Introduction

Applications of pseudo-holomorphic curves to symplectic topology

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This chapter is an introduction to the book. First we will describe some problems in symplectic geometry, or more exactly topology, and the way to solve them using pseudo-holomorphic curves techniques. Then we describe very roughly the contents of the book. For the basic results in geometry, the reader can consult chapters I, II or III.

Unless otherwise mentioned (i.e. apart from Darboux's theorem), all theorems in this chapter are due to Gromov and come from [9] (see also [10]).

1. Examples of problems and results in symplectic topology

1.1. Flexibility and rigidity in symplectic manifolds

A *symplectic manifold* (W, ω) is a manifold W endowed with a closed nondegenerate 2-form ω. The basic examples are $\mathbf{R}^{2n}$ with the 2-form $\omega_0 = \sum dp_i \wedge dq_i$ and the 2-sphere with its usual volume form (or any orientable surface with any volume form).

Locally, such a structure is very flexible: all such structures are isomorphic. More precisely, if we call a diffeomorphism *symplectic* when it preserves the symplectic structures, the basic form of Darboux's theorem (see chapter I) is:

THEOREM 1.1.1 (Darboux). — *Let (W, ω) be a symplectic manifold of dimension $2n$. Let x be any point in W. There exists a neighborhood U of x and a symplectic diffeomorphism of $\left(U, \omega_{|U}\right)$ into an open subset of $(\mathbf{R}^{2n}, \omega_0)$.*

This shows that there exists no *local* invariant in symplectic geometry like, for instance, the curvature in Riemannian geometry: locally, all symplectic manifolds are alike, and there is no difference between suitable open subsets of the 2-sphere and the plane[1].

[1]This allows us to draw local maps of the earth in which the areas are exact if not the lengths and angles.

Of course, globally, things are quite different. For instance, if W is a $2n$-manifold, $\omega^{\wedge n}$ is a volume form (this is the nondegeneracy condition) and thus, any symplectic diffeomorphism preserves the volume: there is no symplectic diffeomorphism from a large ball into a smaller one in $\mathbf{R}^{2n}$.

A basic question is thus to understand if there is a way to distinguish between volume preserving and symplectic diffeomorphisms. Note that "volume geometry" is very flexible, even globally. On the one hand, we have the Moser theorem, which asserts that two volume forms on a closed manifold are diffeomorphic when they give the same total volume to the manifold. On the other hand, if U and V are connected open subsets of $\mathbf{R}^n$ with $\mathrm{vol}(U) < \mathrm{vol}(V)$, there is an embedding $U \to V$ which preserves the volume form. In contrast, the first result we quote in symplectic geometry is a very spectacular rigidity theorem:

THEOREM 1.1.2. — *Let (W, ω) be a symplectic manifold, and let f be a diffeomorphism of W which is a limit, in the compact-open (C^0) topology, of a sequence of symplectic diffeomorphisms. Then f is symplectic.*

Thus a volume preserving diffeomorphism which is not symplectic cannot be a limit of symplectic diffeomorphisms. According to Arnold, who invented the phrase "symplectic topology", this theorem of Gromov proves the existence of symplectic topology [1].

In the same mood, Gromov defined a *symplectic invariant*, much finer than the volume, the *width*, which prevents, for instance, the embedding of a large ball into an infinite thin cylinder. We consider a ball B_R^{2n+2} of radius R in the standard symplectic space $\mathbf{R}^{2n+2} = \mathbf{R}^2 \oplus \mathbf{R}^{2n}$ and a ball B_r^2 of radius r in the *symplectic* summand $\mathbf{R}^2$:

THEOREM 1.1.3. — *Suppose there exists a symplectic embedding of B_R^{2n+2} into the cylinder $B_r^2 \times \mathbf{R}^{2n}$. Then $R \leq r$.*

Similarly, one cannot embed two disjoint small balls in a large one (not too large, of course, but with volume bigger than the sum of the volumes of the small balls):

THEOREM 1.1.4. — *Let U be an open subset of $\mathbf{R}^{2n}$ which contains two disjoint balls of radii r_1 and r_2. Suppose there exists a symplectic embedding of U into a ball of radius R. Then $r_1^2 + r_2^2 \leq R^2$.*

Remarks.

1. C. Viterbo likes to present theorem 1.1.3 as an "uncertainty principle"... in classical mechanics [25].

2. The inequality in theorem 1.1.4 is called a *packing inequality*. For beautiful results and a discussion of packing inequalities, see [17].

3. Another famous and related problem is that of the *symplectic camel*, whose ability to pass through the eye of a needle is discussed, following St Luke[2] **18** 25, in e.g. [19], [7] and [18].

1.2. Flexibility and rigidity for Lagrangian submanifolds

The next flexibility result is Gromov's h-principle, applied here to Lagrangian immersions. Recall that an immersion $f : L \to W$ is *Lagrangian* if $f^\star\omega = 0$ and $\dim L = \frac{1}{2}\dim W$. From such an f, we can extract two topological data: the homotopy class of the map f (such that $f^\star\omega = 0$) and the Lagrangian subbundle TL of $f^\star TW$. The h-principle (see [11] and chapter X for interesting special cases and other references) asserts that the map which associates these two data with f is a weak homotopy equivalence: everything which is not excluded by homotopy theory is allowed.

Using pseudo-holomorphic curves methods, Gromov proved that the situation for Lagrangian embeddings is quite different. A typical consequence is the proof of the Arnold conjecture. Let $\lambda_0 = \sum p_i dq_i$ be the Liouville form on $\mathbf{R}^{2n}$, so that $\omega_0 = d\lambda_0$. Note that $f^\star\omega_0 = 0 \Leftrightarrow f^\star\lambda_0$ is a closed 1-form, and call *exact* any Lagrangian immersion f such that $f^\star\lambda_0$ is an exact 1-form.

THEOREM 1.2.1 (Arnold conjecture). — *There exists no closed exact Lagrangian submanifold in* $\mathbf{R}^{2n}$.

For $n = 1$, this is a consequence of the theorems of Stokes and Jordan. In higher dimensions, this rigidity theorem implies the existence of *exotic* symplectic structures on $\mathbf{R}^{2n}$ ($2n \geq 4$); it is also related to problems of intersections of Lagrangian submanifolds and of fixed points of symplectic diffeomorphisms: any symplectic diffeomorphism of (W, ω) defines a Lagrangian submanifold, namely its graph in $W \times W$ endowed with the symplectic form $\omega \oplus -\omega$.

All of these things will be discussed in great detail in chapter X, so we shall not spend more time on them here. However, we must mention that a lot of activity related to the above conjecture of Arnold—which actually belongs to a whole set of conjectures, stated by Arnold in the sixties—began a long time before theorem 1.2.1 was proved, and took its roots in "Poincaré's last geometric theorem" (or Poincaré-Birkhoff theorem). Once again, we refer the reader to the paper of Arnold [1], and also of Chaperon [3] and to chapter X and the references therein.

2. Pseudo-holomorphic curves in almost complex manifolds

Although some rigidity results can be proved by variational methods, we will emphasise here methods involving pseudo-holomorphic curves, as initiated by Gro-

[2]See also Mat. **19** 24, Mk **10** 25.

mov in [9]. It might seem surprising, but we know of no other way to prove the Arnold conjecture or the existence of exotic structures.

Basically, what happens is the following:

1. Given a symplectic manifold (W, ω), there are plenty of *tame almost complex structures*, that is: sections of $\mathrm{End}(TW)$ such that $J^2 = -\mathrm{Id}$ and such that $\omega(x, Jx)$ is positive on $TW - 0$. But (except in the case of a Kähler structure), there is no natural choice for such a J. The basic idea of Gromov is to consider the whole family of these tame almost complex structures.

2. For a generic almost complex structure, there are no holomorphic functions (even locally), but there are many local pseudo-holomorphic curves. Moreover, many classical properties of ordinary holomorphic curves still hold: see § 2.2 below and chapters V and VI respectively for analytical and geometric properties.

3. In many cases, if there are compact pseudo-holomorphic curves for *some* almost complex structure J_0 tamed by a symplectic form ω, there are such curves for *any* tame symplectic structure J. For instance, the properties of pseudo-holomorphic curves in $\mathbf{CP}^2$ or $\mathbf{CP}^1 \times \mathbf{CP}^1$ for any almost complex structure tamed by the usual (Kähler) symplectic form mimic the classical properties of algebraic curves (holomorphic for the usual complex structure): see theorems 2.2.3 and 2.2.4 below.

2.1. Almost complex structures and pseudo-holomorphic curves

As we mentioned above, an *almost complex* structure on a manifold W is a field of endomorphisms J (a section of $\mathrm{End}(TW)$) such that $J^2 = -\mathrm{Id}$. For example, a complex manifold carries a natural almost complex structure. Namely, if $(z_1, \ldots, z_n)$ is a local holomorphic chart on an open set U and if we write $z_k = x_k + iy_k$, J is given by

$$J\frac{\partial}{\partial x_k} = \frac{\partial}{\partial y_k} \text{ and } J\frac{\partial}{\partial y_k} = -\frac{\partial}{\partial x_k}.$$

However, "most" almost complex structures are not defined by a complex (or holomorphic, or analytic, or *integrable*) structure (see chapter II). In other words, an almost complex structure J is usually not integrable. To see this, define *holomorphic functions* as smooth functions $f : W \to \mathbf{C}$ such that $T_x f \circ J = iT_x f$. For a generic almost complex structure, there are *no* holomorphic functions, even locally.

On the contrary, there are always many pseudo-holomorphic *curves*. A *pseudo-holomorphic curve* f is a map from a Riemann surface S into W such that $J \circ Tf_x = Tf_x \circ i$ (the same compatibility condition as above, but now W is the target space). In local coordinates, this amounts to the differential system

$$\frac{\partial f_j}{\partial y} = \sum_{k=1}^{2n} J_{j,k}(f)\frac{\partial f_k}{\partial x} \qquad (1 \le j \le 2n)$$

of $2n$ equations for $2n$ unknown functions.

If J is analytic, the Cauchy-Kowalevskaya theorem implies that there are (local) analytic solutions. In the smooth case, this is still true, since the differential system is a quasi-linear elliptic system: the system is linear with respect to higher derivatives; when linearised, it gives the usual Cauchy-Riemann equations, which are elliptic. Therefore, pseudo-holomorphic curves enjoy the properties which are expected of the solutions of an elliptic system: regularity properties coming from Sobolev and/or Hölder estimates (chapter V) and also a removable singularity theorem (chapter VII).

To save on typing, we shall often simply write a *J-curve* for a pseudo-holomorphic curve.

Now, any symplectic manifold will have a lot of complex structures, related to the symplectic form in the following way: an almost complex structure J on a manifold W is *tamed* by a symplectic form ω if

$$\omega(X, JX) > 0 \quad \forall X \in TW - 0;$$

if moreover ω is J-invariant, J is said to be *calibrated.*

PROPOSITION 2.1.1. — *The space of almost complex structures on a given symplectic manifold (W, ω) which are tamed (resp. calibrated) by ω is nonempty and contractible. In particular, these spaces are connected.*

See chapter II for details and for a proof of the proposition. Note that, if J is an ω-tame almost complex structure, then

$$\mu(X, Y) = \omega(X, JY) - \omega(JX, Y)$$

is a J-invariant Riemannian metric.

The following elementary property of tame structures turns out to be crucial when studying pseudo-holomorphic curves.

PROPOSITION 2.1.2. — *Let C be a pseudo-holomorphic curve in a closed symplectic manifold (W, ω) equipped with a tame almost complex structure, and let μ be a J-invariant metric. There is a universal bound*

$$\text{area}_\mu(C) \leq A(\omega, J, \mu, [C])$$

involving the geometry of W and the homology class $[C]$ of C.

Proof. — Using compactness, we see there exists some constant $M > 0$ such that

$$\mu(X, JX) \leq M\omega(X, JX) \quad \forall X \in TW.$$

Then

$$\text{area}_\mu(C) = \int_C v_\mu \leq M \int_C \omega.$$

But

$$\int_C \omega = \langle [\omega], [C] \rangle. \square$$

An important consequence of this easy proposition is that a closed pseudo-holomorphic curve is never a boundary.

Remark. — This result is a coarse generalisation of a classical property of compact Kähler manifolds, namely: any Kähler submanifold is volume minimising in its homology class (Wirtinger inequality, see chapter III).

Another basic property of holomorphic curves persists in the pseudo-holomorphic case, namely positivity of intersections. Indeed, one easily sees that, when two smooth pseudo-holomorphic curves in a 4 dimensional almost complex manifold (W, J) meet transversally, the intersection of their homology classes is positive as in the holomorphic case. M. Gromov pointed out and D. McDuff proved (see chapter VI) that the same positivity property holds without the transversality or smoothness assumptions.

2.2. Some existence theorems for pseudo-holomorphic curves

Gromov discovered a general procedure for proving the existence of J-holomorphic curves with certain properties when J is a tame almost complex structure on a manifold which carries some model structure J_0 for whose holomorphic curves the analogous properties are well understood.

Kähler structures. — A good "model" structure might be a Kähler structure: a *Kähler metric* on a manifold W is given by a Riemannian metric g, an integrable complex structure J compatible with J (i.e. J is an isometry of g), such that the skew form

$$\omega(X, Y) = g(JX, Y)$$

is closed. The form ω is then called *the Kähler form.* The condition $\omega(X, JX) > 0$ forces the nondegeneracy of ω, thus a Kähler form is symplectic and tames the complex structure. For more details, see e.g. [20], [13] and chapter II.

There are some basic examples. The first one is, of course, $\mathbf{R}^{2n}$ identified with $\mathbf{C}^n$ and equipped with its standard metric and complex structure. The Kähler form is just the standard symplectic form. Also, in dimension 2, every J is integrable (see chapter II for a discussion and references) and giving J is equivalent, once an orientation is prescribed, to giving a (pointwise) conformal class of Riemannian metrics: Kähler 2-dimensional manifolds are just Riemann surfaces.

The projective space. — The next example is the complex projective space $\mathbf{CP}^n$. Consider the Hopf fibration

$$h : S^{2n+1} \to \mathbf{CP}^n$$

which assigns to $Z = (z_0, \ldots, z_n) \in S^{2n+1} \subset \mathbf{C}^{n+1}$ the point with homogeneous coordinates Z in $\mathbf{CP}^n$. In other words, $h(Z)$ is the line in $\mathbf{C}^{n+1}$ generated by the vector Z, and $\mathbf{CP}^n$ is obtained as the quotient of S^{2n+1} by the S^1-action $(u, Z) \mapsto uZ$. There is a natural Riemannian metric on $\mathbf{CP}^n$: take the standard (induced) metric on the sphere, for which the S^1-action is by isometries, and decide that h is a Riemannian submersion.

2.2.1. Exercise. — Use Fubini's theorem to show that the volume of S^{2n+1} is $2\pi \mathrm{vol}(\mathbf{CP}^n)$. Compute $\mathrm{vol}(\mathbf{CP}^n)$ and show in particular that $\mathrm{vol}(\mathbf{CP}^1) = \pi$. Deduce that, for $n = 1$, the procedure gives the 2-sphere, equipped with a metric of constant curvature 4. Give another (direct) proof of this last fact.

Now the normal space to the sphere at Z is the real line generated by Z and the tangent space to the S^1-orbit is the real line generated by iZ, so that, taking quotients, we get an almost complex structure on $\mathbf{CP}^n$, which is, of course, the one associated with the analytic structure.

Now, let α be the 1-form on S^{2n+1} given by $\alpha_Z(X) = \langle JZ, X \rangle$. Then $d\alpha = h^\star \omega$, where ω is the 2-form on $\mathbf{CP}^n$ given by

$$\omega(X, Y) = g(JX, Y).$$

Thus we have all the necessary structures and the following exercise is straightforward.

2.2.2. Exercise. — Show that g is a Kähler metric.

Remark. — This way of describing the 2-form ω is an example of the basic *symplectic reduction* process, see chapter II.

Curves in "pseudo"-projective spaces. — We can now state two results which are derived from the basic examples of $\mathbf{CP}^2$ and $\mathbf{CP}^1 \times \mathbf{CP}^1$ equipped with the Kähler structures just described[3].

THEOREM 2.2.3. — *Let J be an almost complex structure tamed by the (standard) symplectic form ω on $\mathbf{CP}^2$. Then*

a) If x and y are two distinct points, there exists one and only one J-curve homologous to $\mathbf{CP}^1 \subset \mathbf{CP}^2$ (in other words: of degree 1) through them.

b) If $(x_1, \ldots, x_5)$ are five points such that no three of them lie on a degree 1 J-curve, there is one and only one degree 2 J-curve through them.

In other words, the situation of "pseudo-lines" or "pseudo-conics" in a "pseudo-plane" is the same as that of lines and conics in a plane. In the same way, the following result mimics existence theorems for algebraic curves in $\mathbf{CP}^1 \times \mathbf{CP}^1$ (see [22]).

THEOREM 2.2.4. — *Let J be a tame almost complex structure on $\mathbf{CP}^1 \times \mathbf{CP}^1 = S^2 \times S^2$ equipped with the standard (product) symplectic structure.*

a) Any point (x, y) is contained in a unique J-curve homologous to $S^2 \times \star$.

b) If $a \geq 0$, $b \geq 0$ and $a + b > 0$, the class $a[S^2 \times \star] + b[\star \times S^2]$ is represented by a J-curve.

Now we give a statement for higher dimensions. Here the model situation is that of $S^2 \times V$, equipped with a split symplectic form and a split almost complex structure. We assume that V is *aspherical* (for 2-spheres), namely that $\pi_2(V) = 0$.

THEOREM 2.2.5. — *Let (V, σ) be a compact aspherical symplectic manifold and let J be an almost complex structure on $\mathbf{CP}^1 \times V$ tamed by the product symplectic structure $\omega \oplus \sigma$. Then any point (x, v) of $\mathbf{CP}^1 \times V$ is contained in exactly one J-curve homologous to the fibre $\mathbf{CP}^1 \times \{v\}$ of v.*

The proofs of these results go as follows. Let (W, ω) be one of the symplectic manifolds in the statements ($\mathbf{CP}^2$, $\mathbf{CP}^1 \times \mathbf{CP}^1$ or $\mathbf{CP}^1 \times V$). Consider the set C of pairs (f, J) where J is a tame almost complex structure and f a J-sphere in W in the given homology class. Then C can be given the structure of a manifold modelled on a suitable chain of Hölder or Sobolev spaces (see [12] and [2]).

Let p be the projection of C into the space $\mathcal{J}$ of all tame complex structures. The main point is to show that p is surjective. As is usual in this kind of nonlinear problem, one shows that $p(C)$ is open, closed and nonempty (we already know from 2.1.1 that $\mathcal{J}$ is connected).

1. $p(C)$ is not empty: in the first two cases, take the standard complex structure and algebraic curves, in the third case, use any tame split structure.

[3] Of course, there is a natural Kähler structure on the product of two Kähler manifolds.

2. $p(C)$ is open: basically, one shows that the linearisation of p is surjective by combining a vanishing theorem and an index computation (see [9], [14]) involving the Riemann-Roch theorem (see chapter IV).

3. p is proper, therefore $p(C)$ is closed. As usual, this is the most difficult part. Here, one proves first an equicontinuity property (see chapters V, VII and VIII) which comes from a suitable generalisaton of the Schwarz lemma for holomorphic functions. The main difficulty is to control the possible collapsing of the curves. For instance a sequence of conics may converge to a degenerate conic (i.e. two projective lines intersecting at a point) in the same homology class. In the same way, a sequence of curves of degree[4] $(1,1)$ in $\mathbf{CP}^1 \times \mathbf{CP}^1$ may converge to $S^2 \times \{x\} \cup \{x\} \times S^2$. Indeed, this type of difficulty already occurs in the proof of the conformal mapping theorem (of which assertion (b) of theorem 2.2.4 is some kind of generalisation). It can be proved (see chapter VIII or [26]) that such degeneracies never occur for simple homotopy classes, i.e. classes which do not admit a nontrivial decomposition as a sum of classes containing J-curves: for instance, the class of a conic in $\mathbf{CP}^2$ is not simple, as it can be written as twice the class of a line, the class of the line is simple, as are the degree $(1,0)$ class in $S^2 \times S^2$ and the class of $S^2 \times V$ with the assumptions of theorem 2.2.5.

3. Proofs of the symplectic rigidity results

We apply these existence results to prove the rigidity theorems of § 1.1.

3.1. Symplectic width

We begin by the proof of theorem 1.1.3. Suppose φ is an embedding of the ball B_R^{2n+2} into the cylinder $B_r^2 \times \mathbf{R}^{2n}$. Consider the smaller concentric balls $B_{R'} = B_{R'}^{2n+2}$ for $R' < R$. Since $\varphi\left(\bar{B}_{R'}\right)$ is compact, there is a symplectic embedding of $B_{R'}$ in $B^2(r) \times T^{2n}$: take $T^{2n} = \mathbf{R}^{2n}/L\mathbf{Z}^n$ for L big enough, one infers a symplectic embedding of $B_{R'}$ into $B^2(r) \times T^{2n}$, using the

LEMMA 3.1.1. — *The ball $B^2(r)$ and the sphere of same volume with one point removed, equipped with their standard symplectic structures, are symplectomorphic.* □

Take $R'' < R'$. Let J_0 be the standard almost complex structure of $\mathbf{R}^{2n+2}$. Then, using proposition 2.1.1 we see that the almost complex structure $\varphi_\star J_0$ on $\varphi\left(\bar{B}_{R''}\right)$ can be extended to $S^2(r/2) \times T^{2n}$ as an almost complex structure tamed by $\omega_0' + \omega_0''$. Now we can use theorem 2.2.5: there exists a J-holomorphic curve C homologous

[4] A curve has degree (p,q) if its projections onto the two factors have respective degrees p and q. Equivalently, it is in the same homology class as $p\mathbf{CP}^1 \times \star + q \star \times \mathbf{CP}^1$.

to $S^2 \times \star$ and containing $\varphi(0)$. One has

$$\int_C \omega_0' + \omega_0'' = \pi r^2.$$

If $C' = C \cap \varphi\left(\bar{B}_{R''}\right)$, we have

$$\pi r^2 \geq \int_{C'} \omega_0' + \omega_0'' = \int_{\varphi^{-1}(C')} \omega_0$$

since φ is symplectic. But $\varphi^{-1}(C')$ is a J_0-holomorphic curve through 0, whose boundary is contained in $S_{R''}$. Using the monotonicity lemma for minimal surfaces (see chapter III), we have

$$\operatorname{area}\left(\varphi^{-1}(C')\right) = \int_{\varphi^{-1}(C')} \omega_\circ \geq \pi (R'')^2.$$

Finally, since R'' and R' are arbitrary, one infers that $R \leq r$. □

The arguments for theorem 1.1.4 are very similar, but use 2.2.3. We can suppose that U is symplectically embedded in the standard complex projective space, using the following equivariant version of 3.1.1:

LEMMA 3.1.2. — *The open sets $B_r^{2n} \subset \mathbf{R}^{2n}$ and $\mathbf{CP}^n - \mathbf{CP}^{n-1}$, equipped with their standard symplectic structures, are symplectomorphic, if they have the same total volume.* □

Now we have a symplectic embedding of V into $\mathbf{CP}^2$ equipped with the $U(3)$-invariant symplectic form such that the total volume is $\pi^2 R^4/2 = \operatorname{vol}(B_R^4)$. Then we can proceed just as in the previous theorem. If $B(a, R_1)$ and $B(b, R_2)$ are balls contained in U, push forward the standard complex structure J_0 of $\mathbf{R}^4$ to $(B(a, R_1')) \cup \varphi(B(b, R_2'))$ (for some $R_1' < R_1$ and $R_2' < R_2$, and extend it to $\mathbf{CP}^2$ as a tame almost complex structure J. Let C be a J-holomorphic curve through $\varphi(a)$ and $\varphi(b)$, homologous to $\mathbf{CP}^1$. Then area$(C) = \pi R^2$, and the same area comparison argument applies.

3.2. Rigidity of symplectic diffeomorphisms

The rigidity theorem 1.1.2 is a simple corollary of Darboux's theorem and of the next result:

THEOREM 3.2.1. — *Let $(f_k)_{k \in \mathbf{N}}$ be a sequence of symplectic embeddings of the ball $B(0, R)$ into $\mathbf{R}^{2n}$. If f_k converges for the C^0-topology to a map f which is differentiable at 0, then the linear map $f'(0)$ is symplectic.*

The proof relies on a simple but tricky linear algebra lemma, due to Y. Eliashberg:

LEMMA 3.2.2. — *Let $L \in SL(2n, \mathbf{R}^{2n})$. Equip $\mathbf{R}^{2n}$ with its standard symplectic struture. If L is neither symplectic nor anti-symplectic, there exists a linear symplectic map S and a symplectic basis $(e_1, f_1, \ldots, e_n, f_n)$ such that*

1. *modulo* $\operatorname{Vect}(e_2, \ldots, f_n)$, *one has* $(L \circ S)(e_1) = \lambda e_1$ *and* $(L \circ S)(f_1) = \lambda f_1$, *with* $|\lambda| < 1$,
2. *$L \circ S$ leaves the symplectic orthogonal to $\mathbf{R}e_1 \oplus \mathbf{R}f_1$ globally invariant.*

Proof. — There is a pair (e, f) of vectors such that

$$|\omega(e, f)| = 1$$

and

$$|\omega(Le, Lf)| = \lambda^2 < 1.$$

Indeed, since L est unimodular, $|\omega(Le, Lf)|$ cannot be always strictly bigger than 1.

Let S be a symplectic map such that $S(Le) = \lambda e$ and $S(Lf) = \lambda f$. Now, to get the result, just transpose everything (symplectically!). □

Proof of the theorem. — Since the f_k are symplectic embeddings, they preserve Lebesgue measure. This property is preserved by $\mathcal{C}^0$-limits. We claim that $f'(0)$ is unimodular. Indeed, by composing the f_k on the left and on the right with linear unimodular maps, one is reduced to the case when $f'(0)$ is diagonal with respect to the standard basis. The claim follows by comparing f to $f'(0)$.

Now, it is enough to prove that $f'(0)$ is symplectic or anti-symplectic: using the same argument for the sequence $g_k = f_k \times \mathrm{Id}_{\mathbf{R}^2}$, we see that $f'(0) \times \mathrm{Id}_{\mathbf{R}^2}$ will be also symplectic or anti-symplectic.

Taking coordinates with respect to the symplectic basis of the lemma, and working in a neighbourhood of 0, we have:

$$(f \circ S)(x_1, x_2, \ldots, x_{2n}) = (\lambda x_1, \lambda x_2, \ldots) + o(|x|).$$

(Don't bother about the coordinates of $f \circ S$ after the second one!). Take the Euclidean metric defined by this basis. If ε is small enough, we can find $\varepsilon' < \varepsilon$ such that $f \circ \mathrm{S}$ sends the ball $B^{2n}(0, \varepsilon)$ into $B^2(0, \varepsilon') \times \mathbf{R}^{2n-2}$. Then the same thing is true for $f_k \circ S$, provided that k is big enough. This contradicts theorem 1.1.3.

4. What is in the book... and what is not

4.1. Contents of the book

We begin with two chapters introducing symplectic geometry and almost complex manifolds. The reader will find the very basic definitions and results, a lot of examples (how can one construct symplectic manifolds?) and exercises.

The next chapter is devoted to some facts in Riemannian geometry which will be useful, either directly in the proofs below, or simply to understand what is going on. Chapter IV is a panorama of the theory of linear connections and Chern classes, culminating in the Hirzebruch-Riemann-Roch theorem.

Then we get to the technical part. Chapter V is devoted to analytic aspects of the theory of pseudo-holomorphic curves, chapter VII gives a proof of Gromov's version of the classical Schwarz lemma, which is the main tool in the proof of the Gromov compactness theorem, which is in turn the subject of chapter VIII. We have already explained above how this theorem can be used to prove the existence of (global) pseudo-holomorphic curves satisfying given properties. In chapter IX, pseudo-holomorphic curves are shown to appear also in Riemannian geometry. Chapter VI explains some geometrical aspects of pseudo-holomorphic curves: just as two holomorphic curves in analytic surfaces, two pseudo-holomorphic curves in almost complex manifolds have *positive* intersection.

Chapter X is a panoramic view of Lagrangian submanifolds and related subjects. We have tried to explain all aspects of the situation: homotopic, topologic, and "hard" (that is: using pseudo-holomorphic curves techniques) results and to give as many examples as we know. All of the results evoked in § 1.2 and their interrelationships will be proved—usually in a more general setting.

4.2. What could have been in the book

There is no complex analysis in this book. By this we mean results like filling by holomorphic discs, rational convexity of Lagrangian tori and so forth, as appear in the work of Eliashberg [6] and Duval [4], [5] for instance.

Related to this, there is no contact geometry, although there are many interesting new results in this theory. We refer the reader to the beautiful survey talk by Giroux [8] and the references he gives.

There are no infinite dimensional Morse theory techniques although results like the distinction between the thin cylinder and the sphere above (theorem 1.1.3) would have fitted very well[5] with them. Here, good references are [24] and [21].

But this is a book about pseudo-holomorphic curves. There are also some results closely related with pseudo-holomorphic curves which are also not in the book. For example, a whole set of results on the classification of 4-dimensional symplectic manifolds, which constitute one of the beautiful achievements of the theory, have been obtained by D. McDuff [15] and [16] in the last few years. They do not appear explicitly here, but we hope the reader of the present book will be able to read the original papers.

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