invariants which generalize that described by Arnol'd [1] in one-dimensional case: Arnol'd's invariant is the oriented area bounded by the immersed curve. In the nonoriented case we show that the obvious map from the Legendre bordism group to the Lagrange bordisms is an isomorphism.

0. Notation

Let X be an infinitely differentiable manifold, T*X denote the space of its cotangent bundle, α be the Liouville form (pdq), and $\omega = -d\alpha$ be the canonical symplectic form. $M\lambda$ (respectively $M\tilde{\lambda}$) is the Thom spectrum constructed from the tautological bundle $\lambda_n \to \Lambda_n = U(n)/O(n)$ [respectively $\tilde{\lambda}_n \to \tilde{\Lambda}_n = U(n)/SO(n)$] over the Grassmanians of Lagrangian nonoriented (oriented) subspaces of $\mathbf{R}^{2n}$.

University of Paris-South. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 21, No. 3, pp. 61-64, July-September, 1987. Original article submitted July 23, 1986.

0016-2663/87/2103-0223$12.50 © 1988 Plenum Publishing Corporation

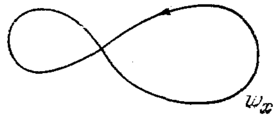

Fig. 1

1. Pontryagin–Thom Construction.

An immersion $f: V \to T^*X$ is called Lagrangian if $\dim V = \dim X$ and $f^*\omega = 0$. In this case the 1-form $f^*\alpha$ is closed; if in addition it is exact, then one says that the immersion f is exact Lagrangian. Exact Lagrangian immersions are obtained by projection of Legendre immersions to the space $J^1(X, \mathbf{R}) \cong T^*X \times \mathbf{R}$.

Lagrangian cobordisms were defined and studied in a very general situation by Arnol'd [1], to which we refer the reader. Here we restrict ourselves to the so-call "cylindrical" case: the cobordisms lie in $T^*(X \times \mathbf{R})$.

The Gromov–Lees theorem [5, 6] lets one apply the Pontryagin–Thom construction to these problems; previously such constructions were used in a more general situation by Eliashberg [4].

To simplify the formulations we shall assume that the manifold X is oriented.

Proposition 1 (cf. [4, 3]). If X is a compact manifold, then the group of oriented (respectively nonoriented) cobordisms of exact Lagrangian immersions in $T^*(X \times \mathbf{R}^m)$ is isomorphic to $L^{-m}(X)$ [respectively $\mathfrak{N}L^{-m}(X)$], where $L^*(\bullet)$ [respectively $\mathfrak{N}L^*(\bullet)$] denotes the generalized cohomology theory defined by the spectrum $M\tilde{\lambda}(M\lambda)$.

This is the same as the groups of Legendre cobordisms in $J^1(X, \mathbf{R})$. We give a more general assertion. By an L-regular homotopy we mean a map $H: V \to [0, 1] \to T^*X$ such that for any t, $H_t = H(\bullet, t)$ is a Lagrangian immersion, and the class of the element $H_t{}^*\alpha$ in the group $H^1(V, \mathbf{R})$ is independent of t. Then the Gromov–Lees theorem lets one identify a class of L-regular homotopies of Lagrangian immersions $f: V \to T^*X$ with a triple consisting of the homotopy class of isomorphisms of complex bundles $((\pi \circ f)^*TX) \otimes \mathbf{C} \to TV \otimes \mathbf{C}$ on V (where $\pi: T^*X \to X$ is the projection of the bundle), the class of the form $f^*\alpha$ in $H^1(V, \mathbf{R})$ and the homotopy class of the composite map $\pi \circ f: V \to X$.

Since, as is easy to see, L-regular homotopies are precisely those homotopies which lift to Lagrangian cobordisms $\tilde{H}(x, t) = (H(x, t), t, u(x, t)) \in T^*X \times \mathbf{R}^2 \cong T^*(X \times \mathbf{R})$, it follows that one has the following proposition.

Proposition 2 (cf. [4, 3]). For a compact manifold X the group of oriented (respectively nonoriented) cobordisms of Lagrangian immersions in $T^*(X \times \mathbf{R}^m)$ is isomorphic to $\mathrm{Lag}^{-m}(X)$ [respectively $\mathfrak{N}\mathrm{Lag}^{-m}(X)$], where $\mathrm{Lag}^*(\bullet)$ [respectively $\mathfrak{N}\mathrm{Lag}^*(\bullet)$] denotes the generalized cohomology theory defined by the spectrum $M\tilde{\lambda} \wedge K(\mathbf{R}, 1)$ [respectively $M\lambda \wedge K(\mathbf{R}, 1)$].

$K(\mathbf{R}, 1)$ is the Eilenberg–MacLane space corresponding to $H^1(\bullet, \mathbf{R})$ and we denote it by K. To calculate the groups of Lagrangian cobordisms in T^*X it is theoretically sufficient to know the homotopy of the spaces $M\lambda$ and $M\tilde{\lambda}$, the homology of K and the cohomology of X.

Reminder (cf. [7]). $H_*(K, \mathbf{Z})$ is isomorphic to $\Lambda_{\mathbf{Z}}\mathbf{R}$ (the exterior algebra over the Z-module R with the natural grading).

We note immediately that K is a rational space; it is also clear that $M\lambda$ is 2-primary [$\mathfrak{N}L^\circ(X)$ is a vector space over $\mathbf{Z}_2$]. Consequently, the natural map

$$\mathfrak{N}L^\circ(X) \to \mathfrak{N}\,\mathrm{Lag}^\circ(X)$$

is an isomorphism.

COROLLARY. Any Lagrangian immersion is nonoriented cobordant to an exact Lagrangian immersion.

It is well known that the oriented case is much more complicated. In fact, if $x \in H^{n-1}(U/SO, \mathbf{R})$ and f is a Lagrangian immersion of an oriented manifold in $T^*\mathbf{R}^n$, then x defines a class $\gamma(f)^* x \in H^{n-1}(V, \mathbf{R})$ and the number $\int_V \gamma(f)^* x \wedge f^*\alpha$ depends only on the cobordism class of f. For example, the oriented area bounded by a plane oriented curve in $\mathbf{R}^2$ is a cobordism invariant (cf. [1]).

In what follows we restrict ourselves to the oriented case and we assume that $X = R^n$, i.e., we consider the groups L_n (in the "exact" case) and Lag_n (in the general case); these groups are the "coefficient" groups of the cohomology theories considered (i.e., the cohomology groups of a point). Cartesian products of immersions define graded ring structures on the sums $L_* = \oplus L_n$, $Lag_* = \oplus Lag_n$. It is shown in [2] how to calculate L_*, while we show here how, knowing L_*, to calculate Lag_*.

2. Calculation of the Ring Lag_*

First of all we note that Proposition 2 can be reformulated as follows: the group Lag_n is equal to the group $L_n(K)$ (the generalized homology of the space K).

THEOREM 1. $Lag_* \cong L_* \otimes H_* (K, \mathbf{Z})$.

Proof. We write $Lag_n \cong L_n \oplus \tilde{L}_n (K)$ (where $\tilde{L}$ is the reduced homology theory). Since the space K is rational, we get from the Atiyah-Hirzebruch spectral sequence that $\tilde{L}_n (K) \cong \bigoplus_{p\geqslant 1} L_{n-p} \otimes H_p (K)$.

Remark. Now we have $Lag_n \cong L_n \oplus \bigoplus_{p\geqslant 1} L_{n-p} \otimes H_p (K) \cong L_n \oplus \bigoplus_{p\geqslant 1} H_{n-p} (U/SO) \otimes H_p (K)$, since for $p \geq 1$, $H_p(K)$ is a vector space over Q, and $L_{n-p} \otimes Q \cong H_{n-p} (U/SO; Q)$ according to a famous theorem of Serre [$L_k \cong \pi_k(M\tilde{\lambda})$].

COROLLARY 1 (cf. [1]). $Lag_1 \cong L_1 \oplus \mathbf{R} \cong \mathbf{Z} \oplus \mathbf{R}$.

In what follows we shall use the isomorphism $\mathbf{R} \to Lag_1$, whose image in L_1 is equal to 0: to any real number x one associates a class of immersions w_x: $S^1 \to R^2$, regularly homotopic to the Whitney immersion (and consequently having Maslov class zero) and surrounding an oriented area equal to x. It follows from Corollary 1 that the homomorphism sought is well-defined by this.

In order to represent the geometric content of the factor $H_*(K)$ better we consider the projection $Lag_n \to H_n(K)$ in more detail. For a Lagrangian immersion f: $V^n \to T^*R^n$ we denote by φ_f a map $V \to K$, defining the class $[f^*\alpha] \in H^1 (V, \mathbf{R}) \cong [V, K]$. There is a commutative diagram

$$\begin{array}{ccc} H_n(V) & \xrightarrow{\Delta} & \Lambda^n H_1(V) \\ {\scriptstyle (\varphi_f)_*}\downarrow & & \downarrow{\scriptstyle \Lambda^n(\varphi_f)_*} \\ H_n(K) & \xrightarrow{\cong} & \Lambda^n H_1(K), \end{array}$$

where the lower isomorphism is defined by the natural comultiplication, and Δ is the composition $H_n (V) \to H_n (V \times \ldots \times V) \cong (\otimes H_* V)_n \oplus T \to \Lambda^n H_1 (V)$ of the diagonal map $V \to V \times \ldots \times V$, the Kunneth isomorphism [which, although it is not natural, but its Tor part is annihilated in $K_*(K)$] and the natural projection. Consequently, the projection $Lag_n \to H_n(K)$ makes correspond to the class of the immersion f an element $\Lambda^n (\varphi_f)_* \Delta [V]$, where $(\varphi_f)_*: H_1 (V, \mathbf{Z}) \to \mathbf{R}$ is exactly the integrated Liouville form $f^*\alpha$ on cycles. For example, if V is a surface of genus g, and $(a_1, \ldots, a_g, b_1, \ldots, b_g)$ is a basis in $H_1(V, Z)$, in which the intersection form is defined by the formulas $(a_i, a_j) = (b_i, b_j) = 0$, $(a_i, b_j) = \delta_{ij}$, then the image of the element $[f] \in Lag_2$ in the group $H_2 (K) = \Lambda^2_{\mathbf{Z}}\mathbf{R}$ is $x_1 \wedge y_1 + \ldots + x_g \wedge y_g$, where $x_i = \int_{a_i} f^*\alpha$, $y_i = \int_{b_i} f^*\alpha$. Since $L_2 = 0$ (cf. [2]), we get from this the following corollary.

COROLLARY 2. $Lag_2 \cong L_1 \otimes \mathbf{R} \oplus H_2 (K) \cong \mathbf{R} \oplus \Lambda^2_{\mathbf{Z}}\mathbf{R}$. This isomorphism makes correspond to any Lagrangian immersion f of a surface V of genus g in T^*R^2 the pair

$$\left(\int_m f^*\alpha, \sum_{i=1}^{g} \left(\int_{a_i} f^*\alpha\right) \wedge \left(\int_{b_i} f^*\alpha\right)\right),$$

where m is a cycle, dual to the Maslov class of the immersion f, and $(a_1, \ldots, a_g, b_1, \ldots, b_g)$ is the basis in $H_1(V, Z)$ described above.

These same arguments quickly prove

COROLLARY 3. The immersion $w_{x_1} \times \ldots \times w_{x_n}$: $S^1 \times \ldots \times S^1 \to T^*\mathbf{R}^n$ is cobordant to zero if and only if the numbers $x_1, \ldots, x_n$ are linearly dependent over Q.

We have reason for considering such special immersions.

THEOREM 2. The map $L_* \otimes H_*(K) \to \mathrm{Lag}_*$, which makes correspond to an element of the form $[f] \otimes (x_1 \wedge \ldots \wedge x_q)$ (where $f: V^p \to T^*\mathbf{R}^p$ is an exact Lagrangian immersion) the class of the immersion $f \times w_{x_1} \times \ldots \times w_{x_q}: V^p \times T^q \to T^*R^{p+q}$, is an isomorphism.

Proof. It follows from Corollary 3 that this map is well-defined and is a group homomorphism. It is clear that the composition $L_p \otimes H_q(K) \to \mathrm{Lag}_{p+q} \to H_p(U/SO) \otimes H_q(K)$ for $q \geq 1$ is an isomorphism, so by virtue of Theorem 1, what is needed follows.

COROLLARY. Any Lagrangian immersion in T^*R^n is cobordant to a disconnected union of Lagrangian immersions of the form $f \times g$: $V^p \times T^q \to T^*R^n$, where f: $V^p \to T^*R^p$ is exact, and g: $T^q \to T^*R^q$ is Lagrangian immersion of the torus such that the corresponding Gauss map of this torus to $\tilde{\Lambda}_q \cong U(q)/SO(q)$ is homotopically trivial.

Thus, all the "inexactness" of immersions can be concentrated in immersions of tori: this is not surprising since it is useful to consider K(Q, 1) as a "rational annulus," and hence K(**R**, 1) as a kind of torus.

The following questions arise from the examples considered in the present paper.

A. Can a Lagrangian imbedding of an oriented manifold be cobordant to zero (in the class of Lagrangian immersions)?

B. Can a Lagrangian imbedding of the torus define a Gauss map which is homotopically constant?

In the case of curves, the answers to these questions are negative.

LITERATURE CITED

1. V. I. Arnol'd, "Lagrangian and Legendre cobordisms," Funkts. Anal. Prilozhen., 14, No. 3, 1-13; No. 4, 8-17 (1980).
2. M. Audin, "Quelques calculs en cobordisme lagrangien," Ann. Inst. Fourier, 35, 159-194 (1985).
3. M. Audin, Cobordismes Lagrangien et Legendriens, Hermann, Paris (1986).
4. Ja. M. Eliashberg, "Cobordisme des solutions de relations differentielles," in: Seminaire Sud-Rhodanien de Geometrie 1, Travaux en Cours, Hermann, Paris (1984).
5. M. L. Gromov, "Topological methods of construction of solutions of differential equations and inequalities," in: International Congress of Mathematicians at Nice, 1970 [Russian translation], Nauka, Moscow (1972).
6. J. A. Lees, "On the classification of Lagrange immersions," Duke Math. J., 43, 217-224 (1976).
7. K. Rozhe, "Homology of affine groups and classifying spaces for semilinear bundles," Funkts. Anal. Prilozhen., 13, No. 4, 47-52 (1979).