Chapter X

Symplectic rigidity: Lagrangian submanifolds

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Introduction

This chapter is supposed to be a summary of what is known today about Lagrangian embeddings. We emphasise the difference between *flexibility* results, such as the h-principle of Gromov applied here to Lagrangian immersions (and also to the construction of examples of Lagrangian embeddings) and *rigidity* theorems, based on existence theorems for pseudo-holomorphic curves.

The most famous result in the family of rigidity theorems is the Arnold conjecture (corollary 2.2.5), according to which there exists no *exact* (see § 2.1) closed Lagrangian submanifold in $\mathbf{C}^n$ and its application to the existence of exotic structures on $\mathbf{R}^{2n}$. An equivalent statement is that the *symplectic area class* (see § 2.1), which belongs to the first de Rham cohomology group of a Lagrangian submanifold, *cannot* vanish. Very analogous are *Maslov class* rigidity results: if it is true that essentially any integral cohomology class can be the Maslov class of a Lagrangian immersion, the situation is quite different for embeddings.

We have tried to give complete proofs—at least of the results which are either easy to prove, or which rely on pseudo-holomorphic curves' methods, as this is the subject of the book. This chapter is based mainly on the results and techniques of Gromov but also on work of the authors, of course. We have also tried to present new results, in the text as well as in exercises, as for instance in § 4. The proofs of the basic results come more or less directly from [24], although very often via the unobtainable "corollaires symplectiques" [44]. We thank Jean-Claude Sikorav, who encouraged us to plunder his paper.

Let us now describe in detail the contents of this chapter.

We begin by recalling in § 1 the h-principle for Lagrangian immersions and by giving many examples of Lagrangian submanifolds and of ways to construct them, mainly as submanifolds of standard symplectic space $\mathbf{C}^n$.

In the next section, we state the main theorem, the rigidity result 2.2.4 and derive some symplectic corollaries, as for example the above mentioned Arnold conjecture and the solution of some Lagrangian intersection problems (in § 2.3).

In § 3, we concentrate on Lagrangian submanifolds of $\mathbf{C}^n$. We show how the previous rigidity results give obstructions to realising certain manifolds as Lagrangian submanifolds of $\mathbf{C}^n$. It is customary (see [26]) to call these obstructions "hard". For the sake of completeness, we recall also the simplest "soft" obstructions. In this way, no sphere (except the circle) can be a Lagrangian submanifold. Actually, the hard techniques are needed only in the case of the 3-sphere, where they are both essential and very poorly used: as it is simply connected, any Lagrangian embedding has to be exact. However we give other examples: a 3-manifold with non zero H^1, no soft obstruction, but which cannot be embedded as a Lagrangian submanifold due to the rigidity results (see theorem 3.3.3).

The next section is devoted to rigidity results in cotangent bundles and applications to mechanics: we look especially at Lagrangian tori in $T^\star T^n$, where the topology is large enough to contain both symplectic and topological invariants, and interesting relations between them. This example is also very interesting in mechanics, particularly if one is willing to look at perturbations of integrable systems.

In § 5, we concentrate on pseudo-holomorphic curves and, finally, we give a proof of the main theorem 2.2.4.

In a short appendix, for the sake of completeness, we construct exotic symplectic structures on $\mathbf{R}^{2n}$. Surprisingly enough, we know of no way to prove that these structures are different from the standard one other than via the rigidity theorems.

1. Lagrangian constructions

In this §, we give some systematic ways of constructing Lagrangian submanifolds, mainly in standard linear symplectic space. We begin by recalling Gromov's h-principle—a flexibility result—for Lagrangian immersions in § 1.1, then show how it may also be used to construct examples of Lagrangian submanifolds. In the following §§, we give as many examples of systematic constructions as we know. Perhaps nothing is really original here (we did not invent either symplectic reduction, Lagrangian cobordisms or wavefronts) but it turns out that folkloric and "well known" examples are not *that* well known, hence the list. We stress the fact that most of these constructions derive, more or less directly, from the symplectic reduction process of Marsden and Weinstein [34] and the Arnold papers [3] on Lagrangian cobordisms.

1.1. Lagrangian immersions in $\mathbf{C}^n$ and the h-principle

Recall from chapter I that $\lambda \subset \mathbf{C}^n$ is a Lagrangian subspace if and only if $\lambda \perp i\lambda$ and $\dim \lambda = n$. Hence an immersion $f : L^n \to \mathbf{C}^n$ is Lagrangian if and only if

$$\forall x \in L, \qquad T_x f\,(T_x L) \perp i T_x f\,(T_x L)\,.$$

In particular, T_xf defines a complex linear isomorphism

$$\begin{array}{ccccc} T_xL & \otimes_{\mathbf{R}} & \mathbf{C} & \longrightarrow & \mathbf{C}^n \\ v & \otimes & (a+ib) & \longmapsto & aT_xf(v)+ibT_xf(v) \end{array}$$

and, varying x, a complex vector bundle isomorphism

$$TL \otimes_{\mathbf{R}} \mathbf{C} \xrightarrow{Tf} L \times \mathbf{C}^n.$$

Such an isomorphism will be called a *U-parallelisation* of L. We have just proved that, in order for L to admit Lagrangian immersion in $\mathbf{C}^n$, it is necessary that it is U-parallelisable. The h-principle for Lagrangian immersions in $\mathbf{C}^n$, also called the Gromov-Lees theorem, asserts that the converse is true (see theorem 1.1.3 below for a precise statement).

Thus, for instance, any n-dimensional *stably* parallelisable manifold admits Lagrangian immersions into $\mathbf{C}^n$: complexifying a stable parallelisation, gives an isomorphism

$$\varphi : (TL \otimes \mathbf{C}) \oplus \left(L \times \mathbf{C}^k\right) \longrightarrow L \times \mathbf{C}^{n+k}.$$

Recall the

PROPOSITION 1.1.1 (see e.g. [29]). — *Let E be a complex n-dimensional vector bundle on a manifold of (real) dimension n. Then E is trivial if and only if it is stably trivial. Moreover, any stable trivialisation is homotopic to the suspension of an instable one.*

This says that φ is homotopic to the sum of an isomorphism

$$\psi : TL \otimes \mathbf{C} \longrightarrow L \times \mathbf{C}^n$$

and the identity.

Using the normal vector field of a codimension 1 embedding, we see that all spheres, all orientable surfaces satisfy $TV\oplus$ trivial line bundle $= V \times \mathbf{R}^{n+1}$. Thus these manifolds are stably parallelisable. Consequently, they can have Lagrangian immersions, and the h-principle asserts thet they do have. The next exercise gives an explicit construction for the sphere.

1.1.2. Exercise. — Let $S^n = \left\{(x,y) \in \mathbf{R}^n \times \mathbf{R} \mid \|x\|^2 + y^2 = 1\right\}$ and let $\mathcal{W} : S^n \to \mathbf{C}^n$ be given by $\mathcal{W}(x,y) = (1+iy)x$. Draw a picture of $\mathcal{W}$ for $n=1$, and show, in general, that $\mathcal{W}$ is a Lagrangian immersion with one double point[1].

[1]The existence of an immersion of S^n with one double point was very important in Whitney's study of immersions in twice the dimension. Hence the name $\mathcal{W}$.

Remark. — One can show that the U-parallelisation of S^n defined by the Lagrange immersion $\mathcal{W}$ is homotopic to (the destabilisation of) the complexification of a stable parallelisation of S^n (see chapter 0 of [5]).

Two Lagrangian immersions $f_0, f_1 : L \to \mathbf{C}^n$ are said to be *regularly homotopic* (implicitly "among Lagrangian immersions") if there exists a map

$$f : L \times [0,1] \longrightarrow \mathbf{C}^n$$

such that f_t is a Lagrangian immersion for any t. Now we can state the Gromov-Lees theorem in $\mathbf{C}^n$.

THEOREM 1.1.3 (Gromov [23], Lees [32]). — *The set of Lagrangian regular homotopy classes of Lagrangian immersions $L^n \to \mathbf{C}^n$ is in one-to-one correspondence with the set of homotopy classes of U-parallelisations of L.*

In other words, giving a Lagrangian immersion of L into $\mathbf{C}^n$ is equivalent to giving a U-parallelisation of L.

1.1.4. Exercise. — Show that if a Lagrangian immersion $f : L^n \to \mathbf{C}^n$ exists, then the set of regular homotopy classes of all Lagrangian immersions $L \to \mathbf{C}^n$ is in one-to-one correspondence with the set $[L, U(n)]$ of homotopy classes of maps $L \to U(n)$.

1.1.5. Exercise. — Let V be any closed surface.

1. Show that the set $[V, U(2)]$ is in one-to-one correspondence with $H^1(V; \mathbf{Z})$ (Hint: recall that V can be represented by a polygon, whose edges are identified in pairs, see [35] for instance).
2. Show that $TV \otimes \mathbf{C}$ is a trivialisable complex vector bundle if and only if the Euler characteristic $\chi(V)$ is even. If this is the case, show that the regular homotopy class of a Lagrangian immersion is well defined by its Maslov class (see the appendix to chapter I).

Let us mention that the "Gromov-Lees theorem" is a very special case of the general h-principle of Gromov. We recall here this principle, for the general case of Lagrangian immersions in symplectic manifolds.

Let $f : L^n \to W^{2n}$ be a Lagrangian immersion into a symplectic manifold. From f, we can extract two topological data: the homotopy class of the map $f : L \to W$ (which is such that the cohomology class of $f^\star\omega$ vanishes) and the Lagrangian subbundle TL of the symplectic bundle $f^\star TW \to L$.

THEOREM 1.1.6 ([23]). — *Let L be a closed manifold and (W, ω) be a symplectic manifold. Then there is a weak homotopy equivalence between the two spaces:*

- *The space of Lagrangian immersions $L \to W$.*
- *The space of all bundle maps $\varphi : TL \to TW$ such that the induced map on the bases $\psi : L \to W$ pulls back the cohomology class of ω to 0 and the induced map on the fibres sends any fibre of TL to a Lagrangian subspace.* □

Recall that a weak homotopy equivalence is a map that induces an isomorphism of homotopy groups. We shall use this statement only at the π_0 level.

When W is $\mathbf{C}^n$, $f : L \to \mathbf{C}^n$ is homotopic to a constant map, so that the first datum is redundant. The statement 1.1.3 is then 1.1.6 at the π_0 level. The condition $\psi^\star[\omega] = 0$ is also automatically satisfied when ω is exact, e.g. when it is the canonical structure on a cotangent bundle.

1.1.7. Exercise. — Let $W = T^\star X$ be a cotangent bundle. Prove that the regular homotopy classes of Lagrangian immersions $L \to W$ are in one-to-one correspondence with the disjoint union

$$\coprod_{f \in [L,X]} U(f)$$

where $U(f)$ is the set of homotopy classes of complex isomorphisms

$$\varphi : TL \otimes \mathbf{C} \longrightarrow f^\star TX \otimes \mathbf{C}.$$

What happens when $L = X = T^2$?

1.2. First examples of Lagrangian embeddings

The study of Lagrangian embeddings is far more difficult (the present chapter is devoted to this problem). Let us begin with a list of examples in $\mathbf{C}^n$.

Of course any embedding of S^1 in $\mathbf{C}$ is Lagrangian. Taking products gives Lagrangian embeddings of any torus T^n into $\mathbf{C}^n$ and in particular of T^2 into $\mathbf{C}^2$. It is easy to see (cf. 3.2.1 below) that no other orientable surface can have a Lagrangian embedding into $\mathbf{C}^2$. Beautiful examples of Lagrangian embeddings of all non orientable surfaces L with $\chi(L) < 0$ and divisible by 4 were constructed by Givental (see [22], [8] and below exercise 1.5.5). The condition that $\chi(V) \equiv 0 \bmod 4$ is necessary (see [7] and § 3.1) but nobody knows what happens for $\chi = 0$: the Klein bottle is open!

Another example, taken from [5], is the "classifying space", the Lagrangian grassmannian Λ_n (see chapter I).

1.2.1. Exercise. — Consider the complex vector space Sym $(n, \mathbf{C})$ of all complex symmetric $n \times n$ matrices, endowed with its natural hermitian (and symplectic) structure. Show that

$$\begin{array}{rcl} \widetilde{\Phi} : U(n) & \longrightarrow & \mathrm{Sym}\,(n, \mathbf{C}) \\ A & \longmapsto & A^t A \end{array}$$

induces a Lagrangian embedding $\Phi : \Lambda_n \to \mathrm{Sym}\,(n, \mathbf{C})$.

1.2.2. Exercise. — Deduce from 1.2.1 that $U(n)/SO(n)$ admits Lagrangian immersions into $\mathbf{C}^{n(n+1)/2}$ and that $SU(n)/SO(n)$ is U-parallelisable and thus admits Lagrangian immersions into $\mathbf{C}^{n(n+1)/2-1}$ (Hint: show that $U(n)/SO(n)$ is diffeomorphic to $S^1 \times SU(n)/SO(n)$).

We have already used the elementary fact that the product of two Lagrangian embeddings is a Lagrangian embedding. Almost as elementary—but more useful—is the following generalisation.

PROPOSITION 1.2.3. — *Assume V and W are compact manifolds, $f : V^n \to \mathbf{C}^n$ is a Lagrangian immersion and $g : W^m \to \mathbf{C}^m$ a Lagrangian embedding ($m \geq 1$). Then, in the Lagrangian regular homotopy class of the immersion $f \times g$, there is a Lagrangian embedding of $V^n \times W^m$ into $\mathbf{C}^{n+m}$.*

Proof. — Consider the *isotropic* immersion obtained as the composition

$$V^n \xrightarrow{\ f\ } \mathbf{C}^n \subset \mathbf{C}^{n+m}$$

and apply the general position lemma:

LEMMA 1.2.4. — *Let $f : V^n \to \mathbf{C}^{n+m}$ ($m \geq 1$) be an isotropic immersion. Then, among isotropic immersions, there exists an approximation of f which is an embedding.* □

We can approximate by an isotropic embedding $f_\varepsilon : V^n \to \mathbf{C}^{n+m}$. A tubular neighbourhood of $f_\varepsilon(V)$ is symplectomorphic to a neighbourhood of the zero section in $T^\star V \times \mathbf{C}^m \to V$ (see chapter I). We can assume that $g(W)$ is contained in a small enough neighbourhood of 0 in $\mathbf{C}^m$ and embed $V \times W$ as a Lagrangian submanifold of this neighbourhood via the zero section and g. □

An easy consequence of 1.2.3, 1.1.1 and of the Gromov-Lees theorem 1.1.3 is the following corollary:

COROLLARY 1.2.5. — *The three following assertions are equivalent:*
(i) V has a Lagrangian immersion in $\mathbf{C}^n$.
(ii) $V \times T^m$ ($m \geq 1$) has a Lagrangian immersion in $\mathbf{C}^{n+m}$.
(iii) $V \times T^m$ ($m \geq 1$) has a Lagrangian embedding in $\mathbf{C}^{n+m}$. □

Here is an application:

1.2.6. Exercise. — Use 1.2.5 and the existence of a Lagrangian embedding of Λ_n (see 1.2.1) to improve 1.2.2 and show that $U(n)/SO(n)$ actually has a Lagrangian *embedding* into $\mathbf{C}^{n(n+1)/2}$.

1.3. Symplectic reduction

It is a classical fact that one can use symplectic reduction to construct Lagrangian immersions in the "small" symplectic manifold starting from Lagrangian immersions in the "big" one (see [3] and [5] for instance). In this §, we use symplectic reduction to construct examples of Lagrangian embeddings. We use the constructions and the notations of chapter II, where the following diagram is considered:

$$
\begin{array}{ccccccc}
M & \overset{f}{\hookrightarrow} & \mu^{-1}(\xi) & \overset{j}{\hookrightarrow} & W^{2N} & \overset{\mu}{\longrightarrow} & \mathfrak{g}^\star \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} & & & & \\
L^n & \overset{i}{\hookrightarrow} & V^{2n} & & & &
\end{array}
\tag{1.3.1}
$$

Here μ is the moment mapping for a G-action on W^{2N}, $\xi \in \mathfrak{g}^\star$ is a regular value such that the induced G_ξ-action on $\mu^{-1}(\xi)$ is free and $p : \mu^{-1}(\xi) \to V^{2n}$ is the corresponding principal G_ξ-bundle. The manifold V is endowed with the reduced symplectic structure σ induced from the symplectic structure ω of W.

Assume that i is a Lagrangian embedding and that $M \to L^n$ is the principal G_ξ bundle induced by i.

Then $j \circ f$ is an *isotropic embedding*: it is obviously the embedding of a dimension $n + \dim G_\xi$ submanifold, and

$$(j \circ f)^\star \omega = f^\star j^\star \omega = f^\star p^\star \sigma = p^\star i^\star \sigma = 0.$$

In particular, if $G_\xi = G$, for instance when G is abelian, or, more generally, if ξ is a fixed point of the coadjoint action, then $\dim M = N$ and $j \circ f$ is the inclusion of a *Lagrangian submanifold.*

At first sight, this does not seem very useful: you begin with a Lagrangian submanifold of something complicated like V and you get a Lagrangian submanifold of something simple. In fact, as the reader has probably already understood, it is very hard to construct Lagrangian submanifolds of the standard symplectic vector space $\mathbf{C}^N$. On the other hand, there are a lot of complex algebraic manifolds which are obtained as symplectic reductions... and they have obvious Lagrangian submanifolds: their real parts, from which one can construct lots of examples.

Circle bundles over real projective spaces. — This example appears in [50] but is a consequence of the remark above in the simplest case where the reduced manifold is the complex projective space: $G = S^1$, $W = \mathbf{C}^{n+1}$, $V = \mathbf{P}^n(\mathbf{C})$ and

$L = \mathbf{P}^n(\mathbf{R})$. We then get a Lagrangian submanifold M of $\mathbf{C}^{n+1}$ which is the circle bundle of the complexified canonical bundle over $\mathbf{P}^n(\mathbf{R})$.

1.3.2. Exercise.

1. Show that M is diffeomorphic to $S^1 \times S^n/(z,x) \sim (-z,-x)$, the Lagrangian embedding into $\mathbf{C}^{n+1}$ being given by $(z,x) \mapsto zx$.
2. Write M as a bundle onto S^1 with fibre S^n. Such a bundle can be understood as $([0,1] \times S^n)/(0,x) \sim (1,\varphi(x))$ where φ is a diffeomorphism of the fibre. Show that our M is given by $\varphi =$ antipodal map, and deduce that for n odd, M is diffeomorphic to $S^1 \times S^n$.
3. With the notations of the appendix to chapter I, show that $\|\mu_M\| = n+1$.

Torus bundles over real Hirzebruch surfaces. — The symplectic reduction process described above can be applied in the case of torus bundles over real toric manifolds. Once complex toric manifolds are described by symplectic reduction, as for instance in [9], one gets a lot of examples. For Hirzebruch surfaces, this was done in chapter II.

Here the diagram (1.3.1) is

$$\begin{array}{ccccccc} M^4 & \xhookrightarrow{f} & \mu^{-1}(\xi) & \xhookrightarrow{j} & \mathbf{C}^4 & \xrightarrow{\mu} & \mathbf{R}^2 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} & & & & \\ L^2 & \xhookrightarrow{i} & W_k & & & & \end{array}$$

The real part of the Hirzebruch manifold W_k is (topologically) the torus $S^1 \times S^1$ (the real part of $\mathbf{P}^1(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})$) if k is even and the Klein bottle $\mathbf{P}^2(\mathbf{R})\#\mathbf{P}^2(\mathbf{R})$ ($\cong \widetilde{\mathbf{P}^2}(\mathbf{R})$, the real part of $\widetilde{\mathbf{P}^2}(\mathbf{C})$) if k is odd (recall from chapter II that $\widetilde{\mathbf{P}^2}(\mathbf{C})$ is $\mathbf{P}^2(\mathbf{C})$ blown up at one point). Thus we get Lagrangian embeddings of T^2-bundles over T^2 or over the Klein bottle into $\mathbf{C}^4$.

Principal bundles over real Grassmannians. — Let us look now at an example with a non-abelian group. Apply the construction above to the complex Grassmann manifolds obtained as symplectic reductions as in chapter II. The diagram (1.3.1) is now:

$$\begin{array}{ccccccc} M & \xhookrightarrow{f} & \mu^{-1}(\xi) & \xhookrightarrow{j} & M_{(n+k)\times k}(\mathbf{C}) & \xrightarrow{\mu} & \mathfrak{u}(k)^\star \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} & & & & \\ G_k\left(\mathbf{R}^{n+k}\right) & \xhookrightarrow{i} & G_k\left(\mathbf{C}^{n+k}\right) & & & & \end{array}$$

and M is the $U(k)$-principal bundle over the real Grassmannian associated with the complexified canonical bundle.

1.4. Surgeries

It is very easy to remove the double points of a Lagrangian immersion... provided that you are allowed to change the source manifold! The surgery operation we use is a variant of a classical topological process which removes an $S^k \times D^l$ in a $(k+l)$-manifold and pastes in a $D^{k+1} \times S^{l-1}$ along the boundary $S^k \times S^{l-1}$ (this is an *index* k surgery). In the context of Lagrangian immersions, the construction is directly connected with Lagrangian cobordisms and thus comes from [3], see also [31] and [42]. Consider, following Arnold, the function

$$S_y(q,a) = \frac{a^3}{3} - aQ(q) + ay$$

where $q \in \mathbf{R}^n$, Q is a nondegenerate quadratic form, $a \in \mathbf{R}$ and $y \in \{-1, 1\}$ is a parameter.

Let L_y be the submanifold (a quadric) of $\mathbf{R}^n \times \mathbf{R}$ defined by

$$L_y = \left\{(q,a) \in \mathbf{R}^n \times \mathbf{R} \mid \frac{\partial S_y}{\partial a} = a^2 - Q(q) + y = 0\right\}$$

and let $f_y : L_y \to \mathbf{C}^n$ be defined by

$$f_y(q,a) = q + i\frac{\partial S_y}{\partial q} = q - ia\frac{\partial Q}{\partial q}.$$

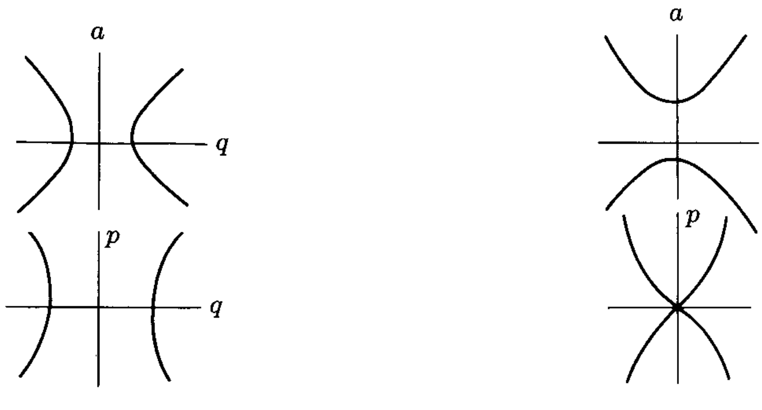
This is a Lagrangian immersion by construction. As the index of the quadratic form varies, we get examples of surgeries of all indices.

1.4.1. Exercise. — Consider the function $S(q,a,y) = \dfrac{a^3}{3} - aQ(q) + ay$ and the $n+1$-manifold with boundary (*elementary cobordism*)

$$L = \left\{(q,a,y) \in \mathbf{R}^n \times \mathbf{R} \times [-1,1] \mid \frac{\partial S}{\partial a} = 0\right\}.$$

Check that $f : (q,a,y) \mapsto (q,y) + i\left(\dfrac{\partial S}{\partial q}, \dfrac{\partial S}{\partial y}\right)$ defines a Lagrangian immersion of L into $\mathbf{C}^{n+1}$ and that the restriction of f to $y = \pm 1$ projects to f_y (thus f is a *Lagrangian cobordism* between f_{-1} and f_1).

Assume now that Q is positive definite. Then f_{-1} is a Lagrangian immersion of the disjoint union of two n-balls, with one double point and f_1 is a Lagrangian embedding of a cylinder $S^{n-1} \times \mathbf{R}$ (see figure 16). It is not hard to embed these local models in order to replace a Lagrangian immersion of a manifold V by another Lagrangian immersion of a manifold V' having one double point less and thus to get an embedding after a finite number of steps. The manifold V' is obtained from V by adding a handle of index 1.



$y = 1$ Figure 16 $y = -1$

In the next application, we construct, following [30], a Lagrangian immersion of the Klein bottle in $\mathbf{C}^2$ which will be shown later to be, in some sense, as close as possible to an embedding (see 2.2.9).

1.4.2. Exercise.

1. Show that there exist two copies S_1 and S_2 of the Whitney sphere (see exercise 1.1.2) which intersect at two points p_1 and p_2, which can be chosen arbitrarly close to each other. Check that these two points have different signs, and that after an index 1 surgery on each of these two points, one gets a Lagrangian immersion j of the Klein bottle into $\mathbf{C}^2$ with two double points.

2. Assume p_1 and p_2 were chosen very close to each other. Consider a small loop $\gamma = \gamma_1 \cup \gamma_2$ where γ_1 joins p_1 to p_2 in S_1 and γ_2 joins p_2 to p_1 in S_2. The loop γ has two corners at p_1 and p_2. The index 1 surgery on p_1 and p_2 transforms γ into a smooth loop γ'. Show that one can assume that γ and γ' lie in $\mathbf{C} \times 0 \subset \mathbf{C}^2$. Compute the Maslov index of γ' by looking at the normal field of lines $X_{\gamma'}$ along γ' defined by $X_{\gamma'}(p) =$ orthogonal complement of $T_p(\gamma')$ in $T_p(j(K^2))$. The Maslov index is the sum of the rotation indices along γ' of $T(\gamma')$ and $X_{\gamma'}$. Deduce that $\mu(j) = a^\star$, where $a^\star$ generates $H^1(K^2, \mathbf{Z})$.

1.5. Wavefronts and Lagrangian surfaces

In this §, we give some examples based on Givental's beautiful paper [22]. The idea is to construct Lagrangian immersions or submanifolds in $\mathbf{R}^4$ by drawing them in some $\mathbf{R}^3$. Here it must be understood that $4 = 2n$ and $3 = n + 1$. Consider canonical coordinates $(q, p) \in \mathbf{R}^{2n}$, add a coordinate z and endow $\mathbf{R}^{2n} \times \mathbf{R}$ with the (contact) 1-form $\alpha = dz - pdq$.

An immersion $f : L \to \mathbf{R}^{2n}$ which is Lagrangian is *exact* (see below § 2.1) if (and only if) there exists a lift $\widetilde{f} : L \to \mathbf{R}^{2n} \times \mathbf{R}$ such that $\widetilde{f}^\star(\alpha) = 0$ (a *Legendre immersion*): choose a primitive h of $f^\star(pdq)$ and set $\widetilde{f}(x) = (f(x), h(x))$.

Note that, as $dz = pdq$ on L, the projection $\pi \circ f : L \to \mathbf{R}^n \times \mathbf{R}$, which forgets the p coordinates, suffices to determine $\tilde{f}$ (and f). This is the *wavefront* of f. For instance the "eye" in figure 17 is the wavefront of the Whitney immersion in $\mathbf{R}^2$ while the "flying saucer", obtained by rotating the eye about the z-axis is that of the Whitney immersion in $\mathbf{R}^4$. Double points of the Lagrangian immersion correspond to points in the front which project to the same $q \in \mathbf{R}^n$ and have parallel tangent spaces.

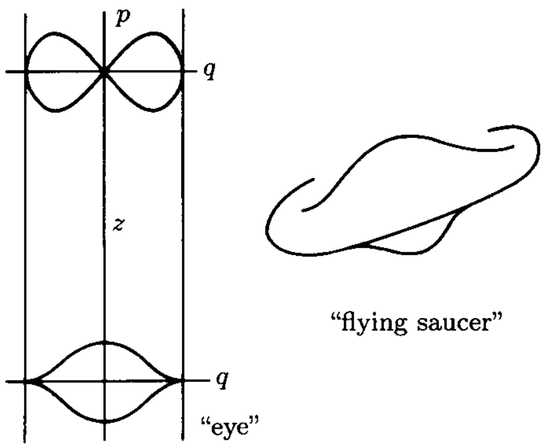


Figure 17

1.5.1. Exercise. — Show that, for a hypersurface in $\mathbf{R}^n \times \mathbf{R}$ to be a wavefront, it is necessary that its tangent space never contains the "vertical" direction (z axis). Interpret the singularities in figure 17 in terms of the Lagrangian immersion. Deduce that the cohomology class dual to the singularities of the wavefront is the Maslov class of the Lagrangian immersion.

If f is a Lagrangian immersion which is not exact, the construction gives a Legendre immersion and a wavefront of the smallest cover of L on which $f^\star(pdq)$ becomes exact. For instance, the zigzag in figure 18 is the wavefront of a Lagrangian immersion of $\mathbf{R}$ into $\mathbf{R}^2$... which is just the standard embedding of S^1. It is still possible to "draw" the Lagrangian submanifold by drawing a significative part (image of a fundamental domain for the deck transformations) of the wavefront (as the bold part in the zigzag).

Rotating the zigzag about the z-axis (figure 18) gives the wavefront of a lagrangian immersion of $\mathbf{R} \times S^1$ in $\mathbf{R}^4$... which defines an embedded Lagrangian torus, so that the "flying saucer with a hole" in figure 18 represents a genuine Lagrangian subtorus of $\mathbf{R}^4$. The following exercises rely on this technique and show how to construct the Givental surfaces according to [8].

1.5.2. Exercise. — Show that figure 19 represents a Lagrangian immersion of an orientable genus g surface with $g-1$ double points, all with the same sign.

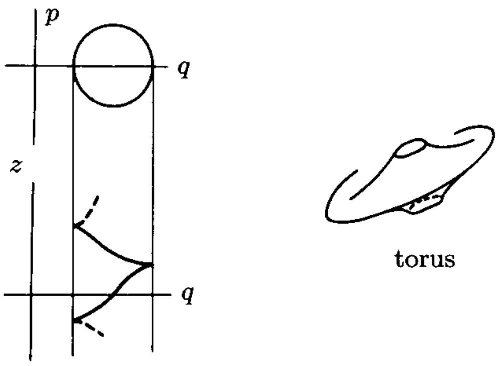


Figure 18

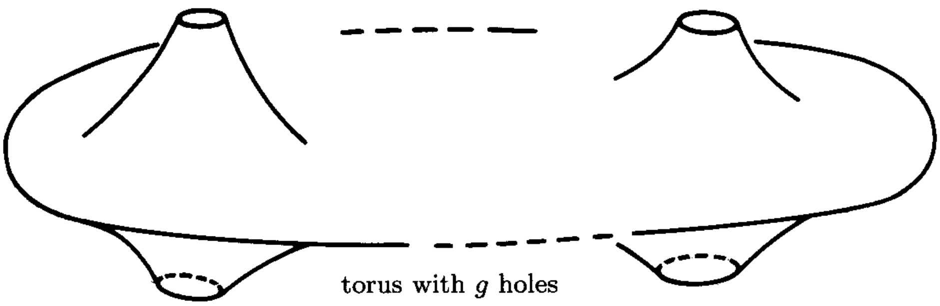


Figure 19

Now consider the Lagrangian submanifold L_y used in § 1.4.

1.5.3. *Exercise (see* [3]*).* — Give formulas for the wavefronts of the immersions $f_y : L_y \to \mathbf{R}^{2n}$. For $n = 1$ and Q definite positive, draw the wavefront of the elementary cobordism.

To solve the next exercises, one must be rather careful with signs. We concentrate on *surfaces*, and use the orientation of $\mathbf{C}^2$ defined by the symplectic form.

1.5.4. *Exercise.* — Show that the double point in the Whitney 2-sphere is positive, and that the double points in figure 19 are all negative.

At this point, it should be obvious that the embedded torus in figure 17 is what one obtains by suppressing the double point of the Whitney immersion.

1.5.5. *Exercise.* — Suppress the $g - 1$ double points of the Lagrangian immersion in figure 19 and show that the result is a Lagrangian embedded non orientable surface V_g. Compute the Euler charcteristic $\chi(V_g)$. Show that one obtains Lagrangian embeddings of all non orientable surfaces with Euler characteristic divisible by 4, except the Klein bottle.

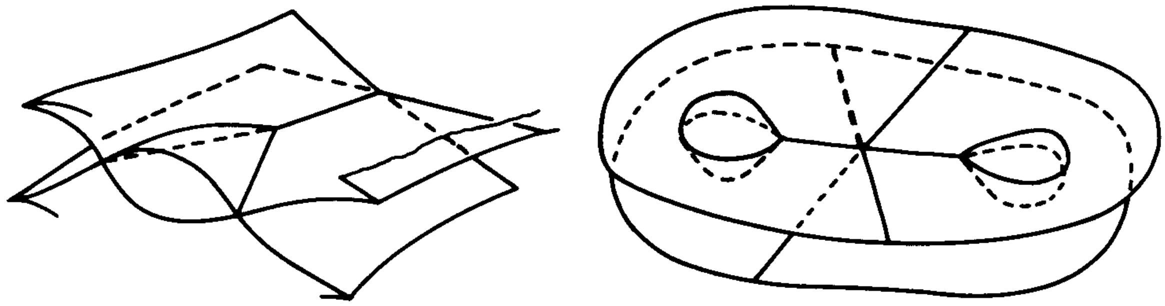


Figure 20

We shall see in 3.2 that the condition that χ is divisible by 4 is necessary. For the Klein bottle, we already did the best we could (up to now) in 1.4.2. Of course, it is easy, from the construction given there and 1.5.3, to get the wavefront of the immersion constructed. A nice definition of Lagrangian non oriented handles, coming from [22], allows to give an alternative construction.

1.5.6. Exercise.

1. Show that the map $\mathbf{R}^2 \to \mathbf{R}^4$ given by

$$(u,v) \longmapsto \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 2uv \\ u^2 - v^2 \\ u \\ v \end{pmatrix}$$

is an exact Lagrangian embedding. Draw its wavefront.

2. Show that the left part of figure 20 represents an embedded non oriented handle. Use it to construct alternative pictures of nonoriented Lagrangian surfaces.

3. Show that the right part of figure 20 is a Lagrangian Klein bottle with two double points. Use exercise 1.5.1 to show that its Maslov class is a generator of $H^1(K;\mathbf{Z})$, thus, according to exercise 1.1.5 the Klein bottle is regularly homotopic to that of exercise 1.4.2.

Remark. — Each half of the Klein bottle in figure 20 is a Möbius strip, obtained by adding a handle to a disc (half of the Whitney sphere). The wavefront of an alternative (exact) Lagrangian Möbius strip is drawn in [3] and is probably the ancestor of all these constructions.

2. Symplectic area and Maslov classes—rigidity in split manifolds

2.1. Symplectic area class, exactness and monotonicity

Let (V,ω) be a symplectic manifold. It is *exact* if the 2-form ω is exact. This is the case for cotangent bundles, for instance, where the symplectic form is the differential of the Liouville form. Suppose that $f : L \to V$ is a Lagrangian immersion in an exact symplectic manifold. Let λ be a primitive of the symplectic form. Of course the 1-form $f^\star\lambda$ is closed on L. Indeed,

$$d(f^\star\lambda) = f^\star d\lambda = f^\star\omega = 0 \; .$$

The class $[f^\star\lambda] \in H^1(L;\mathbf{R})$ is called the *symplectic area class* of the immersion f.

2.1.1. Exercise. — Show that the symplectic area class does not depend on the choice of λ.

2.1.2. Exercise (the Lagrangian suspension). — Let L and X be two n-dimensional manifolds and let

$$\begin{array}{ccc} L\times[0,1] & \xrightarrow{\ f\ } & T^\star X \\ (x,t) & \longmapsto & f_t(x) \end{array}$$

be a regular homotopy of Lagrangian immersions. Show that there exists a Lagrangian immersion of the form

$$\begin{array}{ccc} L\times[0,1] & \xrightarrow{\ F\ } & T^\star X\times\mathbf{C} \\ (x,t) & \longmapsto & (f_t(x), t+ig_t(x)) \end{array}$$

if and only if the symplectic area class $[f_t^\star\lambda_X]$ does not depend on t (Hint: set $h_t(x) = i_{\partial/\partial t}\left(f^\star\lambda_X\right)_{(x,t)} - g_t(x)$ and show that F is Lagrangian if and only if $\frac{\partial}{\partial t}\left(f_t^\star\lambda\right) = dh_t$).

Remark. — This well known result will be very useful in § 2.3. It was essential in the theory of Lagrangian cobordisms and thus comes from[2] [3].

A Lagrangian immersion into an exact symplectic manifold will be called *exact* if its symplectic area class vanishes.

2.1.3. Exercise. — Let α be a 1-form on a closed manifold X. We consider α as a section of the cotangent bundle $T^\star X$, and thus its graph as a submanifold in $T^\star X$. When is it a Lagrangian submanifold?, an exact Lagrangian submanifold?

[2] Actually it is only implicit in [3]; it was written explicitly, in [5] for instance where one can find the solution of the exercise.

2.1.4. Exercise. — Show that the Whitney immersion $\mathcal{W} : S^1 \to \mathbf{C}$ is an exact Lagrangian immersion (see figure 17). Use the Jordan theorem to show that there exists no *exact* Lagrangian embedding of S^1 into $\mathbf{C}$ (see 2.2.5 for the Gromov theorem which generalises that remark to higher dimensions).

2.1.5. Exercise. — Let X, Y and Z be three n-dimensional manifolds. Suppose $f : X \to T^\star Y$ and $g : Y \to T^\star Z$ are two *exact* Lagrangian immersions. Show that their composition (in the sense of the appendix to chapter I) is exact.

In an exact symplectic manifold (V, ω), the integration map

$$\int_\cdot \omega : \pi_2 V \longrightarrow \mathbf{R}$$

is zero. If (V, ω) satisfies this weaker condition, it will be called *weakly exact.*

There is a notion of *weakly exact* Lagrangian submanifold L in any symplectic manifold: one requires that the integration morphism

$$\int_\cdot \omega : \pi_2(V, L) \longrightarrow \mathbf{R}$$

is zero. The following exercises are straightforward but unavoidable.

2.1.6. Exercise. — Check that an exact Lagrangian submanifold in an exact symplectic manifold is weakly exact.

2.1.7. Exercise. — If a connected symplectic manifold contains a weakly exact Lagrangian submanifold, then it is weakly exact.

The next notion relates the two 1-cohomology classes we have. A Lagrangian immersion $f : L \to \mathbf{C}^n$ is called *monotone* if its Maslov class is proportional to the symplectic area class: $\mu = c \cdot \lambda$ for some real number $c > 0$.

2.1.8. Exercise. — Show that the standard Lagrangian torus $L = \{p_j^2 + q_j^2 = 1,\ j = 1, \ldots, n\} \subset \mathbf{C}^n$ is monotone.

2.1.9. Exercise (see [39]). — Show that the Lagrangian immersion $M \to \mathbf{C}^{n+1}$ of exercise 1.3.2 is monotone (Hint: evaluate the Maslov class).

2.2. Rigidity in split manifolds

In this §, we state two main general results of rigidity for both the area and the Maslov classes. Both theorems were originally proved by infinite dimensional techniques: the first one using the pseudo-holomorphic curves approach (this proof is given in § 5) and the second one using calculus of variations for the action functional.

DEFINITION 2.2.1. — *Let (V,ω) be a symplectic manifold without boundary. (V,ω) is geometrically bounded if there exist on V an almost complex structure J and a complete Riemannian metric μ such that the following properties are satisfied:*

1. *J is uniformly tamed by ω, that is: there exist strictly positive constants α and β such that:*
$$\omega(X,JX) \geq \alpha \|X\|_\mu^2$$
and
$$|\omega(X,Y)| \leq \beta \|X\|_\mu \|Y\|_\mu$$
for all $X,Y \in TV$. This obviously implies that ω tames J (that is: $\omega(X,JX) > 0$ for all non zero tangent vector X).

2. *There exist an upper bound for the sectional curvature of (V,μ) and a strictly positive lower bound for the injectivity radius of (V,μ).*

Remark. — If V is compact, condition 1 is equivalent to the fact that ω tames J. Condition 2 is also obviously true in this case, thus any compact symplectic manifold is geometrically bounded.

2.2.2. Other examples.

1. The cotangent bundle $(T^\star M, \omega_M)$ of a closed manifold M, endowed with the standard symplectic structure, is geometrically bounded; in fact, one may choose a ω_M-tame almost complex structure J, homogeneous with respect to uniform dilatations in the fibres, and a metric μ on $T^\star M$ induced by any Riemannian metric on M. This is also true if M is an open subset of a closed manifold, or if M is of finite topological type.

2. Any product of a geometrically bounded manifold (V,ω) with the standard $(\mathbf{C}^n, \omega_0, J_0, \mu_0)$ is geometrically bounded.

DEFINITION 2.2.3. — *Let L be a Lagrangian submanifold of a symplectic manifold (V,ω). The pair (V,L) is geometrically bounded if:*

1. *there exist structures J and μ on V such that (V,ω,J,μ) is geometrically bounded;*

2. *L is properly embedded in V (that is: L has no boundary and is closed as a subset of V);*

3. *the second fundamental form of $L \subset (V,\mu)$ is bounded above, and there exist constants $\delta > 0$ and $K > 0$ such that any pair of points $\ell_1, \ell_2 \in L$ with distance (in V) $d_\mu(\ell_1,\ell_2)$ smaller or equal to δ can be joined by a path in L of μ-length smaller or equal to $K\delta$.*

We shall sometimes abbreviate "geometrically bounded" to "g. bounded". Of course, any compact Lagrangian submanifold without boundary in a g. bounded manifold gives rise to a g. bounded pair. We can now state the main theorem of this chapter. We recall that a Lagrangian distribution on a symplectic manifold (V, ω) is simply a smooth field that associates to each point $x \in V$ a Lagrangian vector subspace of $T_x V$.

THEOREM 2.2.4. — *Let (V, ω) be a product $(V' \times \mathbf{C}, \omega' \oplus \omega_0)$ where (V', ω') is a geometrically bounded weakly exact symplectic manifold. Let L be a geometrically bounded Lagrangian submanifold of V. If the projection of L on the $\mathbf{C}$-factor is bounded, then there exists a loop in L which bounds a holomorphic disc (thus L is not weakly exact).*

Moreover, if $\mathcal{L}'$ is any Lagrangian distribution on V', and if L is as above and monotone, then there exists a loop γ which bounds a holomorphic disc and which is such that

$$1 \leq \|\mu_L(\gamma)\| \leq \dim L + 1$$

(with respect to the Lagrangian distribution $\mathcal{L} = \mathcal{L}' \times \mathbf{R}$ of V). Thus $1 \leq \|\mu_L\| \leq \dim L + 1$.

The proof of this theorem using pseudoholomorphic techniques is postponed to § 5. This theorem shows that the requirement of being embedded impose severe constraints on both the symplectic area and the Maslov classes of a Lagrangian submanifold. This is in sharp contrast with an immersed Lagrangian submanifold, whose behaviour is entirely characterised by the h-principle and thus much more flexible. The formulation of theorem 2.2.4 is quite general and has many useful corollaries. The following was conjectured by Arnold in the 60's and proved by Gromov in 1985 and in [24]:

COROLLARY 2.2.5. — *There exists no closed exact Lagrangian submanifold in $(\mathbf{C}^n, \omega_0)$.* □

Remark. — As a consequence, there is no closed simply connected Lagrangian submanifold in $\mathbf{C}^n$. For instance, for $n \geq 2$, the n-sphere has *no* Lagrangian embedding into $\mathbf{C}^n$. If $n \neq 3$, this is more or less[3] easy to prove using standard algebraic topology methods, but for $n = 3$, corollary 2.2.5 is needed (see § 3.2 for these questions).

Consider now the constraints imposed on the Maslov class of a Lagrangian embedding in $\mathbf{C}^n$. The Maslov class μ_f of any Lagrangian immersion $f : L \to \mathbf{C}^n$ is clearly an invariant of Lagrangian regular homotopy. It turns out, as theorem 2.2.4 shows, that the invariant $\|\mu_f\|$ can distinguish those Lagrangian regular homotopy classes that may contain embeddings. For $n = 1$, we have already seen in the

[3]Depending on the actual value of n.

appendix to chapter I that any even integer is the Maslov class of a Lagrangian immersion $f : S^1 \to \mathbf{C}$ but that Lagrangian *embeddings* have $\|\mu_f\| = 2$. For $n \geq 2$, the following two results are known, the first one being an obvious corollary of theorem 2.2.4:

COROLLARY 2.2.6 ([39]). — *Let L be a closed monotone Lagrangian submanifold of $\mathbf{C}^n$. Then $1 \leq \|\mu_L\| \leq n+1$.* □

The second result was proved by Viterbo using "soft" techniques... in this case infinite dimensional Hamiltonian systems (once again, soft may be very hard!)

THEOREM 2.2.7 (Maslov class rigidity [49]). — *Let L be any closed manifold admitting a Riemannian metric of non negative sectional curvature (a torus for instance). Then for every Lagrangian embedding $f : L \to \mathbf{C}^n$, $1 \leq \|\mu_f\| \leq n+1$.*

Remarks. — There is no known example of a closed Lagrangian submanifold $L \subset \mathbf{C}^n$ that does not satisfy the inequality $1 \leq \|\mu_L\| \leq n+1$.

Corollary 2.2.6 is almost optimal in the following sense: for every pair of integers $2 \leq k \leq n$, there exists a monotone Lagrangian embedding f of a closed manifold into $\mathbf{C}^n$ with $\|\mu_f\| = k$ (see [39]).

Surprisingly enough, there exist no rigidity theorems for higher[4] Lagrangian characteristic classes. Using the h-principle, one can say a lot about the realisability of such or such a class by Lagrangian immersions (see [6]), but nothing is known (at least to us) in the case of embeddings.

2.2.8. Example. — Note that theorem 2.2.4 implies that a simply connected closed manifold L admits no Lagrangian embedding in $(V' \times \mathbf{C}, \omega' \oplus \omega_0)$ if V' is weakly exact. For such an embedding, there would exist a disc D with boundary in L, of strictly positive area. Let D' be a disc in L bounded by ∂D. It would have a non zero symplectic area because V is weakly exact, which is absurd. Does there exist a non weakly exact V' such that $V' \times \mathbf{C}$ contains a closed simply connected Lagrangian submanifold? Here is an example: let ω be an area form on S^2 such that the pull-back of ω by the Hopf fibration $f : S^3(\subset \mathbf{C}^2) \to S^2$ is equal to the restriction of the standard form ω_0 on $\mathbf{C}^2$. Then the graph of $f : S^3 \to S^2$ is a Lagrangian submanifold of $(\mathbf{C}^2 \times S^2, \omega_0 \oplus -\omega)$ since $(gr(f))^\star(\omega_0 \oplus -\omega) = \omega_0 - \omega_0 = 0$.

Now let us generalise the preceeding example: let m, k be two non zero integers, X^{2m+2k} and Z^{2k} be any geometrically bounded symplectic manifolds, with X weakly exact and split, $Y^{m+2k} \subset X$ any geometrically bounded simply connected submanifold whose projection on the $\mathbf{C}$-factor is bounded.

If Z is weakly exact, there is no symplectic map $f : Y \to Z$ (that is such that $f^\star(\omega_0) = \omega_{|Y}$). In particular, one may replace the condition "weakly exact" on Z by

[4]Actually, pseudo-holomorphic curves methods are typically 1 and/or 2-dimensional, and they are the only tool we have up to now for proving rigidity results.

the stronger condition "$\pi_2(Z) = 0$" and obtains: there is no symplectic map from Y to Z if $\pi_1(Y) = 0$ and $\pi_2(Z) = 0$!

Here is a consequence: let Y be a closed coisotropic submanifold of a geometrically bounded split weakly exact manifold X ($\mathbf{R}^{2n}$ for instance). Suppose that the characteristic foliation of Y is globally integrable (that is: Y admits a reduction). Then $\pi_1(Y) = 0$ implies $\pi_1(K) \neq 0$, where K is a leaf of the foliation.

As a final application, the next exercise will show that any (*if* any!) Lagrangian embedding of the Klein bottle must be regularly homotopic to the Lagrangian immersions of K^2 constructed above (1.4.2 and 1.5.6).

2.2.9. *Exercise (see* [30]*).*

1. Suppose that $i : K^2 \longrightarrow \mathbf{C}^2$ is a Lagrangian embedding, and let $\mu(i) = xa^\star$ (where $a^\star$ generates $H^1(K^2; \mathbf{Z})$), $x \in \mathbf{Z}$. Use the Weinstein tubular neighbourhood theorem and the summation formula (appendix to chapter I) to show that i induces a Lagrangian embedding of T^2 in $\mathbf{C}^2$ of Maslov class $2xa_1^\star$. Deduce, using Viterbo's theorem 2.2.7, that $x = \pm 1$, that is : $\mu(i) = \pm a^\star$.

2. Consider the Lagrangian immersion j constructed in 1.4.2. Let $f : T^2 \longrightarrow K^2$ be the orientation twofold covering of K^2, and let $\{a_1, b_1\}$ be a basis of $H_1(T^2; \mathbf{Z})$ with $f_\star(a_1) = 2a$ and $f_\star(b_1) = b$ where b is torsion and a is a torsion-free primitive class. Let $\{a_1^\star, b_1^\star\}$ be the dual basis and $\theta \in H^1_{dR}(T^2)$ the 1-form corresponding to $b_1^\star$ by the de Rham isomorphism. Note that the graph $gr(\theta)$ of $\theta : T^2 \longrightarrow T^\star T^2$ is Lagrangian submanifold and that the composition
$$h : T^2 \xrightarrow{gr(\theta)} T^\star T^2 \xrightarrow{T^\star f} T^\star K^2$$
is a Lagrangian embedding which is a lifting of f. Show that $f^\star(H^1(K^2; \mathbf{Z})) \subset 2H^1(T^2; \mathbf{Z})$ and that $\mu(h) = 0$.

3. Using the result of exercise 1.1.5, prove that the Lagrangian Klein bottles j constructed in 1.4.2 and 1.5.6 are regularly homotopic to i.

2.3. Lagrangian intersections

In the sixties, Arnold stated a set of conjectures in global symplectic geometry, based on Poincaré's last geometric theorem (the Poincaré-Birkhoff theorem), a theorem about fixed points of certain diffeomorphisms of the annulus which preserve the area: that is, about fixed points of some symplectic diffeomorphisms. Using the classical "diagonal trick" explained in chapter I, one easily converts a question about fixed points of symplectic diffeomorphisms into a problem about the intersection of two Lagrangian submanifolds (a very good reference on these conjectures and their mutual relations is the survey by Chaperon [19]). The general conjectures are now theorems of Gromov (here theorem 2.3.6 and corollaries 2.3.9 and 2.3.7). Following Gromov, we give proofs of these results, based on theorem 2.2.4.

The main result in this § asserts that you cannot disconnect certain Lagrangian submanifolds from themselves by deformation. We first need to explain which deformations are allowed.

Exact Lagrangian isotopies. — Let $L \subset (V, \omega)$ be a Lagrangian submanifold, and let $f_t : L \to V$ be an isotopy (among Lagrangian embeddings): f_t is a Lagrangian embedding for all t and $f_0 = \mathrm{Id}$. We shall also consider all the f_t's together as a map

$$f : L \times [0,1] \longrightarrow V.$$

We say that f is an *exact Lagrangian isotopy* if the 1-form

$$i_{\partial/\partial t} f^\star \omega$$

on $L \times [0,1]$ is exact.

2.3.1. Exercise. — Show that f is an exact Lagrangian isotopy if and only if, for any loop α in L and any t, the area of $f(\alpha \times [0,t])$ is zero.

2.3.2. Exercise. — In the neighbourhood of a Lagrangian submanifold, we know that the symplectic form is exact, being isomorphic to that of a cotangent bundle. Show that f is an exact Lagrangian isotopy if and only if the symplectic area class of f_t does not depend on t.

2.3.3. Exercise. — Assume L is compact. Show that a Lagrangian isotopy is exact if and only if, for any given t, the $f_s(L)$ (s close to t) are graphs of *exact* 1-forms dh_s in any symplectic tubular neighbourhood of $f_t(L)$ (isomorphic to $T^\star L$).

Hamiltonian isotopies. — Recall that a diffeomorphism φ of (V, ω) is Hamiltonian if it is of the form $\varphi = \varphi_1$, φ_t being a Hamiltonian isotopy, i.e. the flow of a (time-dependent) Hamiltonian vector field.

If (V, ω) is any symplectic manifold, we endow $V \times V$ with the symplectic form $\omega \otimes -\omega$ so that the diagonal Δ_V is a Lagrangian submanifold.

2.3.4. Exercise. — Show that an isotopy φ_t of V is a Hamiltonian isotopy if and only if $f_t : x \mapsto (x, \varphi_t(x))$ is an exact Lagrangian isotopy of V into $V \times V$.

2.3.5. Exercise (ambient isotopies, see e.g. [19]). — Let L be a compact submanifold of the symplectic manifold (V, ω). Let f_t be an isotopy of L in V. Show that the following two properties are equivalent:

- f_t is an exact Lagrangian isotopy of L,
- L is a Lagrangian submanifold (i.e. f_0 is a Lagrangian embedding) and there exists a compactly supported Hamiltonian isotopy φ_t of (V, ω) such that $f_t = \varphi_t \circ f_0$ for any $t \in [0,1]$.

Now the main results of this § are the following:

THEOREM 2.3.6 ([24]). — *Let (V, ω) be a g. bounded weakly exact symplectic manifold. Let L be a closed weakly exact Lagrangian submanifold and let $f_t : L \to (V, \omega)$ be a Hamiltonian isotopy. Then $f_1(L)$ meets L.*

COROLLARY 2.3.7. — *Let (V, ω) be a weakly exact compact symplectic manifold. Then any Hamiltonian diffeomorphism of V has a fixed point.*

Following Gromov, we prove theorem 2.3.6 by using the isotopy f to construct a Lagrangian immersion of $L \times S^1$ into $(V \times \mathbf{C}, \omega \oplus \omega_0)$, and relate the double points of the later with the points of $L \cap f_1(L)$; we then apply theorem 2.2.4 to conclude. The construction is based on an elegant trick known as the "figure eight trick". Notice first that, according to the Lagrangian suspension process described in exercise 2.1.2, we have a Lagrangian immersion

$$\begin{array}{ccc} L \times [0,1] & \xrightarrow{F} & V \times \mathbf{C} \\ (x,t) & \longmapsto & (f_t(x), t + ig(x,t)) . \end{array}$$

Now, we use the Whitney immersion (the "figure eight"). Consider a symplectic immersion $\varphi : S^1 \times [a,b] \to \mathbf{C}$ giving a tubular neighbourhood (figure 21) of the Whitney sphere: we use an interval $[a,b] = g(L \times [0,1])$ where we can assume that $a < 0 < b$ and consider $S^1 \times [a,b]$ as a part of the (trivial) cotangent bundle $T^\star S^1$.

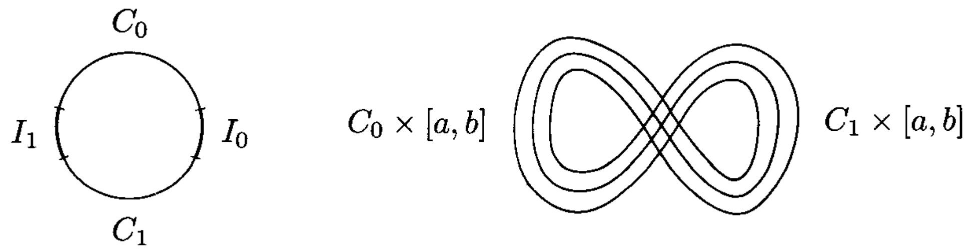


Figure 21

We decompose S^1 as a union of four closed intervals $I_0 \cup C_0 \cup I_1 \cup C_1$ as in figure 21: $\varphi\,|_{(C_0 \cup C_1) \times [a,b]}$ is an embedding. Now, we describe the desired map

$$L \times S^1 \xrightarrow{G} V \times \mathbf{C}$$

on each of the four intervals. To keep notation simple, we rescale and assume that $I_j = C_k = [0,1]$. Then

- On $L \times I_0$, $G(x,t) = (f_0(x), \varphi(t, g_0(x)))$: L does not move.
- On $L \times C_0$, $G(x,t) = (f_t(x), \varphi(t, g(x,t)))$. Here, φ has no double point, so we allow L to move.

- On $L \times I_1$, the situation is the same as it was on $L \times I_0$, and $G(x,t) = (f_1(x), \varphi(t, g(x,1)))$.
- On $L \times C_1$, we just have to come back: $G(x,t) = (f_{1-t}(x), \varphi(1-t, g(x, 1-t)))$.

The smoothing of these maps is left as an exercise to the reader:

2.3.8. *Exercise.* — Construct a smooth map $\rho : S^1 \to [0,1]$ such that

$$G(x,z) = \left(f(x,\rho(z)), \varphi\left(z, h_{\rho(z)}(x)\right)\right)$$

is a smooth map $L \times S^1 \to \mathbf{C}$ with the above properties (the formula comes from [24]).

Now, by its very definition, G is a Lagrangian immersion and its double points in $V \times \mathbf{C}$ are the points in

$$L \cap f_1(L) \times 0 \subset V \times 0 \subset V \times \mathbf{C}.$$

If $L \cap f_1(L) = \emptyset$ we thus have a Lagrangian embedding, which is weakly exact because L was (and because the Whitney map is exact). This concludes the proof. □

Remark. — The same proof gives a more general result: we may assume that L is non compact provided the isotopy has compact support; of course, the conclusion is that L and $f_1(L)$ meet inside the support.

We apply the previous result to the Lagrangian intersection problem in a cotangent bundle: any *exact* Lagrangian submanifold must meet the zero section:

COROLLARY 2.3.9. — *Let L be a Lagrangian submanifold in the cotangent bundle $T^\star X$ of a closed manifold. If L is exact, it intersects the zero section.*

Proof. — Consider the isotopy F_t given by scalar multiplication in the fibres $(q,p) \mapsto (q, e^t p)$. If L misses the zero section, $F_t(L) \cap L = \emptyset$ for t large enough. Now $F_t^\star \omega_X = e^t \omega_X$ thus F_t induces an isotopy of L, which is Lagrangian exact if L is exact. Now the symplectic form ω_X itself is exact thus L is weakly exact (see 2.1.6), and we get a contradiction to theorem 2.2.4. □

Proof of corollary 2.3.7. — We use the "diagonal trick" (see chapter I) which converts results on Lagrangian submanifolds into results on fixed points of symplectic diffeomorphisms. Let φ_t be the Hamiltonian isotopy, $\varphi = \varphi_1$ the diffeomorphism. Consider

$$F_t = \operatorname{Id} \times \varphi_t : V \longrightarrow V \times V,$$

and endow $V \times V$ with the symplectic form $\omega \oplus (-\omega)$ so that the diagonal, which is also $F_0(V)$, is a Lagrangian submanifold. The symplectic manifold (V, ω) is weakly exact if and only if its Lagrangian submanifold Δ_V is. We may apply 2.3.6 thus getting points in $\Delta_V \cap F_1(V)$. Of course, they correspond to fixed points of φ. □

Remark. — One can relax the compactness condition, replacing it by compactness of the support of the isotopy.

3. Soft and hard Lagrangian obstructions to Lagrangian embeddings in $\mathbf{C}^n$

3.1. Lagrangian and totally real embeddings

We begin with a straightforward observation.

PROPOSITION 3.1.1. — *In order for there to exist a Lagrangian embedding of V^n into $\mathbf{C}^n$, it is necessary that there exists, in the same regular homotopy class of (ordinary) immersions, both a Lagrangian immersion and an (ordinary) embedding.*

The example of S^3 (see below and § 2.2) shows that the converse is false. However, it becomes true if one relaxes somewhat the constraint of being Lagrangian. The "soft" version of a Lagrangian embedding is a *totally real*[5] embedding: instead of requiring that

$$T_x f\,(T_x L) \perp iT_x f\,(T_x L)$$

you just insist that those two subspaces be *transversal*. This is equivalent to requiring that $T_x f\,(T_x L)$ contain no non trivial *complex* subspace, or equivalently no complex line, hence the name. This really is a softer notion: there exists an analogue of the Whitney lemma (elimination of double points) which gives the converse to 3.1.1.

THEOREM 3.1.2 ([23]). — *Any Lagrangian or totally real immersion which is regularly homotopic to an embedding is regularly homotopic to a totally real embedding, among totally real immersions.* □

Note that there is no homotopic distinction between an (exact) Lagrangian immersion and a totally real immersion (this is an obvious consequence of the Gromov-Lees theorem 1.1.3).

[5]Totally real embeddings are interesting in their own right, especially in complex analysis (see e.g. [46]).

3.2. Soft obstructions

Now, if $f : V^n \to \mathbf{C}^n$ is any immersion, there is a simple and classical (due to Whitney) obstruction to the existence of an embedding in the same regular homotopy class: you just count appropriately (if V is oriented and n is even, any transversal double point carries a natural sign) the number of double points (assumed to be transversal). You get a number

$$d(f) \in \begin{cases} \mathbf{Z} & \text{if } n \text{ is even and } V \text{ orientable} \\ \mathbf{Z}/2 & \text{otherwise.} \end{cases}$$

It is classical (and very easy to prove in the orientable case) that

PROPOSITION 3.2.1. — *Let $n = 2k$ be even, and let $f : V^n \to \mathbf{C}^n$ be a Lagrangian immersion with normal crossings. Then*

$$d(f) = (-1)^{k+1} \frac{\chi(V)}{2}$$

(equality mod 2 *if V is non orientable).*

As usual, χ is the Euler characteristic[6]. The general result is due to Whitney, at least in the oriented case, and relates $d(f)$ to the Euler characteristic of the *normal* bundle of f. As f is Lagrangian, this is isomorphic to the tangent bundle (the sign comes from the consideration of orientations). Hence 3.2.1 (for the non-orientable case, see [7] for a complete proof). □

3.2.2. Exercise. — Carefully taking orientations into account, check that the formula in 3.2.1 is compatible with the suppression of double points allowed by the surgery operation of § 1.4.

3.2.3. Exercise. — Let V be an oriented surface of genus g. What is the minimal number of double points of a Lagrangian immersion of V? Compare with 1.5.2.

The situation in odd dimensions ($n = 2k + 1$) is more complicated. Very often, the Kervaire semi-characteristic

$$\widehat{\chi}_{\mathbf{Z}/2}(V) = \sum_{i=0}^{k} \dim H^i(V; \mathbf{Z}/2) \mod 2$$

plays the same role as χ, but this is not so easy to prove: "soft" obstructions may come from... hard algebraic topology. We shall not give details here—after all, this

[6]Note that $\chi(V)$ is automatically even when a Lagrangian immersion exists: it is plus or minus the Euler characteristic of the normal bundle, and normal bundles of manifolds are known to have rather few Stiefel-Whitney classes, due to Wu formulas.

is a book on pseudo-holomorphic curves' methods. The reader is invited to consult [7] for precise statements and proofs. The easy-to-state consequences are grouped in the following theorem.

THEOREM 3.2.4. — 1. *A closed connected n-manifold V admits totally real embeddings if and only if it is U-parallelisable and*

(a) *if n is even, $\chi(V) = 0$ (equality* mod 4 *if V is nonorientable)*

(b) *if $n \equiv 1 \mod 4$ and V is orientable, $\widehat{\chi}_{\mathbf{Z}/2}(V) = 0$.*

2. *If V is a non parallelisable stably parallelisable manifold, it has no totally real embedding.* □

The applications of this theorem to the *non* existence of Lagrangian embeddings are obvious.

As a conclusion to this §, let us discuss in some detail the case of the sphere S^n. If n is even, 3.2.1 shows that there is no totally real, hence no Lagrangian embedding. If $n > 1$ is odd, it is a classical and basic result in differential topology (Smale-Hirsch theory of immersions) that there are exactly two regular homotopy classes of immersions $S^n \to \mathbf{R}^{2n}$, that of the standard embedding $S^n \subset \mathbf{R}^{n+1} \subset \mathbf{R}^{2n}$ and that of $\mathcal{W}$. Assume now that there exists a Lagrangian embedding. Then the standard embedding must be regularly homotopic to a Lagrangian immersion, in particular its normal bundle (which obviously is trivial) must be isomorphic to TS^n... Due to the non parallelisability of almost all the spheres, we conclude that $n = 1$ (we know S^1 has Lagrangian embeddings into $\mathbf{C}$), 3 or 7.

If $n = 3$ or 7, one looks carefully at the forgetful map from the space of Lagrangian immersions to the set of ordinary immersions, which turns out, homotopically, to be a map $\pi_n(U(n)) \to \pi_n(O/O(n))$, that is $\mathbf{Z} \to \mathbf{Z}/2$, induced by the natural inclusions and projection[7] $U(n) \subset O(2n) \subset O = O(\infty) \to O/O(n)$. This map is onto for $n = 3$, but not for $n = 7$. In other words, any immersion is homotopic to a Lagrangian immersion for $n = 3$, but not for $n = 7$. Due to the existence of the Lagrangian Whitney immersion $\mathcal{W}$ (see 1.1.2), S^7 has *no* Lagrangian (or totally real) embedding in $\mathbf{C}^7$.

We can conclude that S^3 has a totally real embedding[8]. But we need all the Gromov machinery (see § 2.2) to conclude that S^3 has no Lagrangian embedding in standard[9] $\mathbf{C}^3$!

Remark. — The same arguments show that all orientable closed 3-dimensional manifolds have totally real embeddings into $\mathbf{C}^3$.

[7] According to the Smale-Hirsch theory, the h-principle for immersions, $\pi_n(O/O(n))$ is in one-to-one correspondence with the set of regular homotopy classes of immersions $S^n \to \mathbf{R}^{2n}$.

[8] An explicit one, i.e. formulas, is given in [1].

[9] But there are symplectic structures on $\mathbf{C}^3$ in which you can find a Lagrangian S^3, see [37].

3.3. Hard obstructions to Lagrangian embeddings

We just saw that there is *no* soft obstruction to the existence of Lagrangian embeddings of closed orientable 3-manifolds into $\mathbf{C}^3$. Indeed, each such manifold admits a totally real embedding into $\mathbf{C}^3$ while the algebraic topology cannot distinguish between the totally real and Lagrangian cases. However the rigidity results of § 2 allow us to produce further obstructions which work in dimension 3 as well.

Lagrangian obstructions via the symplectic area class. — According to 2.2.5 every closed Lagrangian submanifold of $\mathbf{C}^n$ has a non-vanishing symplectic area class, and hence a non-trivial first cohomology group. Therefore we have the following propositon:

PROPOSITION 3.3.1. — *Let L^n be a closed manifold with $H^1(L, \mathbf{Z}) = 0$. Then L does not admit a Lagrangian embedding into $\mathbf{C}^n$.* □

For instance, as already discussed, the 3-dimensional sphere S^3 does not admit a Lagrangian embedding into $\mathbf{C}^3$. This is also the case for the homogeneous space $SU(n)/SO(n)$ (see 1.2.2).

Lagrangian obstructions via the Maslov class. — In the present section we describe a sequence of *flat* manifolds K^n $(n \geq 3)$ which do not admit a Lagrangian embedding into $\mathbf{C}^n$ (see [41]). These manifolds can be considered as "multidimensional Klein bottles". Our approach is based on the Maslov class rigidity for flat manifolds (see 2.2.7).

Let T^n be the torus with coordinates $x_1, \ldots, x_n$ (mod 1). Consider the map

$$\alpha : (x_1, \ldots, x_n) \mapsto (x_1 + \frac{1}{2n-2}, x_3, \ldots, x_n, -x_2).$$

3.3.2. Exercise. — Show that α generates a group, say G, of transformations of the torus which is isomorphic to $\mathbf{Z}/(2n-2)$ and acts freely. Denote by K^n the quotient T^n/G and by $\theta : T^n \to K^n$ the natural projection. Show that $H^1(K^n; \mathbf{Z}) \cong \mathbf{Z}$. Show that K^3 is orientable.

Our main result in this § is the following theorem:

THEOREM 3.3.3. — *The manifold K^n does not admit a Lagrangian embedding into $\mathbf{C}^n$.*

Before proving theorem 3.3.3, let us notice that at least the case of K^3 cannot be treated with the methods described above. Indeed, K^3 is orientable and hence there are no obstructions of an algebraic topological nature. Moreover, non-vanishing of the symplectic area class cannot be applied because $H^1(K^3) \neq 0$ as shown in 3.3.2.

Suppose that a Lagrangian embedding, say f, does exist. Let $\tilde{\theta} : T^\star T^n \to T^\star K^n$ be the covering induced by θ, and let $\pi : T^\star K^n \to K^n$ be the natural projection.

3.3.4. Exercise. — Show that there exists $p_0 \in \mathbf{R}^n$ such that the restriction of $\tilde{\theta}$ to a torus $L = \{p = p_0\} \subset T^\star T^n$ is *an embedding* .

Denote this restriction by g. Note that g is a Lagrangian embedding since L is a Lagrangian submanifold. Let

$$A : H^1(K^n; \mathbf{Z}) \to H^1(L; \mathbf{Z})$$

be the homomorphism induced by πg.

3.3.5. Exercise. — Show that A *expands* in the sense that: $\|Aa\| = (2n - 2)\|a\|$ for all $a \in H^1(K^n, \mathbf{Z})$.

3.3.6. Exercise. — Show that the Maslov class of g vanishes (Hint: use the fact that L is *transversal* to the fibres of the bundle $T^\star T^n$).

We are now in a position to finish the proof. Let $h : L \to \mathbf{C}^n$ be a composition (in the sense of the appendix to chapter I) of f and g. Then the summation formula implies that $\mu_h = A\mu_f$. Since K^n and L are *flat*, it follows from 2.2.7 that $\|\mu_h\| \leq n + 1$, and $\|\mu_f\| \geq 1$. On the other hand 3.3.5 implies that $\|\mu_h\| = (2n - 2)\|\mu_f\|$. Therefore we obtain that $n + 1 \geq 2n - 2$ which is impossible for $n > 3$. In order to treat the case $n = 3$ recall that the manifold K^3 is orientable and therefore in this case $\|\mu_f\| \geq 2$ (we saw in the appendix to chapter I that $\|\mu_f\|$ is even). Previous arguments give us that

$$n + 1 \geq (2n - 2) \cdot 2,$$

and we get a contradiction with $n = 3$. □

4. Rigidity in cotangent bundles and applications to mechanics

In the present section we establish constraints on various invariants of Lagrangian submanifolds of cotangent bundles. A specific feature of this problem is that non-linear methods, which allow us to attack the rigidity phenomenon in linear spaces and split manifolds, do not generally work for cotangent bundles. Fortunately, in many important cases one can get around this difficulty with the help of quite elementary (but clever!) constructions which reduce rigidity questions in cotangent bundles to rigidity questions in split manifolds.

We start our discussion with embedded Lagrangian tori in $T^\star T^n$ (see § 4.1). There are at least two reasons for this. Firstly, in this case one can clearly see non-trivial interrelations between symplectic and topologic invariants of Lagrangian embeddings. Secondly, Lagrangian tori arise in various qualitative problems of mechanics (see § 4.3 below). In order to treat the problem we need certain techniques which are described below in a more general context. In § 4.2 we give an account of Maslov class rigidity in cotangent bundles. We use a simple algebraic formalism in which a crucial role is played by the summation formula for the composition of Lagrangian embeddings.

We then give applications to mechanics: Hamiltonian mechanics provides us with a wide variety of different types of dynamic behaviour. However one can distinguish a class of systems with the simplest dynamics, the so called integrable systems, whose phase spaces (up to measure zero) are foliated by invariant Lagrangian tori carrying quasi periodic motion. An important problem is to understand what happens to these tori when perturbed. In §§4.3 and 4.4 we discuss several recent results in this direction in the context of classical mechanical systems on $T^\star T^n$.

4.1. Lagrangian tori in $T^\star T^n$

In the following theorem we sum up our knowledge on embedded Lagrangian tori in $T^\star T^n$.

THEOREM 4.1.1. — *Let $L \subset T^\star T^n$ be an embedded Lagrangian torus, and let $A : H^1(T^n;\mathbf{R}) \to H^1(L;\mathbf{R})$ be the homomorphism induced by the restriction of the natural projection to L. Then*

1. *L is either homologous to the zero section with suitable orientation or homologous to zero;*
2. *in the first case the Maslov class of L vanishes;*
3. *in the second case the symplectic area class λ of L does not vanish and the Maslov class μ satisfies $2 \leq \|\mu\| \leq n+1$. Moreover, neither λ nor μ lies in the image of A.*

Comments and proof. — The first statement is due to Arnold (see [4]) who realised that it is closely related to the Lagrangian intersections problem (see the discussion in §2.3). We present his argument below. Certain generalisations to exact Lagrangian submanifolds of cotangent bundles can be found in [31]. Statement 2 was proved in [49] (see also [39], [31] and [41] for alternative approaches and generalisations). Below we use a method invented by Lalonde and Sikorav in [31]. To our knowledge, the last statement never appeared in the literature before (except the Maslov class computations in the case $n = 2$, see [39]).

Proof of statement 1. — We use an indirect argument. Assume that L is homologous to a non-trivial multiple of the zero section.

4.1.2. Exercise. — Show that this assumption implies that there exists a non-trivial covering $T^\star T^n \to T^\star T^n$ such that all lifts of L are disjoint Lagrangian tori homologous to the zero section.

Let L' and L'' be two such lifts. Then they differ by a deck transformation, say Q, which in canonical coordinates (p,q) can be written $Q(p,q) = (p, q + k \bmod 1)$, for some constant vector k.

4.1.3. Exercise. — Show that Q is a time-one map of a Hamiltonian flow on T^*T^n (Hint: choose the Hamiltonian function to be linear in p).

We now have two disjoint Lagrangian tori L' and L'' in T^*T^n homologous to the zero section, and a time-one map Q of a Hamiltonian flow such that $Q(L') \cap L'' = \emptyset$. But this is impossible due to the theorem on Lagrangian intersections (see § 2.3 above). This contradiction proves the desired statement. □

Proof of statement 2. — Note that if L is homologous to the zero section then A is an isomorphism. Hence the statement will follow from the first assertion in 4.2.9 below. □

Proof of statement 3. — Note that if L is homologous to zero then A has a non-trivial kernel. All assertions about the Maslov class now follow from 4.2.9 below.

It remains to show that λ does not lie in the image of A (of course this will imply that $\lambda \neq 0$). Obviously it is sufficient to find a loop, say γ, on L which bounds a disc with positive symplectic area.

4.1.4. Exercise. — Use our assumption on L in order to show that there exists a covering $\tau : V = T^*(T^{n-1} \times \mathbf{R}^1) \to T^*T^n$ such that every lift of L is compact.

Denote such a lift by L'. Note that V is symplectomorphic to $T^*T^n \times \mathbf{C}$ and hence is a split manifold. Therefore 2.2.4 gives us a loop, say γ' on L' which bounds a disc of positive symplectic area. Obviously the loop $\tau(\gamma')$ has the desired properties. This completes the proof. □

4.2. The Maslov class rigidity in cotangent bundles

As we have already mentioned, a basic tool for understanding Maslov class rigidity in cotangent bundles is the summation formula for the composition of Lagrangian embeddings (see the appendix to chapter I). We will use (essentially) ideas developed in [31]: an iteration method based on the summation formula for Maslov classes. Let us fix some notation. All manifolds considered below are connected and without boundary. We write π_X for the natural projection of the cotangent bundle T^*X. Given two manifolds X and Y and a homomorphism $A : H^1(X;\mathbf{Z}) \to H^1(Y;\mathbf{Z})$, we denote by $F(Y,X;A)$ the set of all Lagrangian embeddings $f : Y \to T^*X$ with $(\pi_X f)^* = A$. The main objects of interest are the sets $I(Y,X;A)$ consisting of all elements of $H^1(Y, \mathbf{Z})$ which can be represented by the Maslov class of an embedding from $F(Y, X; A)$. An important algebraic invariant of such a set is the value

$$\|I\| = \sup_{v \in I} \|v\| \in [0; +\infty].$$

We set $\|\emptyset\| = -1$.

Reparametrisations. — Let us describe the behaviour of the sets $I(Y, X; A)$ under diffeomorphisms of manifolds Y and X. We denote by D_X the group of all automorphisms of $H^1(X, \mathbf{Z})$ generated by diffeomorphisms of X.

PROPOSITION 4.2.1. — *For all $C \in D_Y$, $C\left(I(Y, X; A)\right) = I(Y, X; C(A))$.*

Proof. — Note that every diffeomorphism φ of Y defines a bijective map

$$F(Y, X; A) \to F(Y, X; \varphi^\star A)$$

which takes a Lagrangian embedding f into its composition with φ. Moreover, $\mu_{f\varphi} = \varphi^\star \mu_f$. The proposition follows. □

PROPOSITION 4.2.2. — *For all $C \in D_X$, $I(Y, X; AC) = I(Y, X; A)$.*

Proof. — Note that every diffeomorphism ψ of X admits a unique lift to a (fibrewise linear) symplectic diffeomorphism, say ψ' of $T^\star X$. Therefore ψ defines a bijective map

$$F(Y, X; A) \to F(Y, X; A\psi^\star)$$

which takes each Lagrangian embedding f to its composition with ψ'. The desired assertion follows from the fact that $\mu_{\psi' f} = \mu_f$. □

4.2.3. Exercise. — Show that $I(Y, Y; \mathrm{Id})$ and $I(Y, X; 0)$ are D_Y-invariant sets.

The cocycle property. — Recall that the *sum* of two subsets of a lattice is a set consisting of all pairwise sums of their elements. By definition, $I + \emptyset = \emptyset$.

Our main technical tool is given by the following "cocycle property" which is just a reformulation of the summation formula of chapter I.

PROPOSITION 4.2.4. — *If Z is compact, then $I(Z, Y; B) + BI(Y, X; A) \subset I(Z, X; BA)$.*

Proof. — If h is a composition of Lagrangian embeddings $f \in F(Z, Y; B)$ and $g \in F(Y, X; A)$ then $h \in F(Z, X; BA)$ and moreover $\mu_h = \mu_f + B\mu_g$. □

Geometric properties. — Recall that each $I(Y, X; A)$ is a subset of an integral lattice $H^1(Y; \mathbf{Z})$.

PROPOSITION 4.2.5. — *The set $I(Y, X; A)$ is central symmetric.*

Proof. — Let σ be the standard anti-symplectic involution of $T^\star X$, that is $\sigma(\xi) = -\xi$ for every co-vector ξ. One can easily check that for each $f \in F(Y, X; A)$ the composition σf belongs to $F(Y, X; A)$ and $\mu_{\sigma f} = -\mu_f$. The proposition follows. □

4.2.6. Exercise. — Prove that if X is compact then $I(X, X; \mathrm{Id})$ is star-shaped, that is $\mathbf{Z}v \subset I(X, X; \mathrm{Id})$ provided $v \in I(X, X; \mathrm{Id})$. In particular, either $I(X, X; \mathrm{Id}) = \{0\}$ or $\|I(X, X; \mathrm{Id})\| = +\infty$ (Hint: use the cocycle property 4.2.4).

Rigidity in cotangent bundles via rigidity in $\mathbf{C}^n$. — From now on we assume that X is a compact n-dimensional manifold satisfying the following conditions:

1. the group D_X acts transitively on the set of primitive elements of $H^1(X; \mathbf{Z})$;
2. X admits a Lagrangian embedding into $\mathbf{C}^n = T^\star \mathbf{R}^n$;
3. moreover, $I(X, \mathbf{R}^n; 0)$ does not contain 0 and $\|I(X, \mathbf{R}^n; 0)\| < +\infty$.

A basic example of such a manifold is the torus T^n (see theorem 4.1.1).

Our first observation is that in this situation the structure of the set $I(X, \mathbf{R}^n; 0)$ is quite simple.

4.2.7. Exercise. — There exists a finite set M of positive integers such that $I(X, \mathbf{R}^n; 0)$ consists of all $v \in H^1(X; \mathbf{Z})$ with $\|v\| \in M$ (Hint: use that $I(X, \mathbf{R}^n; 0)$ is D_X-invariant (see 4.2.3) and the fact that the primitive elements are exactly the elements of norm 1).

Now we are in position to formulate the main result.

THEOREM 4.2.8. — *For every homomorphism A of $H^1(X; \mathbf{Z})$ the following holds:*

1. $I(X, X; A) \subset I(X, \mathbf{R}^n; 0)$ *provided A has a non-trivial kernel;*
2. *If A is a monomorphism, $I(X, X; A)$ is equal to $\{0\}$ or empty;*
3. $I(X, X; A) \cap \mathbf{Q} \cdot \mathrm{Image}(A)$ *is $\{0\}$ or empty.*

Proof. — The cocycle property 4.2.4 implies that

$$I(X,X;A) + AI(X,\mathbf{R}^n;0) \subset I(X,\mathbf{R}^n;0).$$

If A has a non-trivial kernel then 4.2.7 implies that the set $AI(X,\mathbf{R}^n;0)$ contains 0 because $\operatorname{Ker} A$ contains at least one primitive element and therefore some multiple of that element must belong to $I(X,\mathbf{R}^n;0)$, and assertion 1 follows. Note that if A is a monomorphism then $\mathbf{Q} \cdot \operatorname{Image}(A)$ covers the whole of $H^1(X;\mathbf{Z})$. Therefore assertion 2 is an immediate consequence of assertion 3.

We prove assertion 3 by using an indirect argument. Assume that there exist primitive elements e_1, e_2 of $H^1(X;\mathbf{Z})$ and positive integers p, q such that $Ae_1 = pe_2$ and $qe_2 \in I(X,X;A)$. Set $m = \max M$, and note that $me_1 \in I(X,\mathbf{R}^n;0)$ due to 4.2.7. We obtain

$$qe_2 \in I(X,X;A),$$

$$A(me_1) = mpe_2 \in AI(X,\mathbf{R}^n;0),$$

and hence the cocycle property gives us that $(q+mp)e_2 \in I(X,\mathbf{R}^n;0)$. This is a contradiction to the assumption $\|I(X,\mathbf{R}^n;0)\| < m+1$. □

Let us apply the previous theorem to the case $X = T^n$.

COROLLARY 4.2.9.

1. *If A is an isomorphism then $I(T^n,T^n;A) = \{0\}$.*
2. *If A has a non-trivial kernel then $2 \leq \|\mu\| \leq n+1$ for all $\mu \in I(T^n,T^n;A)$.*
3. *Moreover, in the last case, $I(T^n,T^n;A) \cap \mathbf{Q} \cdot \operatorname{Image}(A) = \emptyset$.*

Proof. — Statement 1 follows from 4.2.8 (second condition) and the fact that $I(T^n,T^n;A)$ is not empty (it contains a suitable parametrisation of the zero section). Statement 2 follows from 4.2.8 (first condition) and the Maslov class rigidity theorem 2.2.7. Statement 3 follows from 4.2.8 (third condition) and the fact that $I(T^n,T^n;A)$ does not contain 0 due to statement 2. □

Exactness and monotonicity. — Propositions 4.2.1, 4.2.2, 4.2.3, 4.2.4, 4.2.5, 4.2.6 and their proofs remain true with no change if one considers *exact* Lagrangian embeddings instead of general ones (that is the sets $I_{ex}(Y,X;A)$ consisting of all elements from $H^1(X,\mathbf{Z})$ representing the Maslov classes of exact Lagrangian embeddings from $F(Y,X;A)$). The reason is that that *the class of exact Lagrangian embeddings is closed under the composition operation* as we already mentioned in § 2.1. The situation with regard to 4.2.7, 4.2.8 and 4.2.9 is quite different since there are no exact Lagrangian embeddings of a compact manifold into $\mathbf{C}^n$! Nevertheless one can overcome this difficulty using *monotone* Lagrangian embeddings into $\mathbf{C}^n$ (see [41]). As an illustration we present the following exercise:

4.2.10. Exercise. — Let $X = T^n \times S^2$. Show that $I_{ex}(X, X; \mathrm{Id}) = \{0\}$. Due to the "exact" version of 4.2.6 it is sufficient to show that $\|I_{ex}(X, X; \mathrm{Id})\| < +\infty$. Suppose that this is not true, then there is an exact Lagrangian embedding, say f from $F(X, X; \mathrm{Id})$ with $\|\mu_f\| > n$. Recall now that X admits a monotone Lagrangian embedding, say g into $\mathbf{C}^{n+2}$ (see 2.1.8). Moreover, using the action of D_X one can choose g in such a way that the vectors μ_g and μ_f are proportional with a positive coefficient. Consider now the composition h of f and g. One can check that h is monotone and that $\|\mu_h\| > n + 1$, in contradiction with 2.2.6.

4.2.11. Exercise. — Let X be as in the previous exercise. Show that $I(X, X; \mathrm{Id}) = \{0\}$ (Hint: show that each embedding from $F(X, X; \mathrm{Id})$ is Lagrangian isotopic to an exact one, and apply 4.2.10).

4.3. Applications to mechanics: from order to chaos

In the following we use canonical coordinates (p, q) on $T^\star T^n$. We say that a Hamiltonian function H on $T^\star T^n$ satisfies the *Legendre condition* if it is strictly convex with respect to momenta: $H_{pp} > 0$. Note that the Legendre condition appears in most physically important examples. Consider for instance the motion of a free particle. The corresponding Hamiltonian function is just the kinetic energy $H(p, q) = (A(q)p, p)/2$, where each matrix $A(q)$ is symmetric and positive definite.

4.3.1. Exercise. — Set $(a^{ij}) = A^{-1}$. Show that trajectories of the Hamiltonian flow generated by H project to geodesics of the Riemannian metric $a^{ij} dq_i dq_j$ on T^n.

It turns out that the Legendre condition has a nice geometric meaning which is crucial for our considerations. Roughly speaking, if a Hamiltonian function satisfies this condition, then the corresponding Hamiltonian flow *twists* every Lagrangian subspace into the positive direction with respect to the vertical Lagrangian distribution $\mathcal{F} = \{dq = 0\}$. Let us express this *twisting property* in a more precise way. In what follows we identify every tangent space $T_x T^\star T^n$ with the standard symplectic vector space $\mathbf{R}^{2n}$, and we write $\Lambda = \Lambda_n$ for the Lagrangian Grassmann manifold. We denote by α the subspace $\{dq = 0\}$, and by Λ^α the set off all Lagrangian subspaces from Λ which are not transversal to α.

PROPOSITION 4.3.2 (Twisting property). — *Let g^t be the Hamiltonian flow generated by a Hamiltonian function H which satisfies the Legendre condition. Then for all $x \in T^\star T^n$, $\lambda \in \Lambda^\alpha$ the vector*

$$\frac{d}{dt}\left(g^t_\star(x)\lambda\right)\Big|_{t=0} \in T_\lambda \Lambda$$

is α-positive.

Proof. — Given the definition of an α-positive vector (see the appendix to chapter I), we have to check that for every non-zero vector $\xi \in \lambda \cap \alpha$ the following inequality holds:

$$\omega\left(\xi, \frac{d}{dt}\left(g_\star^t(x)\xi\right)\Big|_{t=0}\right) > 0.$$

In (p,q)-coordinates such a vector ξ can be written as $(a,0)$. Linearising the Hamiltonian equations we obtain

$$\frac{d}{dt}\left(g_\star^t \xi\right)\Big|_{t=0} = (-H_{qp}a, H_{pp}a),$$

and the desired inequality can be reformulated as $(H_{pp}a, a) > 0$. But this follows from the Legendre condition[10]. □

The simplest integrable system on $T^\star T^n$ is given by a Hamiltonian function $H(p,q) = (p,p)/2$ associated with the Euclidean metric on the torus (cf. 4.3.1). The corresponding Hamiltonian flow can be written as

$$(p,q) \mapsto (p, q+pt),$$

and hence each torus $\{p = \text{ const}\}$ is invariant under the dynamics.

A striking fact which follows from the celebrated Kolmogorov-Arnold-Moser theory (see e.g. [2], appendix 8) is that most of these tori survive under a *small* perturbation, namely a set of large measure in the phase space is still foliated by invariant tori of the perturbed system. Let us emphasise two properties of those invariant tori which are exhibited by the KAM-theory:

- each such torus is homologous to the zero section;
- in some angular coordinates φ mod 1 on such a torus the dynamics is given by a linear shift $\varphi \mapsto \varphi + \sigma t$, where the components $(\sigma_1, \dots, \sigma_n)$ of σ are independent over $\mathbf{Z}$.

A smooth embedded n-dimensional torus which is invariant under a Hamiltonian flow on $T^\star T^n$ and satisfies these two conditions is an *essential invariant torus*. One might expect that the geometry of invariant tori becomes very complicated when the perturbation becomes very large. An obstruction to such a behaviour is given by the following theorem, the proof of which will be given in § 4.4.

THEOREM 4.3.3. — *If a Hamiltonian function satisfies the Legendre condition, every essential invariant torus of its Hamiltonian flow is the graph of a smooth section of the cotangent bundle.*

[10] This twisting property makes sense in any symplectic manifold endowed with a Lagrangian distribution. Functions with this property are investigated in [16], [17].

Let us make some comments. The first result of this kind was obtained by Birkhoff for invariant curves of area preserving maps of the annulus (see [13], and also [27], [21]) and is usually known as *Birkhoff's second theorem.* Our formulation for the multidimensional case comes from [40] (see also [15], [12] for the case $n = 2$, and [28] for some other interesting extensions of the Birkhoff's theory). We refer the reader to [16] for a detailed discussion and far reaching generalisations.

As we shall show in § 4.4, theorem 4.3.3 gives us a tool to understand the *invariant tori breaking mechanism.* In order to formulate the result, we specify our deformation of the Euclidean metric on the torus. For simplicity, we restrict our attention to the case $n = 2$ (the multidimensional case can be treated in exactly the same manner). Let γ be an embedded closed curve on T^2 which bounds a disc, say B, and fix a point K inside the disc. Let G_ε, ($\varepsilon \geq \varepsilon_0$) be a Riemannian metric on the torus such that

$$\text{distance}(K, \gamma) > \varepsilon \cdot \text{length}(\gamma).$$

In other words, increasing the parameter ε corresponds to a bump growing up inside the disc B and whose top point is K. The proof of the next theorem will be given in § 4.4.

THEOREM 4.3.4 ([14], see also [10] and [40]). — *There exists a critical value of the parameter, say ε_{cr}, such that for all $\varepsilon > \varepsilon_{cr}$ the Hamiltonian flow associated with the metric G_ε has no essential invariant tori.*

Remark. — Thus tori disappear. The question of the dynamical properties of the system is far from being understood. Nevertheless, there are some very interesting results in dimension 2: more complicated invariant sets appear, the Aubry-Mather Cantor sets (see [14], [10] and [20] in which Donnay constructed a metric on T^2 whose geodesic flow is ergodic—a classical property of *negatively curved manifolds*).

4.4. Symplectic geometry and variational properties of invariant tori

The following simple observation is crucial for proving the results of the previous section.

PROPOSITION 4.4.1 ([28]). — *Every essential invariant torus of a Hamiltonian flow on T^*T^n is Lagrangian.*

Proof. — Let φ mod 1 be angular coordinates on an essential invariant torus L such that in these coordinates the restriction of the Hamiltonian flow to L is given by an irrational shift. Denote by Ω the restriction of the symplectic form to L. Since Ω is preserved by the dynamics, and since every trajectory is dense on L we conclude that Ω is a *translation invariant* form with respect to φ-coordinates. Moreover, Ω is exact since the symplectic form is. Hence Ω vanishes, and L is a Lagrangian torus. □

Now we are in position to prove 4.3.3. The proof given below incorporates a nice argument which we learned from G. Paternain (see [38]). We use the notations of § 4.3.

Proof of 4.3.3. — Let L be an essential invariant torus of a Hamiltonian flow g^t, which is generated by a Hamiltonian function H satisfying the Legendre condition. Note first of all that it is sufficient to check that L is *transversal* to the Lagrangian distribution $\mathcal{F} = \{dq = 0\}$. Assume for contradiction that there exists a point $x \in L$ such that the tangent subspace $\lambda = T_x L$ belongs to Λ_n^α.

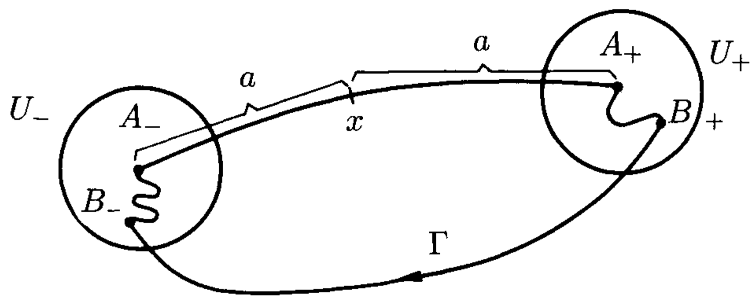


Figure 22

Our plan is to construct a loop, say γ, on L with positive value of the Maslov class. By 4.3.2, the vector

$$\frac{d}{dt}\left(g_\star^t(x)\lambda\right)\Big|_{t=0}$$

is α-positive and hence transversal to Λ_n^α. Therefore there exists some positive a such that the for every $t \in [-a, a]$ the subspace

$$g_\star^t(x)\lambda = T_{g^t(x)}L$$

is transversal to α (see figure 22). Set $A_+ = g^a(x), A_- = g^{-a}(x)$. Let U_+, U_- be small neighbourhoods in L of the points A_+ and A_- respectively such that for each point, say y, of these neighbourhoods the tangent subspace $T_y L$ is still transversal to α. Note that every trajectory on L is dense, and hence there exists a segment of trajectory, say Γ, whose end points B_+ and B_- belong to U_+ and U_- respectively. Now join B_- with A_- by a smooth path inside U_-, A_- with A_+ by a piece of the trajectory of the point x for $t \in [-a, a]$, and A_+ with B_+ by a smooth path inside U_+. Adding the segment Γ we obtain a loop γ on L. Note that the corresponding loop $z \mapsto T_z L$ $(z \in \gamma)$ in Λ_n has a non-empty intersection with Λ_n^α. Moreover, our construction together with 4.3.2 imply that its tangent vector in each intersection point is α-positive. Therefore the value of the Maslov class of L on γ is positive as noticed in the appendix to chapter I. On the other hand L is an embedded Lagrangian torus homologous to the zero section and hence its Maslov class vanishes by theorem 4.1.1. This contradiction proves the theorem. □

Before we give the proof of 4.3.4, let us recall that a fundamental property of a geodesic is that locally it minimises the length. A geodesic is called (globally) *minimal* if any segment minimises length in the homotopy class of paths with the same endpoints (see chapter III).

4.4.2. Exercise. — Show that for ε sufficiently large there is no complete globally minimal geodesic of the metric G_ε which passes through the top of the bump K (Hint: if ε is sufficiently large then a globally minimal geodesic will prefer to go round the bump!).

We now give the proof of 4.3.4 in two exercises. A basic observation is the following:

4.4.3. Exercise. — Let L be a smooth Lagrangian section of $T^\star T^n$ which is invariant under a Hamiltonian flow associated to a Riemannian metric on T^n and which is contained in a regular level $\{h = \text{ cont}\}$. Show that each trajectory which lies on L projects to a globally minimal geodesic (Hint: use a classical Weierstrass theorem of the variational calculus (see [33] for details)).

4.4.4. Exercise (proof of 4.3.4*).* — Show that 4.3.4 is a consequence of the two previous exercises together with 4.3.3 and 4.4.1.

4.4.5. Exercise. — Find an estimate for the critical value ε_{cr} of the parameter.

5. Pseudo-holomorphic curves: proof of the main rigidity theorem

5.1. The Riemann-Hilbert problem

In order to prove theorem 2.2.4, we will first establish a theorem on the existence and unicity of solutions of the equation $\overline{\partial}_J f = g$ on *geometrically bounded* almost complex manifolds containing a given totally real submanifold. This is a generalisation of the so-called Riemann-Hilbert problem.

The simplest form of the Riemann-Hilbert problem consists, given a map $g : D^2 \to \mathbf{C}^n$ on the closed unit disc $D^2 \subset \mathbf{C}$, in finding a solution $f : D^2 \to \mathbf{C}^n$ of the equation $\overline{\partial} f = g$ which sends the boundary ∂D^2 into the totally real subspace $\mathbf{R}^n = \langle x_1, \dots, x_n \rangle \subset \mathbf{C}^n = \langle x_1, y_1, \dots, x_n, y_n \rangle$. This problem always admits a solution in appropriate Hölder spaces (and therefore in $\mathcal{C}^\infty$ spaces by elliptic regularity). More precisely, for any fixed real non integral number $r > 0$, the map

$$\overline{\partial} : \mathcal{C}^{r+1}(D^2, \partial D^2, 1; \mathbf{C}^n, \mathbf{R}^n, 0) \longrightarrow \mathcal{C}^r(D^2, \mathbf{C}^n)$$

is onto, where the domain is the space of maps of Hölder class $r + 1$ from D^2 to $\mathbf{C}^n$ which send ∂D^2 in $\mathbf{R}^n$ and 1 on 0, and where the target space is the space of maps from D^2 to $\mathbf{C}^n$ of Hölder class r (see the classical reference [47], pp. 56 and 355-356).

This problem can be generalised in the following way (see [24], [43], we follow here [44]). Let (V, J) be an almost complex manifold. Denote by i the complex structure on $\mathbf{C}$ (multiplication by i). For a $\mathcal{C}^1$ map $f : D^2 \to V$, define

$$\overline{\partial}_J f = \frac{1}{2}\left(\frac{\partial f}{\partial x} + J\frac{\partial f}{\partial y}\right) = \frac{1}{2}(df + Jdfi)\left(\frac{\partial}{\partial x}\right)$$

which is a section of the vector bundle $f^\star(TV)$ on D^2. It will be more convenient to consider this section as a section of the bundle $pr_2^\star TV \to D^2 \times V$ defined over the graph of f (here $pr_2 : D^2 \times V \to V$ is the projection on the second factor). The generalisation of the Riemann-Hilbert problem is then:

GENERAL PROBLEM. — *Given a totally real submanifold L of an almost complex manifold (V, J) and a global section g of $pr_2^\star TV$, find $f : (D^2, \partial D^2) \to (V, L)$ homotopic to a point in $\pi_2(V, L)$ and such that $\overline{\partial}_J f = g|_{\mathrm{graph}(f)}$ (which we will simply denote by $\overline{\partial}_J f = g$).*

Note that this problem does not always admit a solution; for instance, if L is a closed, totally real submanifold embedded in $\mathbf{C}^n$ endowed with the standard complex structure, and if $g : D^2 \times \mathbf{C}^n \to \mathbf{C}^n$ is equal to a constant g_0, there is no solution when $\|g_0\|$ is sufficiently large (see the harmonicity argument in the proof of theorem 2.2.4).

The following theorem (which provides a solution to the Riemann-Hilbert problem under certain hypotheses) is proved by Sikorav in [44].

THEOREM 5.1.1. — *Let V be endowed with any complete Riemannian metric μ. The general problem above has a solution for all $\mathcal{C}^\alpha$ bounded (with respect to μ) sections g of Hölder class $\mathcal{C}^\alpha$ (for $\alpha > 0$) if the following conditions hold:*

1. *if $g \equiv 0$, the only solutions of $\overline{\partial}_J f = g$ which belong to $\mathcal{C}^{\alpha+1}(D^2, \partial D^2; V, L)$ are the constants;*
2. *for any $B > 0$, the family $\{f \in \mathcal{C}^{\alpha+1}(D^2, \partial D^2; V, L) | \left\|\overline{\partial}_J f\right\|_{\mathcal{C}^\alpha} \leq B\}$ is equicontinuous.*

Moreover, one may impose $f(1) = \ell_0$ for any given point $\ell_0 \in L$.

It may be embarassing to try to apply this theorem directly, because the second condition is not easily verifiable in practice. Sikorav shows that this condition holds true under some hypotheses on the geometry of V and L. His proof does not make use of the compactness theorem for J-holomorphic discs and, for this reason, the hypotheses on V and L needed to establish the surjectivity of $\overline{\partial}_J f = g$ are stronger than those needed by Gromov in [24] theorem 2.3.B. We will now prove the stronger version of the theorem due to Gromov, using a compactness theorem.

THEOREM 5.1.2. — *Let (V, ω) be a symplectic manifold and $L \subset V$ be a Lagrangian submanifold such that (V, L) is g. bounded (with respect to structures J and μ on V). If there is no non-constant J-holomorphic 2-sphere in V and no non-constant J-holomorphic disc with boundary in L, then for any C^∞ bounded section g of $pr_2^\star TV \to D_2 \times V$ and any point $\ell_0 \in L$, there exists a solution to the equation $\overline{\partial}_J f = g$ which sends ∂D^2 in L and the point 1 to the point ℓ_0, and such that f is homotopic to a point in $\pi_2(V, L, \ell_0)$.*

Remarks.

1. Since ω tames J, any non-constant J-holomorphic disc $f : (D^2, \partial D^2) \to (V, L)$ satisfies $\int_{D^2} f^\star \omega > 0$. Therefore, the hypotheses of the theorem concerning J-holomorphic discs and spheres are satisfied if L is a *weakly exact* Lagrangian submanifold (recall that this means that the integration map $\int \omega$ from $\pi_2(V, L)$ to $\mathbf{R}$ is zero). Thus, if (V, L) is g. bounded and L is weakly exact, there always exists a solution to the Riemann-Hilbert problem. This is the case, for example, when $(V, \omega, J, \mu) = (\mathbf{C}^n, \omega_0, J_0, \mu_0)$ and L is the image of $\mathbf{R}^n \subset \mathbf{C}^n$ by any bounded symplectic diffeomorphism.
2. The conclusion of theorem 5.1.2 is a Fredholm alternative (see the proof below): suppose (V, L) is g. bounded and V contains no non-trivial J-holomorphic sphere, then either the map $\overline{\partial}_J$ is "surjective" or there exists a non-trivial J-holomorphic disc in $\pi_2(V, L)$. Of course, this alternative may be useful in both directions: when we know that there is no non-trivial J-holomorphic disc in $\pi_2(V, L)$, we conclude that $\overline{\partial}_J$ is surjective as in the example above; when we know that $\overline{\partial_J}$ cannot be surjective, we conclude that there exists a non-trivial J-holomorphic disc in $\pi_2(V, L)$ and hence that L is not (weakly) exact (this applies to any closed (compact and without boundary) Lagrangian submanifold of $\mathbf{C}^n$, see below).

Proof of theorem 5.1.2. — We first bring the problem into a Fredholm framework with a given operator $\overline{\partial}_J$: we do not yet have a well defined operator since the section $\overline{\partial}_J f$ of $f^\star TV$ is defined only on a subset of $D^2 \times V$ which depends on f. For $\ell_0 \in L$ and a non integral $r > 0$ fixed, let us define:

- The set F^{r+1} of Hölder class C^{r+1} maps $f : (D^2, \partial D^2, 1) \to (V, L, \ell_0)$ which are homotopic to $f_0 \equiv \ell_0$ in $\pi_2 (V, L, \ell_0)$,
- the set G^r of C^r sections of $pr_2^\star TV \to D^2 \times V$ which are C^r bounded,
- the set $H^{r+1} = \Big\{(f, g) \in F^{r+1} \times G^r \text{ such that } \overline{\partial}_J f = g\Big\}$,
- the projection map $\Delta^r : H^{r+1} \to G^r$ defined by $\Delta^r(f, g) = g$.

The following four properties hold for any totally real submanifold L of an almost complex manifold (V, J) (see chapter V).

1. G^r is a complex Banach vector space, F^{r+1} is an almost complex Banach manifold with tangent space

$$T_f F^{r+1} = \{\mathcal{C}^{r+1} \text{ sections of } (f^\star(TV), (f|_{\partial D^2})^\star TL) \text{ which vanish at } 1\},$$

and the charts of F^{r+1} are given by the exponential map of a Riemannian metric on V for which L is totally geodesic (that is such that any geodesic tangent to L at any point of L lies entirely in L). Finally, H^{r+1} is a $\mathcal{C}^\infty$ submanifold of $F^{r+1} \times G^r$. Indeed, let $f_0 \in F^{r+1}$ and $\mathcal{U}(f_0)$ be a $\mathcal{C}^0$-small neighbourhood of f_0. For any $f \in \mathcal{U}(f_0)$, denote by $\Lambda^r(f)$ the space of $\mathcal{C}^r$-sections of $pr_2^\star(TV)$ over the graph of f. Let V be endowed with any $\mathcal{C}^\infty$ connection compatible with J; then the parallel transport induced by this connection gives a complex isomorphism:

$$PT(f, f_0) : \Lambda^r(f) \to \Lambda^r(f_0)$$

for any $f \in \mathcal{U}(f_0)$. On $\mathcal{U}(f_0) \times G^r$, define $\Phi : \mathcal{U}(f_0) \times G^r \longrightarrow \Lambda^r(f_0)$ by

$$\Phi(f, g) = PT(f, f_0)\left(\overline{\partial}_J f - g|_{\text{graph}(f)}\right).$$

It is clear that $\Phi^{-1}(0) = H^{r+1} \cap (\mathcal{U}(f_0) \times G^r)$, that Φ is a smooth map, and that 0 is a regular value of Φ because $\Phi(f, .) : G^r \to \Lambda^r(f_0)$ is affine and onto. Finally, the kernel of the associated linear map, $K(f) = \{g \mid g|_{\text{graph}(f)} = 0\}$ admits a closed supplement in G^r. Since these are the sufficient conditions for the inverse image of a point by a smooth map between Banach manifolds to be a smooth submanifold, we conclude that H^{r+1} is a $\mathcal{C}^\infty$ submanifold.

2. Δ^r is a Fredholm map of index 0. In fact, Δ^r is locally equivalent to a quasi-linear differential operator of order 1 whose linearisation is of the form

$$(\text{Id} + K) \circ \overline{\partial} : \mathcal{C}^{r+1}(D^2, \partial D^2, 1; \mathbf{C}^n, \mathbf{R}^n, 0) \longrightarrow \mathcal{C}^r(D^2; \mathbf{C}^n)$$

where K is a compact operator. Since $\overline{\partial}$ is an isomorphism, $(\text{Id} + K) \circ \overline{\partial}$ is Fredholm of index 0 (see chapter V).

3. Δ^r is regular at $(\ell_0, 0)$(see chapter V).

4. Elliptic regularity: if f is of class $\mathcal{C}^1$ and if $\overline{\partial}_J f = g$ with g of class $\mathcal{C}^r$, then f is of class $\mathcal{C}^{r+1}$ (see chapter V or [36]).

5.2. Proof of theorem 5.1.2

Let us now prove theorem 5.1.2: given elliptic regularity, it is sufficient to prove that Δ^r is surjective for any non integral $r > 0$. The non-existence of non-constant

J-holomorphic disc in $\pi_2(V, L)$ means that $\Delta^{-1}(0) = (\ell_0, 0)$, and together with the third property above, this implies that 0 is a regular value of Δ^r. In order to prove the theorem, it is therefore enough to show that Δ^r is proper. By Smale's generalisation (see [45]) of Sard's theorem to Fredholm maps between Banach manifolds, one can deduce (see figure 23):

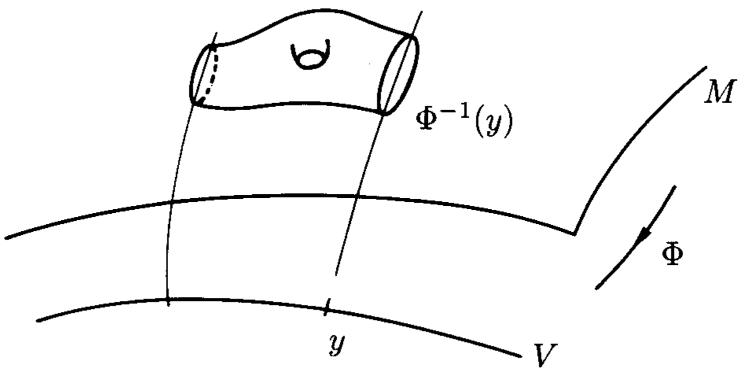


Figure 23

THEOREM 5.2.1. — *Let $\Phi : M \to V$ be a C^∞ proper Fredholm map between Banach manifolds of nonnegative index. Then, for any regular value $y \in V$, $\Phi^{-1}(y)$ is a smooth submanifold (either empty, or of dimension* ind Φ*), and the non oriented cobordism class of $\Phi^{-1}(y)$ does not depend on the choice of the regular value y of V. Denoting by $\gamma(\Phi)$ this invariant in the ring of non oriented cobordism, we have: Φ is onto when $\gamma(\Phi) \neq 0$.* □

Let us apply this theorem to Δ^r, assuming that it is proper. Since 0 is a regular value whose inverse image is a single point, $\gamma(\Delta^r) \neq 0$, and hence Δ^r is onto. Let us describe this more explicitely: let $g \in G^r$ be a regular value of Δ^r and let $\gamma : [0,1] \to G^r$ be a path from 0 to g, everywhere transversal to Δ^r. Since Δ^r is a proper Fredholm map of index 0, $(\Delta^r)^{-1}(\gamma)$ is a non empty compact submanifold of real dimension ind $\Delta^r + 1 = 1$, whose boundary is included in $(\Delta^r)^{-1}(0) \cup (\Delta^r)^{-1}(g)$. In fact, $\partial(\Delta^r)^{-1}(g)$ equals $(\Delta^r)^{-1}(0) \cup (\Delta^r)^{-1}(g)$ because if there were a point p belonging say to $(\Delta^r)^{-1}(g)$ but not to $\partial(\Delta^r)^{-1}(\gamma)$, the differential of Δ^r would send the vector tangent to $(\Delta^r)^{-1}(\gamma)$ at p on the zero vector at $\gamma(1)$, hence the dimension of ker Δ^r at p would be strictly positive. Since ind $\Delta^r = 0$, this would contradict the regularity of g. Thus $(\Delta^r)^{-1}$ is a non oriented cobordism between $(\Delta^r)^{-1}(0)$ and $(\Delta^r)^{-1}(g)$, therefore their mod 2 cardinalities are equal. We conclude that $(\Delta^r)^{-1}(g)$is not empty. This shows that all regular values of Δ^r are in the image of Δ^r, and since this is true by definition for non regular values as well, Δ^r is onto. □

It remains to prove:

PROPOSITION 5.2.2. — *Δ^r is proper.*

Proof. — Suppose that Δ^r is not proper: then there exists a sequence (f_n, g_n) $(n \in \mathbf{N})$ which contains no convergent subsequence in H^{r+1} but whose projection $(g_n)_{n\in\mathbf{N}}$ has a convergent subsequence. Denoting by the same symbols $(g_n)_{n\in\mathbf{N}}$ this subsequence, we thus have a sequence $(f_n) \in (F^{r+1}, \| \|_{C^{r+1}})$ with no convergent subsequence, and a sequence (g_n) which C^r-converge to a value $g \in G^r$ such that $\overline{\partial}_J f_n = g_n$.

We now show that each section g of $pr_2^* TV$ induces an almost complex structure J_g on $D^2 \times V$ with the properties described in the following lemma, due to Gromov (we do not specify the differentiability class, since the lemma clearly holds true for any class of maps).

LEMMA 5.2.3. — *There exists a unique structure J_g on $D^2 \times V$ such that the germs of J_g-holomorphic sections* $\mathrm{Id} \times f$ *of $pr_1 : D^2 \times V \to D^2$ correspond exactly to the solutions of $\overline{\partial}_J f = g$ on V and such that the restriction of J_g to any fibre $s \times V$ of $D^2 \times V \to D^2$ coincides with the original structure J on $V = s \times V$.*

Proof. — In the exact sequence of bundles over $D^2 \times V$:

$$0 \longrightarrow \mathrm{Hom}_{\mathbf{C}}(TD^2, TV) \xrightarrow{\text{incl}} \mathrm{Hom}_{\mathbf{R}}(TD^2, TV) \xrightarrow{\pi}$$

$$\mathrm{Hom}_{\mathbf{R}}(TD^2, TV)/\mathrm{Hom}_{\mathbf{C}}(TD^2, TV) \longrightarrow 0$$

the projection π has a right inverse whose image is the set $\mathrm{Hom}_{\text{anti-}\mathbf{C}}(TD^2, TV)$ of real homomorphisms C such that $C \circ i = -J \circ C$. Hence,

$$\mathrm{Hom}_{\mathbf{R}} = \mathrm{Hom}_{\mathbf{C}} \oplus \mathrm{Hom}_{\text{anti-}\mathbf{C}},$$

since

$$A = \frac{1}{2}(A - J \circ A \circ i) \oplus \frac{1}{2}(A + J \circ A \circ i)$$

(of course $\overline{\partial}_J$ is simply the projection on the second factor of this decomposition, that is the projection on the anti-complex part). Since D^2 is of complex dimension 1, an anti-complex homomorphism is determined by its value on $\partial/\partial x$ and this defines an identification of $\mathrm{Hom}_{\text{anti-}\mathbf{C}}(TD^2, TV)$ with $pr_2^*(TV)$.

Let g be a section of $pr_2^*(TV)$, which we consider as a section of the bundle $\mathrm{Hom}_{\text{anti-}\mathbf{C}}(TD^2, TV)$. We define the endomorphism J_g of $T(D^2 \times V)$ at $(s, v) \in D^2 \times V$ by

$$J_g | T_v V = J \text{ and } J_g | T_s D^2 = i + 2g_{s,v} \circ i.$$

The fact that g is anti-complex obviously implies that $J_g^2 = -\mathrm{Id}$. Finally, the germ of a section $\mathrm{Id} \times f$ of $pr_1 : D^2 \times V \to D^2$ defined near $s \in D^2$ is J_g-holomorphic at s if and only if $J_g(d(\mathrm{Id} \times f)(s)) = d(\mathrm{Id} \times f)(s) \circ i$. But

$$\begin{aligned} J_g(d(\mathrm{Id} \times f)(s)(X)) &= J_g(X \oplus df(s)(X)) \\ &= iX \oplus (2g(iX) + Jdf(X)) \end{aligned}$$

is equal to $d(\mathrm{Id} \times f)(s)(iX) = iX \oplus df(iX)$ for all $X \in T_s D^2$ exactly when $g(Z) = \frac{1}{2}(df + Jdfi)(Z)$ for all $Z \in T_s D^2$, that is exactly when $g = \overline{\partial}_J f$. □

It is clear that the properties stated in the lemma determine an almost complex structure on $D^2 \times V$ in a unique way. Hence, the two sequences (f_n) and (g_n) induce a sequence of J_{g_n}-holomorphic sections $(\mathrm{Id} \times f_n)$ of $D^2 \times V \to D^2$. The sequence (J_{g_n}) C^r converges to an almost complex structure J_g but the sequence $(\mathrm{Id} \times f_n)$ of maps from D^2 to $D^2 \times V$ C^{r+1} diverges. We will deduce from this the existence of a non constant *rational* J-holomorphic curve in V (that is: a non constant J-holomorphic map S^2, in other words a J-holomorphic sphere) or the existence of a non-constant J-holomorphic disc in V whose boundary lies in L. In both cases, this will contradict the hypotheses of the main theorem. We need the following theorem of Gromov on the compactness of holomorphic discs (see chapter VIII for these compactness properties):

THEOREM 5.2.4. — *Let (M, ω_M) be a symplectic manifold, g. bounded for J_M and μ_M. Assume J_M is of class C^r. Let $W \subset M$ be a Lagrangian submanifold such that the pair (M, W) is g. bounded. Let*

$$\varphi_n : \left(D^2, \partial D^2, 1\right) \longrightarrow (M, W, w_0)$$

be a sequence of (i, J_M)-holomorphic discs of class C^{r+1} whose areas are bounded above. Then there exists a subsequence which either C^{r+1} converges to a (i, J_M)-holomorphic disc or weakly converges to a J_M-holomorphic cusp-disc whose image in M is the union of k_1 non-constant J_M-rational curves and k_2 non-constant J_M-holomorphic discs with boundary in W, where $k_1 \geq 0, k_2 \geq 1$, and $k_1 + k_2 \geq 2$. □

This theorem is still true for a sequence $(J_M)_n \to J_M$ of almost complex structures C^r-converging to an almost complex structure J_M and a sequence φ_n of holomorphic discs (for i and $(J_M)_n$), provided that all the $((J_M)_n, \omega_M, \mu_M)$ for n sufficiently large and (J_M, ω_M, μ_M) are geometrically bounded with respect to the same constant α_M. Then the limit disc or cusp-disc is J_M-holomorphic.

To apply this theorem to the sequence $(\mathrm{Id} \times f_n)$ of J_{g_n}-holomorphic discs in $D^2 \times V$, we verify the hypotheses of the theorem. Here $M = D^2 \times V$, $W = \partial D^2 \times L$, and $\mu_M = \mu_0 \oplus \mu_1$ where μ_0 is the standard metric on D^2 and $\mu_1 = \mu$ is the metric given on V. Since $(D^2, \partial D^2)$ is obviously geometrically bounded with respect to (i, μ_0, ω_0), the conditions in the definition of "geometrically bounded" which depend only on the metric structure are satisfied for the pair $(D^2 \times V, \partial D^2 \times L)$. Therefore, it is enough to find a symplectic form ω_M on $M = D^2 \times V$ and a positive constant α_M such that

$$\omega_M(X, J_{g_n} X) \geq \alpha_M \left\|X\right\|^2_{\mu_M}$$

for all tangent vectors of $D^2 \times V$ and all sufficiently large n.

We set $\omega_M = (C\omega_0) \oplus \omega$, where ω_0 is the standard form on D^2, ω is the given form on V, and C is a constant depending on $\left\|g_n\right\|_{C^0}$ which we will now compute[11].

[11] The reader may consider the following estimations as a (rather technical) exercise and compute the constant C by her or himself.

Let G be an upper bound for all $\|g_n\|_{C^0}$. Then:

$$\begin{aligned}\omega_M(X, J_{g_n}X) &= C\omega_0 \oplus \omega(X_0 \oplus X_1, iX_0 \oplus (2g_n(iX_0) + JX_1)) \\ &= C\|X_0\|^2 + \omega(X_1, JX_1) + 2\omega(X_1, g_n(iX_0)) \\ &\geq C\|X_0\|^2 + \alpha\|X_1\|^2 + 2\omega(X_1, g_n(iX_0)) \\ &\geq C\|X_0\|^2 + \alpha\|X_1\|^2 - 2\beta G\|X_0\|\,\|X_1\|\end{aligned}$$

because $|\omega(X_1, g_n(iX_o))| \leq \beta\|X_1\|\,\|g_n(iX_0)\| \leq \beta\|X_1\|\,\|g_n\|_{C^0}\|X_0\|$. Thus, obviously,

$$\omega_M(X, J_{g_n}X) \geq \frac{\alpha}{2}\|X\|^2_{\mu_M}$$

if C is sufficiently large. Actually, if

$$\frac{\|X_1\|}{\|X_0\|} \leq \frac{4\beta G}{\alpha},$$

then:

$$\begin{aligned}\omega_M(X, J_{g_n}X) &\geq \left(C - 2\beta G\frac{\|X_1\|}{\|X_0\|}\right)\|X_0\|^2 + \alpha\|X_1\|^2 \\ &\geq \left(C - \frac{8\beta^2G^2}{\alpha}\right)\|X_0\|^2 + \alpha\|X_1\|^2 \\ &\geq \frac{\alpha}{2}\|X\|^2_{\mu_M}\end{aligned}$$

with

$$C = \frac{8\beta^2G^2}{\alpha} + \frac{\alpha}{2}.$$

When

$$\frac{\|X_1\|}{\|X_0\|} \geq \frac{4\beta G}{\alpha},$$

one gets

$$\begin{aligned}\omega_M(X, J_{g_n}X) &\geq \left(\alpha - 2\beta G\frac{\|X_0\|}{\|X_1\|}\right)\|X_1\|^2 + C\|X_0\|^2 \\ &\geq \frac{\alpha}{2}\|X_1\|^2 + C\|X_0\|^2 \geq \frac{\alpha}{2}\|X\|^2_{\mu_M}.\end{aligned}$$

Therefore, $M = D^2 \times V$ is geometrically bounded with $\mu_M = \mu_0 \oplus \mu_1$, $\omega_M = C\omega_0 \oplus \omega$ where

$$C = \frac{8\beta^2G^2}{\alpha} + \frac{\alpha}{2},$$

and with any of the given almost complex structures J_{g_n} or J_g.

Since every $f_n : D^2 \to V$ is contractible in $\pi_2(V, L)$, all discs $\mathrm{Id} \times f_n$ belong to the same homotopy class in $\pi_2(D^2 \times V, \partial D^2 \times L)$. Their ω_M-area is $\int_{D_2} C\omega_o + \int_{D^2} f_n^\star \omega = C\pi$ for all n, and their μ_M-area is uniformly bounded above. Writing $\rho = \mathrm{Id} \times f_n$,

$$\begin{aligned}
\mu_M\text{-area}(\rho) &= \int_{D^2} \left\| \frac{\partial \rho}{\partial X} \wedge \frac{\partial \rho}{\partial Y} \right\|_{\mu_M} \\
&\leq \frac{1}{2} \int_{D^2} \left(\left\| \frac{\partial \rho}{\partial X} \right\|^2_{\mu_M} + \left\| \frac{\partial \rho}{\partial Y} \right\|^2_{\mu_M} \right) \\
&\leq \frac{1}{\alpha} \int_{D^2} \left(\omega_M \left(\frac{\partial \rho}{\partial X}, J_{g_n} \frac{\partial \rho}{\partial X} \right) + \omega_M \left(\frac{\partial \rho}{\partial Y}, J_{g_n} \frac{\partial \rho}{\partial Y} \right) \right) \\
&\leq \frac{2}{\alpha} \int_{D^2} \rho^\star \omega_M
\end{aligned}$$

because ρ is J_{g_n}-holomorphic. Hence:

$$\mu_M\text{-area}(\mathrm{Id} \times f_n) \leq \frac{2C\pi}{\alpha}.$$

The compactness theorem thus applies to the C^{r+1} divergent sequence $\mathrm{Id} \times f_n$ and therefore there exists a J_g-cusp-disc whose components contain at least one non-constant J_g-holomorphic sphere S or two non constant J_g-holomorphic discs D with boundary in $\partial D^2 \times L$. But since this cusp-disc is a weak limit of sections of $D^2 \times V \to D^2$, S cannot be transverse to the fibres of the projection $D^2 \times V \to D^2$ and the same is true for D; that is to say, S (or D) lies entirely in a fibre of the projection. There exists therefore either a non-constant J-rational curve in V or a non-constant J-holomorphic disc in V whose boundary lies in L. This concludes the proof of theorem 5.1.2. □

5.3. Proof of theorem 2.2.4

Let (V, ω) be a product $(V' \times \mathbf{C}, \omega' \oplus \omega_0)$ where (V', ω') is a weakly exact geometrically bounded symplectic manifold. Let L be a g. bounded Lagrangian submanifold of V. If L is bounded with respect to the second factor (that is if the image of L by $pr_2 : V' \times \mathbf{C} \to \mathbf{C}$ is bounded), we show that there exists a loop in L that bounds a holomorphic disc (with respect to an ω-tamed almost complex structure).

First, $(V' \times \mathbf{C}, \omega' \oplus \omega_0)$ is also geometrically bounded. It is easy to see that, given any totally real submanifold L in $V' \times \mathbf{C}$ which is bounded with respect to $\mathbf{C}$, the equation $\overline{\partial}_J f = g$ has no solution when the factor g_2 of $g = g_1 \oplus g_2$ is equal to a constant c of sufficiently large norm. In fact, any solution f to the equation $\overline{\partial}_J f = g_1 \oplus c$ satisfies $\overline{\partial}(pr_2 o f) = g_2 = c$ and therefore $f_2 = pr_2 \circ f : D^2 \to \mathbf{C}$ is harmonic. The Poisson formula then gives

$$\overline{\partial}_J f_2(0) = -\frac{1}{2\pi} \int_0^{2\pi} e^{i\theta} f_2(e^{i\theta}) d\theta;$$

so $|c| = \left|\overline{\partial}_J f_2(0)\right| \leq \sup_{0 \leq \theta \leq 2\pi} \left| f_2(e^{i\theta}) \right| \leq \mathrm{diam} f_2(\partial D^2) \leq \mathrm{diam}(pr_2(L))$.

Thus, there is no solution to $\overline{\partial}_J f = g_1 \oplus c$ when $|c| > \mathrm{diam}(pr_2(L))$. On the other hand, if the totally real submanifold L is Lagrangian and g. bounded, the Fredholm alternative of 5.1.2 shows that if $\overline{\partial}_J f = g$ has no solution for some g, there must exist either a holomorphic sphere or a holomorphic disc with boundary on L. Since V is weakly exact, only the second case can occur.

To prove the second part of theorem 2.2.4, let us quickly review the proof above from a slightly more general point of view. For each class $\alpha \in \pi_2(V, L)$, define $F^{r+1}_\alpha, H^{r+1}_\alpha$ and Δ^r_α as follows:

- F^{r+1}_α is the set of $\mathcal{C}^{r+1}$ Hölder maps $f : (D^2, \partial D^2) \to (V, L)$ such that $[f] = \alpha$,
- $H^{r+1}_\alpha = \{(f, g) \in F^{r+1}_\alpha \times G^r \text{ such that } \overline{\partial}_J f = g\}$,
- $\Delta^r_\alpha(f, g) = g$.

Note that since V is weakly exact, the symplectic area of $\alpha \in \pi_2(V, L)$ depends only on its image by $\pi_2(V, L) \xrightarrow{\partial} \pi_1(L) \xrightarrow{H} H_1(L; \mathbf{Z})$ where ∂ is the boundary homomorphism and H the Hurewicz map. Observe also that the Maslov index of L along $H\partial(\alpha)$ relative to the Lagrangian distribution $\mathcal{L}$ in V is independent of the choice of $\mathcal{L}$ because $H\partial(\alpha)$ is contractible in V (so we need only know that such a distribution exists). We will refer to this Maslov index as $\mu_L(H\partial\alpha)$ or $\mu_L(\alpha)$ indifferently.

Here, F^{r+1}_α, H^{r+1}_α are again Banach manifolds and $\Delta^r_\alpha : H^{r+1}_\alpha \to G^r$ is a Fredholm map of index $\mathrm{ind}(\Delta^r_\alpha) = \mu_L(\alpha) + n$.

The Poisson formula argument showed that there exists some value $g \in G^r$ such that there is no solution to the equation $\overline{\partial}_J f = g$ with $f \in F^{r+1}_0$. Let γ be a path in G^r from $\gamma(0) = 0$ to $\gamma(1) = g$ transverse to the Fredholm projections $\Delta^r_\alpha : H^{r+1}_\alpha \to G^r$ for all α. The proof above shows that the restriction of Δ^r to $(\Delta^r)^{-1}(\gamma)$ is not proper. Therefore, there exists a sequence $g^t \in \mathrm{Im}\gamma$ converging to $g_\infty \in \mathrm{Im}\gamma$ and a sequence f_t with $\overline{\partial}_J f_t = g_t$ such that the sequence $\{f_t\}$ has no convergent subsequence. By the compactness theorem and lemma 5.2.3, this implies (extracting a subsequence of f_t if needed) that the sequence f_t weakly converges to a union of k_1 non-constant rational curves in V with one disc f_∞ with boundary in L such that $\overline{\partial}_J f_\infty = g_\infty$, and $k_2 - 1$ non-constant holomorphic discs with boundary in L. Since V is weakly exact, we have therefore one disc f_∞ and $k = k_2 - 1 > 0$ holomorphic discs $f_1, \ldots, f_k$ with boundary in L. Denote by $\alpha_\infty, \alpha_1, \ldots, \alpha_k \in \pi_2(V, L)$ the homotopy classes of $f_\infty, f_1, \ldots, f_k$.

5.3.1. Exercise. — Using the index formula, show that the fact that γ is transversal to $\Delta^r_{\alpha_\infty}$ and the fact that $\left(\Delta^r_{\alpha_\infty}\right)^{-1}(g_\infty)$ is not empty together imply that $\mu_L(\alpha_\infty) \geq -n - 1$.

Since the discs $f_\infty, f_1, \dots, f_k$ occur as a weak limit of a contractible disc, we have

$$\alpha_\infty + \alpha_1 + \cdots + \alpha_k = 0,$$

so that $\mu_L(\alpha_\infty) + \sum_{i=1}^k \mu_L(\alpha_i) = 0$. But each disc f_i is holomorphic and non-constant, so their symplectic area $\lambda(\alpha_i)$ is strictly positive, and by monotonicity $\mu_L(\alpha_i) > 0$ for all $1 \leq i \leq k$. Therefore

$$\sum_{i=1}^{k} \mu_L(\alpha_i) \leq n+1$$

implies that $1 \leq \mu_L(\alpha_i) \leq n+1$ for at least one i. This concludes the proof. □

Appendix: Exotic structures on $\mathbf{R}^{2n}$

A.1. Exotic structures

A symplectic form ω on $\mathbf{R}^{2n}$ is *exotic* if there exist *no* symplectic embedding $(\mathbf{R}^{2n}, \omega) \hookrightarrow (\mathbf{R}^{2n}, \omega_0)$ in the standard structure. The fact that the proof of the Arnold conjecture (corollary 2.2.5) implies the existence of exotic structures on $\mathbf{R}^{2n}$ for $n \geq 2$ is now considered to be folklore. Nevertheless, it appeared in the written culture only in 1984 in a paper by Viterbo [48]. It is very surprising that we do not know any other method of constructing exotic structures. There are some variants (see e.g. [11]), but basically, you construct a symplectic structure which admits a closed exact Lagrangian submanifold. We already mentioned such a construction, due to M.-P. Muller in § 3.2. Here we content ourselves[12] with a simple proof of:

THEOREM A.1.1. — *For any $n \geq 2$, there exist on $\mathbf{R}^{2n}$ exotic symplectic structures.*

A.2. Proof of existence

It is enough to construct an immersion $F : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ that sends a given closed Lagrangian submanifold L of $(\mathbf{R}^{2n}, \omega_0)$ onto an immersed exact Lagrangian submanifold W of $(\mathbf{R}^{2n}, \omega_0)$. Indeed, $F^\star\omega_0$ is then such that L is a closed exact Lagrangian submanifold of $(\mathbf{R}^{2n}, F^\star\omega_0)$. Hence $(\mathbf{R}^{2n}, F^\star\omega_0)$ is exotic because a symplectic embedding

$$\varphi : (\mathbf{R}^{2n}, F^\star\omega_0) \to (\mathbf{R}^{2n}, \omega_0)$$

would send a primitive λ of $F^\star\omega_0$ (a form whose exterior derivative is $F^\star\omega_0$) to a form $\varphi^\star(\lambda)$ on Im φ whose derivative is ω_0. We would then have $d(\lambda_0 - \varphi^\star(\lambda)) = 0$ on Im φ and $\lambda_0 = \varphi^\star(\lambda) +$ exact form on Im φ, so that$\lambda_0|_{\varphi(L)}$ would be exact which contradicts the inexistence of a closed exact Lagrangian submanifold of $(\mathbf{R}^{2n}, \omega_0)$.

To construct the immersion $F : (\mathbf{R}^{2n}, L) \to (\mathbf{R}^{2n}, W)$, we take the simplest example of a closed Lagrangian submanifold, that is $L = T^n \subset \mathbf{R}^{2n}$. Let $f_t : S^1 \to$

[12] The construction in [37] is actually harder... but gives something more precise, as the title of the paper shows.

$\mathbf{R}^2$ be a regular homotopy from the standard inclusion to an exact (Lagrangian) immersion $f_1 : S^1 \to \mathbf{R}^2$ (that is f_1 has for instance two double points of opposite signs and the total area enclosed by Im f_1 is algebraically zero). The product of n copies of this regular homotopy gives a regular homotopy $F_t : T^n \to \mathbf{R}^{2n}$. Now, given any closed smooth submanifold X of a closed manifold Y and any manifold Z of dimension larger than dim Y, Smale's h-principle for immersions (see e.g. [25]) implies that the restriction map

$$r : \mathrm{Imm}(Y, Z) \to \mathrm{Imm}(X, Z)$$

between spaces of smooth immersions is a Serre fibration: that is, given any homotopy $H_t : A \to \mathrm{Imm}(X, Z)$ defined on a CW-complex A and any r-lift $\widetilde{H_0} : A \to \mathrm{Imm}(Y, Z)$ of H_0, there exists an r-lift of the whole homotopy, $\widetilde{H_t} : A \to \mathrm{Imm}(Y, Z)$, which coincides with the given lift at time $t = 0$. Let us apply this with $X = T^n$, $Y = S^{2n-1}$, $Z = \mathbf{R}^{2n}$ (note that $n < 2n - 1$ because $n \geq 2$), $A = \{pt\}$, $H_t = F_t : T^n \to \mathbf{R}^{2n}$, and $\widetilde{H_0}$ the standard inclusion. Hence F_t can be extended to a regular homotopy $\widetilde{F_t} : S^{2n-1} \to \mathbf{R}^{2n}$ and so $\widetilde{F_1}$ extends F_1 (this is all we need). Let D^{2n-1} be a disc smoothly embedded in S^{2n-1}, such that T^n lies in D^{2n-1} (such a disc exists because $n < 2n - 1$), and let U be a tubular neighbourhood of D^{2n-1} in $\mathbf{R}^{2n}$. Then the restriction $\widetilde{F_1} : D^{2n-1} \to \mathbf{R}^{2n}$ of $\widetilde{F_1} : S^{2n-1} \to \mathbf{R}^{2n}$ can be extended to an immersion $G : U \to \mathbf{R}^{2n}$. The composition of G with an immersion $\varphi : \mathbf{R}^{2n} \to U$ which is the identity on a smaller disc $D_1^{2n-1} \subset D^{2n-1}$ containing T^n, gives an immersion $G \circ \varphi : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ sending T^n onto an immersed exact Lagrangian submanifold of $(\mathbf{R}^{2n}, \omega_0)$.

Remark. — Note that by theorem 2.2.4, the exotic structure ω on $\mathbf{C}^n$ constructed in the above proof is not symplectomorphic to $(\mathbf{C}^{n-1} \times \mathbf{C}, \omega' \oplus \omega_0)$ for any geometrically bounded factor $(\mathbf{C}^{n-1}, \omega')$.

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