

Part A

Lagrangian Submanifolds

Michèle Audin

Introduction

This text is an introduction to Lagrangian and special Lagrangian submanifolds. Special Lagrangian submanifolds were invented twenty years ago by Harvey and Lawson [18]. They have become very fashionable recently, after the work of McLean [25], leading to the beautiful speculations of Strominger, Yau and Zaslow [32] and the remarkable papers of Hitchin [19, 20] and Donaldson [11].

My aim here is mainly to present as many examples as possible. I have taken some time to explain why we know so many Lagrangian and so few special Lagrangian submanifolds and immersions. There are mainly two reasons:

- To be Lagrangian is, eventually, a *linear* property. On the other hand, the property to be special Lagrangian is, in dimension 3 and more, non linear.
- The moduli space of Lagrangian submanifolds that are close to a given one is an infinite dimensional manifold, while the corresponding moduli space of special Lagrangian submanifolds is finite dimensional.

This will be apparent in the number and nature of the examples I describe in these notes.

To prepare these lectures, in addition to the papers mentioned above, I have used standard textbooks on manifolds and vector fields as [22], on symplectic geometry as [4, 7, 24, 30] and on complex manifolds and Hodge theory as [8, 15].

I have used standard notation but, although this text pretends to be written in English, I have kept a preference for (transparent) French standards, for instance $\mathbf{P}^n(\mathbf{K})$ for the projective space of dimension n over the field \mathbf{K} and tA for the transpose of a matrix A .

I thank Étienne Mann, Édith Socié, Thomas Vogel and Jean-Yves Welschinger for their comments and their help during the preparation of these notes. Special thanks to Mihai Damian, Alicia Jurado and Sébastien Racanière.

Chapter I

Lagrangian and special Lagrangian immersions in \mathbf{C}^n

In this chapter, I define Lagrangian and special Lagrangian immersions in \mathbf{C}^n . To begin with, I explain that \mathbf{C}^n is the standard *real* vector space endowed with a non degenerate alternated bilinear form (§I.1) and use this “symplectic structure” to define Lagrangian subspaces and immersions (§§ I.2, I.3 and I.4). Later, I use the complex structure as well, to define *special* Lagrangian immersions (§I.5).

I.1 Symplectic form on \mathbf{C}^n , symplectic vector spaces

I.1.a Symplectic vector spaces

Consider the vector space \mathbf{C}^n with the Hermitian form

$$\langle Z, Z' \rangle = \sum_{j=1}^n \bar{Z}_j Z'_j$$

(note that it is anti-linear in the first entry and linear in the second). Decompose it in real and imaginary parts:

$$\langle Z, Z' \rangle = (Z, Z') - i\omega(Z, Z').$$

The real part is the standard scalar product (Euclidean structure) of $\mathbf{C}^n = \mathbf{R}^n \times \mathbf{R}^n$,

$$(Z, Z') = \sum_{j=1}^n (X_j X'_j + Y_j Y'_j) = X \cdot X' + Y \cdot Y',$$

a symmetric non degenerate (real) bilinear form. The imaginary part defines a (real) bilinear form

$$\omega = \sum_{j=1}^n (X'_j Y_j - X_j Y'_j) = X' \cdot Y - X \cdot Y'$$

that is *alternated*, this meaning that $\omega(Z, Z) = 0$ for all Z . Equivalently, ω is skew-symmetric, that is,

$$\omega(Z', Z) = -\omega(Z, Z').$$

To write these formulas, I have decomposed the complex vectors of \mathbf{C}^n as

$$Z = X + iY, \quad X, Y \in \mathbf{R}^n$$

and I have used the scalar product $X \cdot Y$ of \mathbf{R}^n . The form ω is non degenerate too, as

$$\omega(X, Y) = 0 \text{ for all } Y \Rightarrow X = 0.$$

More generally, on a real vector space E , a *symplectic form* is a non degenerate alternated bilinear form. A vector space endowed with a symplectic form is said to be a *symplectic vector space*.

I.1.b Symplectic bases

Fix a complex unitary basis (e_1, \dots, e_n) of \mathbf{C}^n . Put $f_j = -ie_j$, so that

$$(e_1, \dots, e_n, f_1, \dots, f_n)$$

is a basis of the *real* vector space \mathbf{C}^n . Compute ω on the vectors of this basis:

$$\omega(e_i, e_j) = \text{Im}\langle e_i, e_j \rangle = \text{Im} \delta_{i,j} = 0,$$

also

$$\omega(f_i, f_j) = \text{Im}\langle ie_i, ie_j \rangle = \text{Im}\langle e_i, e_j \rangle = 0$$

and eventually

$$\omega(e_i, f_j) = \text{Im}\langle e_i, -ie_j \rangle = \text{Re}\langle e_i, e_j \rangle = \delta_{i,j}.$$

Inspired by these properties, we say that a basis $(e_1, \dots, e_n, f_1, \dots, f_n)$ of a symplectic vector space is a *symplectic basis* if

$$\omega(e_i, f_j) = \delta_{i,j} \text{ and } \omega(e_i, e_j) = \omega(f_i, f_j) = 0 \text{ for all } i \text{ and } j.$$

There are symplectic bases in all symplectic spaces, thanks to the following proposition.

Proposition I.1.1. *Let ω be a symplectic form on a finite dimensional vector space E . There exists a basis $(e_1, \dots, e_n, f_1, \dots, f_n)$ of E such that $\omega(e_i, f_j) = \delta_{i,j}$ and $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$.*

Proof. As ω is non degenerate, it is not identically zero so that one can find two vectors e_1 and f_1 such that $\omega(e_1, f_1) = 1$. One then checks that the restriction of ω to the orthogonal complement (with respect to ω) of the plane $\langle e_1, f_1 \rangle$ is non degenerate. One eventually concludes by induction on the dimension — once noticed that an alternated bilinear form on a 1-dimensional vector space is zero. \square

In particular, the dimension of E is an even number and this is the only invariant of the isomorphism type of (E, ω) . If E has dimension $2n$, then E with its symplectic form is isomorphic to \mathbf{C}^n with the form ω . This result can be called a “linear Darboux theorem”, in reference with the forthcoming (Darboux) theorem II.3.6.

More generally, an alternated bilinear form has a *rank*, that is the dimension of the largest subspace on which it is non degenerate, and is an even number.

Matrices

In a symplectic basis, the matrix of the symplectic form is

$$J = \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix}.$$

Notice that the matrix J satisfies

$$J^2 = -\text{Id}.$$

As the matrix of an endomorphism, this is a *complex structure*. In the symplectic basis of \mathbf{C}^n associated with the canonical (complex) basis (e_1, \dots, e_n) , J is nothing other than the matrix of multiplication by i .

I.1.c The symplectic form as a differential form

One can write ω as a differential form

$$\omega = \sum_{j=1}^n dy_j \wedge dx_j.$$

This is an *exact* differential form (the differential of a degree 1-form):

$$\omega = d\left(\sum_{j=1}^n y_j dx_j\right) = d(Y \cdot X).$$

The form $\lambda = Y \cdot dX$ is called *Liouville form* (see §II.1 below).

I.1.d The symplectic group

This is the group of isometries of ω . A transformation g of \mathbf{C}^n is *symplectic* if it satisfies

$$\omega(gZ, gZ') = \omega(Z, Z') \text{ for all } Z, Z' \in \mathbf{C}^n.$$

Call $\text{Sp}(2n)$ the symplectic group of the space \mathbf{C}^n of dimension $2n$. Consider all the groups $\text{O}(2n)$, $\text{GL}(n; \mathbf{C})$, $\text{U}(n)$ and $\text{Sp}(2n)$ as subgroups of $\text{GL}(2n; \mathbf{R})$.

Proposition I.1.2. *The following equalities hold*

$$\text{Sp}(2n) \cap \text{O}(2n) = \text{Sp}(2n) \cap \text{GL}(n; \mathbf{C}) = \text{O}(2n) \cap \text{GL}(n; \mathbf{C}) = \text{U}(n).$$

Proof. Let us characterize our subgroups of $\text{GL}(2n; \mathbf{R})$:

- (1) $g \in \text{GL}(n; \mathbf{C})$ if and only if g is \mathbf{C} -linear, that is, if and only if

$$g(iZ) = ig(Z) \text{ for all } Z.$$

For a matrix A , this is to say that $AJ = JA$.

- (2) $g \in \text{Sp}(2n)$ if and only if g preserves ω , that is, if and only if $\omega(gZ, gZ') = \omega(Z, Z')$ for all Z and Z' . For a matrix A , this is

$${}^tAJA = J.$$

- (3) $g \in \text{O}(2n)$ if and only if $(gZ, gZ') = (Z, Z')$. For a matrix A , this is ${}^tAA = \text{Id}$.

One then checks that two of these conditions imply the third:

- (2) and (3) imply that

$$\langle gZ, gZ' \rangle = \langle Z, Z' \rangle$$

thus that $g \in \text{U}(n) \subset \text{GL}(n; \mathbf{C})$.

- (3) and (1) imply that

$$\omega(gZ, gZ') = \omega(gZ, -ig(iZ')) = (gZ, g(iZ')) = (Z, iZ') = \omega(Z, Z')$$

thus that $g \in \text{Sp}(2n)$.

- in the same way, (1) and (2) imply (3).

In matrix terms, the intersection $\text{Sp}(2n) \cap \text{O}(2n)$ is the set of matrices

$$\begin{pmatrix} U & -V \\ V & U \end{pmatrix} \in \text{GL}(n; \mathbf{C}) \subset \text{GL}(2n; \mathbf{R})$$

such that

$$\begin{cases} {}^tUV = {}^tVU \\ {}^tUU + {}^tVV = \text{Id}. \end{cases}$$

This is exactly the condition that $U + iV$ be a unitary matrix. \square

I.1.e Orthogonality, isotropy

Write F^\perp for the Euclidean orthogonal of the real subspace F of \mathbf{C}^n and F° for its symplectic (that is, with respect to ω) orthogonal. As ω is non degenerate, one has

$$(F^\circ)^\circ = F \quad \text{and} \quad \dim F + \dim F^\circ = 2n = \dim_{\mathbf{R}} \mathbf{C}^n.$$

Notice however that a subspace and its orthogonal may have a non trivial intersection. The restriction of the non degenerate form ω to a subspace is not always a non degenerate form, in contradiction with what happens in the Euclidean case (which is due to the positivity of the scalar product). In other words, all the subspaces of a symplectic space do not have the same behaviour with respect to the symplectic form. See Exercises I.6 and I.7.

One says that a subspace F is *isotropic* if $F \subset F^\circ$, *co-isotropic* if $F \supset F^\circ$. For instance, a (real) line is always isotropic, as it lies in its orthogonal which is a (real, co-isotropic) hyperplane. Notice that F is isotropic if and only if F° is co-isotropic. Notice also that the dimension of an isotropic subspace is at most equal to n , half the dimension of \mathbf{C}^n .

I.2 Lagrangian subspaces

I.2.a Definition of Lagrangian subspaces

The isotropic subspaces of maximal dimension n are *Lagrangian*. For instance, $\mathbf{R}^n \subset \mathbf{C}^n$ is a Lagrangian subspace. More generally, a subspace generated by “one half” of a symplectic basis is Lagrangian. Conversely, if F is an isotropic subspace of dimension $k \leq n$, it is possible to complete any basis (e_1, \dots, e_k) of F in a symplectic basis and thus to obtain Lagrangian subspaces containing F .

Let us use now the complex multiplication in \mathbf{C}^n to state:

Lemma I.2.1. *A real subspace P of \mathbf{C}^n is Lagrangian if and only if $P^\perp = iP$.*

Proof. This is a straightforward computation:

$$\begin{aligned} \omega(Z, Z') = 0 &\Leftrightarrow \operatorname{Im}\langle Z, Z' \rangle = 0 \\ &\Leftrightarrow \operatorname{Re}\langle Z, iZ' \rangle = 0 \\ &\Leftrightarrow \langle Z, iZ' \rangle = 0. \end{aligned}$$

□

Lemma I.2.2. *Let P be a Lagrangian subspace of \mathbf{C}^n and let (x_1, \dots, x_n) be an orthonormal basis of this real subspace. Then (x_1, \dots, x_n) is a complex unitary basis of \mathbf{C}^n . Conversely, if (x_1, \dots, x_n) is a unitary basis of \mathbf{C}^n , the real subspace it spans is Lagrangian.*

Proof. If (x_1, \dots, x_n) is an orthonormal basis of the Lagrangian P , the previous lemma says that the basis $(x_1, \dots, x_n, ix_1, \dots, ix_n)$ is an orthonormal basis of the real space \mathbf{C}^n , thus that (x_1, \dots, x_n) is a complex basis of \mathbf{C}^n . Moreover, one has

$$\langle x_i, x_j \rangle = (x_i, x_j) - i\omega(x_i, x_j) = \delta_{i,j} - 0,$$

thus this is a unitary basis. The converse is even more obvious. \square

I.2.b The symplectic reduction

This is a simple but useful operation, essentially contained in the next lemma.

Lemma I.2.3. *Let P be a Lagrangian subspace and F be a co-isotropic subspace of \mathbf{C}^n , such that*

$$P + F = \mathbf{C}^n.$$

Then the restriction of the projection

$$P \cap F \subset F \longrightarrow F/F^\circ$$

is injective, the space F/F° is symplectic and the image of $P \cap F$ is a Lagrangian subspace.

Proof. The symplectic form of \mathbf{C}^n clearly induces a non degenerate form on F/F° , as F° is the kernel of the restriction of ω to F . The kernel of the composition

$$P \cap F \subset F \longrightarrow F/F^\circ$$

is

$$\begin{aligned} P \cap F \cap F^\circ &= P \cap F^\circ && F \text{ being co-isotropic, } F \supset F^\circ \\ &= (P^\circ + F)^\circ, && \text{since } (A + B)^\circ = A^\circ \cap B^\circ, \\ &= (P + F)^\circ && \text{as } P \text{ is Lagrangian, } P = P^\circ \\ &= (\mathbf{C}^n)^\circ && \text{because } P + F = \mathbf{C}^n \\ &= 0 && \text{as } \omega \text{ is non degenerate.} \end{aligned}$$

The map is thus injective. Eventually $P \cap F$ is isotropic and has dimension

$$\dim P \cap F = \dim P + \dim F - \dim(P + F) = \dim F - n,$$

half the dimension of the symplectic space F/F° , that is

$$\dim F/F^\circ = \dim F - (2n - \dim F) = 2(\dim F - n).$$

\square

See more generally Exercise I.9.

I.3 The Lagrangian Grassmannian

We consider now the set Λ_n of all Lagrangian subspaces of \mathbf{C}^n .

I.3.a The Grassmannian Λ_n as a homogeneous space

Look again at lemma I.2.2. If P_1 and P_2 are two Lagrangian subspaces of \mathbf{C}^n , choose an orthonormal basis for each. We thus have two unitary bases of \mathbf{C}^n . There exists a unitary transformation (an element of the unitary group $U(n)$) that maps the basis of P_1 on that of P_2 ... and thus *a fortiori* the Lagrangian P_1 on the Lagrangian P_2 .

In other words, the group $U(n)$ acts transitively on the set of Lagrangian subspaces of \mathbf{C}^n . The stabilizer of the Lagrangian \mathbf{R}^n is the group $O(n)$ of orthonormal basis changes in \mathbf{R}^n . We have defined this way a bijection

$$U(n)/O(n) \longrightarrow \Lambda_n$$

with the help of which we identify the two sets. Notice that this provides Λ_n with a topology, namely that of $U(n)/O(n)$, the quotient topology of the topology of the matrix group $U(n)$.

Example I.3.1. As all lines are isotropic, the space Λ_1 is the space of real lines in $\mathbf{C} = \mathbf{R}^2$, namely the projective space $\mathbf{P}^1(\mathbf{R})$. The unitary group $U(1)$ is a circle and the orthogonal group $O(1)$ is the group with two elements $\{\pm 1\}$.

As the unitary group $U(n)$ is compact (being closed and bounded in the space of matrices) and path-connected (exercise), the space Λ_n is a compact path-connected topological space.

I.3.b The manifold Λ_n

Let us firstly describe a neighbourhood of $P \in \Lambda_n$ in Λ_n . Put

$$U_P = \{Q \in \Lambda_n \mid Q \cap (iP) = 0\}.$$

This is an open subset: using a unitary matrix, one can assume that $P = \mathbf{R}^n$, but then $U_{\mathbf{R}^n}$ is the image in Λ_n of the (saturated) open subset of $U(n)$ consisting of all the unitary bases the real parts of whose vectors form a basis of \mathbf{R}^n . This is, clearly, a neighbourhood of P .

Lemma I.3.2. *The open set U_P is homeomorphic to the real vector space of all symmetric endomorphisms of P .*

Proof. The subspaces Q that intersect iP only at 0 are the graphs of the linear maps $\varphi : P \rightarrow iP$. It is more convenient to call $i\varphi$ the linear map, so that φ is a linear map from P to itself. Write now that Q is Lagrangian, namely that

$$\forall x, y \in P, \quad \omega(x + i\varphi(x), y + i\varphi(y)) = 0.$$

We have

$$\begin{aligned}\omega(x + i\varphi(x), y + i\varphi(y)) &= -\operatorname{Im}\langle x + i\varphi(x), y + i\varphi(y) \rangle \\ &= \omega(x, y) + \omega(\varphi(x), \varphi(y)) + (\varphi(x), y) - (x, \varphi(y)) \\ &= (\varphi(x), y) - (x, \varphi(y)),\end{aligned}$$

P being Lagrangian. The subspace Q is Lagrangian if and only if the last expression vanishes for all x and y in P , namely if and only if φ is symmetric¹. We have thus defined a bijection that maps $\mathfrak{0}$ to P

$$\begin{aligned}\operatorname{End} \operatorname{Sym}(P) &\longrightarrow U_P \\ \varphi &\longmapsto \text{graph of } i\varphi\end{aligned}$$

and is clearly a homeomorphism. \square

Remark I.3.3. Consider for instance the “vertical” Lagrangian $i\mathbf{R}^n \subset \mathbf{C}^n$. We see that Λ_n is a disjoint union

$$\Lambda_n = \Lambda_n^0 \cup \Sigma_n$$

where Σ_n is the set of all Lagrangians that are not transversal to $i\mathbf{R}^n$ and Λ_n^0 is identified with the space of $n \times n$ real symmetric matrices.

We intend to prove now that the open sets U_P define the structure of a manifold on Λ_n . Notice firstly that any n -dimensional subspace Q of $\mathbf{R}^n \times \mathbf{R}^n$ may be represented by a rank- n matrix

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix}, \text{ with } 2n \text{ lines and } n \text{ columns,}$$

the column vectors of which form a basis of Q . Two matrices Z and Z' describe the same subspace if and only if there exists an $n \times n$ invertible matrix $g \in \operatorname{GL}(n; \mathbf{R})$, such that $Zg = Z'$.

Lemma I.3.4. *The subspace Q is Lagrangian if and only if the two matrices X and Y are such that*

$${}^tXY = {}^tYX.$$

Proof. Let $u, u' \in \mathbf{R}^n$ and let z, z' be the corresponding vectors in Q :

$$z = \begin{pmatrix} X \\ Y \end{pmatrix} u, \quad z' = \begin{pmatrix} X \\ Y \end{pmatrix} u'.$$

Note that Xu, Yu, Xu' and Yu' are vectors of \mathbf{R}^n . We compute:

$$\begin{aligned}\omega(z, z') &= \omega((Xu, Yu), (Xu', Yu')) \\ &= (Xu) \cdot (Yu') - (Yu) \cdot (Xu') \text{ (scalar product in } \mathbf{R}^n) \\ &= {}^t u {}^t XY u' - {}^t u {}^t YX u' \text{ (as } U \cdot V = {}^t UV) \\ &= {}^t u ({}^t XY - {}^t YX) u'.\end{aligned}$$

\square

¹See also Exercise I.8.

Remark I.3.5. If Q is the graph of a linear map $\mathbf{R}^n \rightarrow i\mathbf{R}^n$, it can be represented by a matrix $Z = \begin{pmatrix} \text{Id} \\ A \end{pmatrix}$. The relation in lemma I.3.4 simply expresses the fact that the matrix A is symmetric.

Consider more generally a subset J of $\{1, \dots, n\}$ and the Lagrangian subspace P_J of $\mathbf{R}^n \times \mathbf{R}^n$ spanned by $\{(e_j)_{j \in J}, (ie_j)_{j \notin J}\}$. Denote U_{P_J} by U_J (for simplicity). Any element of U_J is described by a unique matrix Z such that, if we extract from Z the matrix containing the lines j (for $j \in J$) and $j + n$ (for $j \notin J$), we get the identity matrix. The 2^n open sets U_J clearly cover Λ_n . Moreover, as we have said it, each of them can be identified with the subspace $\text{Sym}(n; \mathbf{R})$ of $n \times n$ symmetric matrices. The U_J 's, with their identification with $\text{Sym}(n; \mathbf{R})$ are coordinate charts. Change of coordinates are given by

$$\begin{array}{ccc} \text{Sym}(n; \mathbf{R}) & \xrightarrow{\varphi_J^{-1}} & U_J \cap U_{J'} & \xrightarrow{\varphi_{J'}} & \text{Sym}(n; \mathbf{R}) \\ A & \longmapsto & Z_J(A) = Z_{J'}(B) & \longmapsto & B \end{array}$$

where $Z_J(A)$ is the matrix obtained from $\begin{pmatrix} \text{Id} \\ A \end{pmatrix}$ by mapping the first n lines on the lines j (for $j \in J$) and $j + n$ (for $j \notin J$). The matrix $Z_{J'}(B)$ is obtained by multiplying $Z_J(A)$ by the inverse matrix of the (invertible!) matrix of the lines corresponding to J' in $Z_J(A)$. The coordinate change $A \mapsto B$ is clearly smooth (it is actually rational, thus analytic).

Proposition I.3.6. *The Grassmannian Λ_n is a compact and connected manifold of dimension $\frac{n(n+1)}{2}$.* □

I.3.c The tautological vector bundle

Consider the space

$$E_n = \{(P, x) \in \Lambda_n \times \mathbf{C}^n \mid x \in P\}.$$

Together with its projection on Λ_n , this is a rank- n vector bundle over Λ_n . The fiber of E_n at $P \in \Lambda_n$ is the Lagrangian subspace P itself, a reason why this bundle is qualified as “tautological”.

The property expressed in Lemma I.2.1, namely $P^\perp = iP$, is translated, in terms of the bundle E_n , in the fact that $E_n \otimes_{\mathbf{R}} \mathbf{C}$, the complexified bundle, is trivial (has a canonical trivialization). The (global) trivialization is the isomorphism of complex vector bundles

$$\begin{array}{ccc} E_n \otimes_{\mathbf{R}} \mathbf{C} & \longrightarrow & \Lambda_n \times \mathbf{C}^n \\ (P, x \otimes (a + ib)) & \longmapsto & (P, (a + ib)x). \end{array}$$

I.3.d The tangent bundle to Λ_n

The canonical identification of the open subset U_P with the space of symmetric endomorphisms of P allows to identify the tangent bundle of Λ_n with the bundle $\text{End Sym}(E_n)$. It is also possible to describe this bundle from the tangent bundle of $U(n)$. The group $U(n)$ is described as a submanifold of the space of all complex matrices by the equation ${}^t\bar{A}A = \text{Id}$, so that we have

$$T_A U(n) = \{X \in \text{GL}(n; \mathbf{C}) \mid {}^t\bar{A}X + {}^t\bar{X}A = 0\}.$$

Call $\mathfrak{u}(n)$ the vector space $T_{\text{Id}} U(n)$ of skew-Hermitian matrices. There is an isomorphism

$$\begin{array}{ccc} T_A U(n) & \longrightarrow & \mathfrak{u}(n) \\ X & \longmapsto & {}^t\bar{A}X \end{array}$$

identifying the tangent bundle $TU(n)$ with the trivial bundle $U(n) \times \mathfrak{u}(n)$ — as any Lie group, $U(n)$ is parallelizable. Consider the Lagrangian \mathbf{R}^n , image in Λ_n of the identity matrix Id . One can write

$$T_{\mathbf{R}^n}(U(n)/O(n)) = \mathfrak{u}(n)/\mathfrak{o}(n),$$

this is the quotient of the vector space of anti-Hermitian matrices by that of skew-symmetric real matrices. We thus identify

$$T_{\mathbf{R}^n}\Lambda_n = i\text{Sym}(n; \mathbf{R}),$$

as the real part of a skew-Hermitian matrix is skew-symmetric and its imaginary part is symmetric.

Let P be any Lagrangian subspace. Choose a unitary matrix A such that $P = A \cdot \mathbf{R}^n$. As we have identified the quotient $\mathfrak{u}(n)/\mathfrak{o}(n)$ with the subspace $i\text{Sym}(n; \mathbf{R})$ of $\mathfrak{u}(n)$, we identify the quotient $T_{[A]}\Lambda_n$ with a subspace of $T_A U(n)$:

$$\begin{array}{ccc} i\text{Sym}(n; \mathbf{R}) & \xrightarrow{X \mapsto A \cdot X} & T_{[A]}\Lambda_n \\ \downarrow & & \downarrow \\ \mathfrak{u}(n) & \longrightarrow & T_A U(n). \end{array}$$

We derive an isomorphism

$$\begin{array}{ccc} i\text{Sym}(n; \mathbf{R}) & \longrightarrow & T_P\Lambda_n \\ X & \longmapsto & A \cdot X. \end{array}$$

Remark I.3.7. This isomorphism depends on the choice of A , this is why it does not follow that Λ_n is parallelizable (it is actually not, as soon as $n \geq 2$).

I.3.e The case of oriented Lagrangian subspaces

One can also consider the space $\tilde{\Lambda}_n$ of *oriented* Lagrangian subspaces. Replacing “orthonormal basis” by “positive orthonormal basis” in what precedes, we get an identification of $\tilde{\Lambda}_n$ with $U(n)/SO(n)$.

I.3.f The determinant and the Maslov class

The “determinant” mapping

$$\det : U(n) \longrightarrow S^1$$

descends to the quotient by $SO(n)$ and, in the same way, its square

$$\det^2 : U(n) \longrightarrow S^1$$

to the quotient by $O(n)$. This allows to compute the fundamental groups of Λ_n and $\tilde{\Lambda}_n$.

Proposition I.3.8. *The fundamental group of Λ_n (resp. $\tilde{\Lambda}_n$) is isomorphic to \mathbf{Z} . The covering $\tilde{\Lambda}_n \rightarrow \Lambda_n$ shows $\pi_1(\tilde{\Lambda}_n)$ as an index-2 subgroup in $\pi_1(\Lambda_n)$.*

Proof. Recall first that the group $SU(n)$ is simply connected. This can be proved by induction on n : $SU(1)$ is a point and $SU(n+1)$ acts transitively on the unit sphere S^{2n+1} of \mathbf{C}^{n+1} with stabilizer $SU(n)$, so that the exact sequence

$$\pi_1 SU(n) \longrightarrow \pi_1 SU(n+1) \longrightarrow \pi_1 S^{2n+1}$$

gives the result. As the determinant mapping

$$\det : U(n) \longrightarrow S^1$$

is a fibration with fiber $SU(n)$, it induces an isomorphism

$$\det_* : \pi_1 U(n) \longrightarrow \pi_1(S^1).$$

The fiber of the determinant mapping $\tilde{\Lambda}_n \longrightarrow S^1$ is $SU(n)/SO(n)$, which is simply connected, thus

$$\det_* : \pi_1 \tilde{\Lambda}_n \longrightarrow \pi_1 S^1$$

is an isomorphism. What is left to prove is a consequence of the fact that the diagram

$$\begin{array}{ccc} \tilde{\Lambda}_n & \longrightarrow & \Lambda_n \\ \det \downarrow & & \downarrow \det^2 \\ S^1 & \xrightarrow{z \mapsto z^2} & S^1 \end{array}$$

is commutative. □

“The” generator of $\pi_1\Lambda_n$ is called the *Maslov class*. One also calls “Maslov class” the cohomology class

$$\mu \in H^1(\Lambda_n; \mathbf{Z})$$

that it defines by duality. Using the notation of Remark I.3.3, it can be shown that μ is the dual class to the integral homology class represented by Σ_n (see [1, 12]).

I.4 Lagrangian submanifolds in \mathbf{C}^n

We are going now to globalize the notion of Lagrangian subspace, considering submanifolds of \mathbf{C}^n whose tangent space at any point is Lagrangian. We will not really need actual submanifolds, but maps

$$f : V \longrightarrow \mathbf{C}^n$$

from some n -dimensional manifold to \mathbf{C}^n , the tangent mapping of which

$$T_x f : T_x V \longrightarrow \mathbf{C}^n$$

is an injection for any point x of V , with image a Lagrangian subspace. It is then said that f is a *Lagrangian immersion*.

For instance, any immersion of a curve (real manifold of dimension 1) in \mathbf{C} is a Lagrangian immersion. Any product of Lagrangian immersions is a Lagrangian immersion (into the product target space), we thus obtain Lagrangian immersions of tori (products of circles). Our next aim is to describe examples of Lagrangian submanifolds and immersions in \mathbf{C}^n and to give a necessary (and sufficient) condition for a given manifold to have a Lagrangian immersion into \mathbf{C}^n .

I.4.a Lagrangian submanifolds described by functions

We consider firstly graphs.

Proposition I.4.1. *The graph of a map $F : \mathbf{R}^n \rightarrow (i)\mathbf{R}^n$ is a Lagrangian submanifold if and only if F is the gradient of a function $f : \mathbf{R}^n \rightarrow \mathbf{R}$.*

Proof. The tangent space to the graph at the point $(x, F(x))$ is the graph of $(dF)_x$, the differential of F at the point x . This graph is a Lagrangian subspace if and only if $(dF)_x$ is a symmetric endomorphism (see the proof of Lemma I.3.2). The matrix $\partial F_i / \partial x_j$ is symmetric for all x if and only if the differential form $\sum F_i dx_i$ over \mathbf{R}^n is closed or, equivalently, exact:

$$F_i = \frac{\partial f}{\partial x_i}, \text{ namely } F = \nabla f.$$

□

See, more generally, Proposition II.2.1.

The Lagrangian submanifolds obtained as graphs have a very specific property: the projection of the Lagrangian submanifold on \mathbf{R}^n is a diffeomorphism. We would like to consider more general Lagrangian immersions, for instance immersions of compact manifolds. Here is a way to construct Lagrangian immersions using the reduction process of §I.2.b. We start from a Lagrangian submanifold² $L \subset \mathbf{C}^{n+k}$. We want to construct a Lagrangian immersion into \mathbf{C}^n . To write \mathbf{C}^n as F/F° , we choose the co-isotropic subspace $F = \mathbf{C}^n \oplus \mathbf{R}^k$, the orthogonal of which is $F^\circ = 0 \oplus \mathbf{R}^k$. We suppose that the submanifold L is “transversal to F ” in the sense that, for all x ,

$$T_x L + F = \mathbf{C}^{n+k}.$$

The Lagrangian subspace $T_x L$ thus satisfies the assumption of the reduction lemma (Lemma I.2.3). Hence the composition

$$T_x L \cap F \subset F \longrightarrow F/F^\circ = \mathbf{C}^n$$

is the injection of a Lagrangian subspace.

Consider now the intersection V of the submanifold L with F . With the transversality assumption we have made on L , V is an n -dimensional submanifold of F (a consequence of the inverse function theorem) whose tangent space $T_x V$ is the intersection of $T_x L$ with F . Thus, the reduction lemma asserts, at the level of each tangent space, that, for all x in $V = L \cap F$, we have the injection of a Lagrangian subspace

$$T_x V \longrightarrow \mathbf{C}^n.$$

In other words, the composition

$$V = L \cap F \subset F \longrightarrow F/F^\circ = \mathbf{C}^n$$

is a Lagrangian immersion.

Remark I.4.2. Even if one starts from a Lagrangian submanifold, what we get in general is only an immersion.

Generating functions

We generalize the “graph” construction, using the reduction process as explained. Let us start with a nice and useful example.

Example I.4.3 (The Whitney immersion). Consider the unit sphere in \mathbf{R}^{n+1}

$$S^n = \left\{ (x, a) \in \mathbf{R}^n \times \mathbf{R} \mid \|x\|^2 + a^2 = 1 \right\}$$

and the map

$$\begin{aligned} f : S^n &\longrightarrow \mathbf{C}^n \\ (x, a) &\longmapsto (1 + 2ia)x. \end{aligned}$$

²Or a Lagrangian immersion.

The tangent space to the sphere is

$$T_{(x,a)}S^n = \{(\xi, \alpha) \in \mathbf{R}^n \times \mathbf{R} \mid x \cdot \xi + a\alpha = 0\}$$

and the tangent mapping to f is

$$\begin{aligned} T_{(x,a)}f : T_{(x,a)}S^n &\longrightarrow \mathbf{C}^n \\ (\xi, \alpha) &\longmapsto \xi + 2i(a\xi + \alpha x). \end{aligned}$$

The map $T_{(x,a)}f$ is injective for all $(x, a) \in S^n$: if $T_{(x,a)}f(\xi, \alpha) = 0$, then $\xi = 0$ and $\alpha x = 0$; if $x = 0$, we have $a = \pm 1$ and the equality $x \cdot \xi + a\alpha = 0$ gives $\alpha = 0$. Thus we have $\xi = 0$ and $\alpha = 0$, so that f is an immersion. Moreover, we have

$$\begin{aligned} \omega(\xi + 2i(a\xi + \alpha x), \xi' + 2i(a\xi' + \alpha'x)) &= 2(\xi \cdot (a\xi' + \alpha'x) - \xi' \cdot (a\xi + \alpha x)) \\ &= 2(\alpha'\xi \cdot x - \alpha\xi' \cdot x) = 0 \end{aligned}$$

so that the image of $T_{(x,a)}f$ is an isotropic subspace of dimension n , a Lagrangian subspace. In conclusion, the map f is a Lagrangian immersion. It has a unique double point (North and South poles of the sphere are mapped to 0). In dimension 1, this is a “figure eight”. Below (in §I.4.b) we will draw pictures in dimensions 1 and 2.

Obviously, the Whitney sphere is not the graph of a map from \mathbf{R}^n to \mathbf{R}^n . Let us show that it can nevertheless be described from the graph of a map defined on a larger space. We start from a function

$$f : \mathbf{R}^n \times \mathbf{R}^k \longrightarrow \mathbf{R}.$$

As we have seen it above, the graph of ∇f is a Lagrangian subspace of \mathbf{C}^{n+k} . We reduce \mathbf{C}^{n+k} as in §I.2.b using the co-isotropic subspace $F = \mathbf{C}^n \oplus \mathbf{R}^k$. Here we intersect the graph of ∇f with F , namely we consider

$$V = \left\{ (x, a) \in \mathbf{R}^n \times \mathbf{R}^k \mid \frac{\partial f}{\partial a_1} = \dots = \frac{\partial f}{\partial a_k} = 0 \right\} \subset \mathbf{R}^n \times \mathbf{R}^k.$$

The transversality assumption above is equivalent to the assumption that V is a submanifold of $\mathbf{R}^n \times \mathbf{R}^k$, in other words that the map

$$\begin{aligned} \mathbf{R}^n \times \mathbf{R}^k &\longrightarrow \mathbf{R}^k \\ (x, a) &\longmapsto \left(\frac{\partial f}{\partial a_1}, \dots, \frac{\partial f}{\partial a_k} \right) \end{aligned}$$

is a submersion along V . In terms of partial derivatives, this is to say that the matrix

$$\left(\left(\frac{\partial^2 f}{\partial a_i \partial a_j} \right)_{\substack{1 \leq i \leq k \\ 1 \leq j \leq k}} \left(\frac{\partial^2 f}{\partial x_i \partial a_j} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq k}} \right)$$

has maximal rank k . In terms of tangent subspaces, this is to say that the Lagrangian subspaces that are tangent to the graph of ∇f are transversal to the co-isotropic subspace F . The reduction lemma I.2.3 says that the map

$$\begin{aligned} V &\longrightarrow \mathbf{R}^n \times \mathbf{R}^n = \mathbf{C}^n \\ (x, a) &\longmapsto \left(x, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right) \end{aligned}$$

is a Lagrangian immersion.

Example I.4.4 (The Whitney immersion, again). With $k = 1$ and $f(x, a) = a \|x\|^2 + \frac{a^3}{3} - a$, we get

$$\frac{\partial f}{\partial a} = \|x\|^2 + a^2 - 1 = 0,$$

an equation which describes the sphere $S^n \subset \mathbf{R}^n \times \mathbf{R}$, and $\partial f / \partial x = 2ax$ gives the Whitney map.

Example I.4.5 (Unfolding). Unfoldings are deeply related with Lagrangian submanifolds (see [2]). I will not explain here the general theory but rather show an example. Let $P \in \mathbf{R}[X]$ be a degree- $(n+1)$ polynomial

$$P(X) = X^{n+1} + x_1 X^{n-1} + \dots + x_{n-1} X$$

where $x_1, \dots, x_{n-1} \in \mathbf{R}$. These coefficients are going to vary, this is the reason why they are named as variables. Call P_x the polynomial corresponding to $x = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$ and consider the map

$$\begin{aligned} f : \mathbf{R}^{n-1} \times \mathbf{R} &\longrightarrow \mathbf{R} \\ (x_1, \dots, x_{n-1}, a) &\longmapsto P_x(a) \end{aligned}$$

to which we apply the previous techniques. The manifold V is

$$\begin{aligned} V &= \left\{ (x, a) \in \mathbf{R}^{n-1} \times \mathbf{R} \mid \frac{\partial f}{\partial a}(x_1, \dots, x_{n-1}, a) = 0 \right\} \\ &= \left\{ (x, a) \in \mathbf{R}^{n-1} \times \mathbf{R} \mid P'_x(a) = 0 \right\}, \end{aligned}$$

this is the set of critical points of P_x (zeroes of its derivative P'_x) when x varies. The condition that V actually be a submanifold is that the matrix of partial derivatives

$$\left(\left(\frac{\partial^2 f}{\partial a^2} \right), \left(\frac{\partial^2 f}{\partial x_i \partial a} \right)_{1 \leq i \leq n-1} \right)$$

has rank 1. But

$$\frac{\partial f}{\partial a} = P'_x(a) = (n+1)a^n + (n-1)x_1 a^{n-2} + \dots + x_{n-1}$$

so that $\partial^2 f / \partial x_{n-1} \partial a$ is identically 1. Thus V is indeed a submanifold. The Lagrangian immersion is

$$V \longrightarrow T^*\mathbf{R}^{n-1}$$

$$(x, a) \longmapsto \left(x, \frac{\partial P_x}{\partial x_1}(a), \dots, \frac{\partial P_x}{\partial x_{n-1}}(a) \right).$$

For instance, starting from the family

$$P_x(X) = X^4 + x_1 X^2 + x_2 X,$$

we get

$$V = \{(x_1, x_2, a) \in \mathbf{R}^3 \mid 4a^3 + 2x_1 a + x_2 = 0\}$$

and the Lagrangian immersion from V into $\mathbf{R}^2 \times \mathbf{R}^2$ is the map

$$(x_1, x_2, a) \longrightarrow (x_1, x_2, a^2, a).$$

Figure I.1 shows V with its projection on the plane \mathbf{R}^2 of coefficients (x_1, x_2) . The cusp curve is the discriminant of the family of degree-3 polynomials, the set of points x such that P'_x has a multiple root. It is obtained here as the set of critical values of the projection $V \rightarrow \mathbf{R}^2$. Over such a point x in the space of coefficients are the (one or three) roots of the polynomial P'_x .

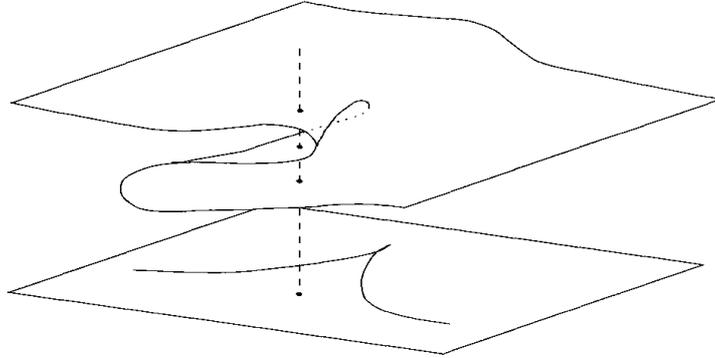


Figure I.1: The discriminant of degree-3 polynomials

I.4.b Wave fronts

Exact Lagrangian immersions

If $f : V \rightarrow \mathbf{C}^n$ is a Lagrangian immersion, the 2-form $f^*\omega$ is zero, so that $d(f^*\lambda) = 0$ and $f^*\lambda$ is a closed 1-form on V . If, for some reason, for instance because $H_{DR}^1(V) = 0$, this form is exact, there exists a function

$$F : V \longrightarrow \mathbf{R}$$

such that $f^*\lambda = dF$. The immersion f is qualified as *exact Lagrangian immersion*. The mapping

$$F \times f : V \longrightarrow \mathbf{C}^n \times \mathbf{R}$$

has the property³

$$(f \times F)^* \left(dz - \sum_{j=1}^n y_j dx_j \right) = 0.$$

Wave fronts

Instead of looking at the Lagrangian immersion f , consider the projection

$$\begin{array}{ccc} V & \xrightarrow{f \times F} & \mathbf{C}^n \times \mathbf{R} & \longrightarrow & \mathbf{R}^n \times \mathbf{R} \\ & & (X + iY, z) & \longmapsto & (X, z). \end{array}$$

We will assume here that, at a general point of the Lagrangian, the tangent space is transversal to the subspace of coordinates Y . The image of the Lagrangian immersion is then a hypersurface of $\mathbf{R}^n \times \mathbf{R}$. This hypersurface is the *wave front*. Of course, it will in general be singular. Precisely, at a point of V where V is not a graph over \mathbf{R}^n , the projection $X + iY \mapsto X$ is singular. However, as $(f \times F)^* \left(dz - \sum_{j=1}^n y_j dx_j \right) = 0$, at every point of the wave front, there is a tangent hyperplane, the hyperplane

$$z = \sum_{j=1}^n Y_j x_j \text{ in the space } \mathbf{R}^n \times \mathbf{R} \text{ of } (x, z) \text{ coordinates}$$

at the point image of (X, Y, z) . Notice that, as the coefficient of z in this equation is non zero, the hyperplane is always transversal to the z -axis. Conversely, if a singular hypersurface of $\mathbf{R}^n \times \mathbf{R}$ has at every point a tangent hyperplane that is transversal to the z -axis, this hyperplane has a unique equation of the form $z = \sum Y_j x_j$ and it is possible to reconstruct a (maybe singular) Lagrangian submanifold from the “slopes” Y_j .

We begin with an example of dimension 1, that of the Whitney immersion again. Notice that this is indeed an exact Lagrangian immersion: the restriction of the Liouville form ydx to the curve is exact because $\int ydx = 0$ (the “algebraic” area surrounded by the curve is zero). A primitive of ydx is easily found. The curve is parametrized by $t \mapsto (\cos t, \sin 2t)$ and

$$ydx = -2 \sin^2 t \cos t = -\frac{2}{3} d(\sin^3 t).$$

³The manifold $\mathbf{C}^n \times \mathbf{R}$ is a “contact manifold” and $F \times f$ is a “Legendrian immersion” lifting the Lagrangian immersion F .

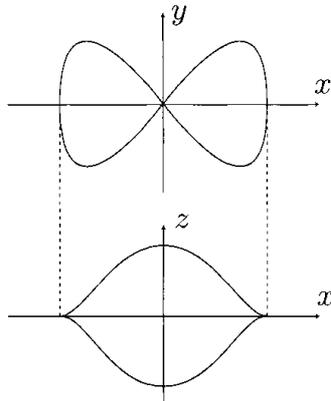


Figure I.2: Eye

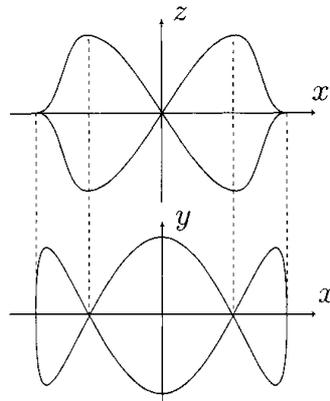


Figure I.3: Crossbow

A map to the (x, z) space is thus

$$t \mapsto (x, z) = \left(\cos t, -\frac{2}{3} \sin^3 t \right).$$

This is depicted on Figure I.2, in an old-fashion “descriptive geometry” mood. It can be seen that the singular points of the (x, z) curve correspond to the tangents to the (x, y) curve that are vertical, and that the double point of the latter corresponds to the two tangents to the wavefront (the “eye”) at points with the same x coordinate that are parallel.

Figure I.3 represents an example in which we start from the wave front (a “crossbow”) to reconstruct the Lagrangian. From the wave front, it is seen that the Lagrangian curve has two double points and two “vertical” tangents.

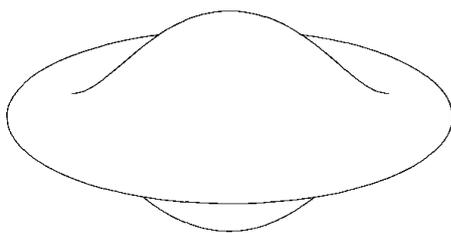


Figure I.4: Flying saucer

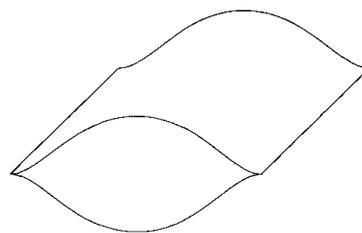


Figure I.5: Cylinder

One could wonder what it is useful for to replace an immersed curve by a singular one. Notice that, in higher dimensions, the wave front is a hypersurface in $\mathbf{R}^n \times \mathbf{R}$ and it replaces a submanifold of the same dimension n in $\mathbf{R}^n \times \mathbf{R}^n$. Even for $n = 2$, this is very useful as this allows to represent exact Lagrangian surfaces of \mathbf{R}^4 by (singular) surfaces in a dimension-3 space. Here are some beautiful examples. Rotate the eye (Figure I.2) about the z -axis to get the flying saucer depicted on Figure I.4. The corresponding Lagrangian surface in $\mathbf{R}^2 \times \mathbf{R}^2$ is a Lagrangian

immersion of the dimension-2 sphere in \mathbf{C}^2 with a double point. In Exercise I.14, one checks that this is, indeed, the Whitney immersion... eventually drawn in dimension 2!

Figure I.5 represents a cylinder constructed on the eye, namely a Lagrangian immersion of a cylinder, product of a figure eight with an interval, with two whole lines of singular points.

Singularities

Wave fronts are, as we have said it, singular hypersurfaces. We have seen, in dimension 1, cusps, in dimension 2, lines of cusps, but this can be more complicated, as Exercise I.15 shows it.

Wave fronts of non exact Lagrangian immersions

Wave fronts are so nice that it is a pity not to have them for all Lagrangian immersions. In dimension 1, the problem is to represent by wave fronts curves that do not surround a zero area. Consider for instance the standard (round) circle in \mathbf{C} . As $\int y dx \neq 0$, it seems that nothing can be done. Look, however, at the parametrization

$$t \longmapsto (\cos t, \sin t).$$

It gives

$$y dx = -\sin^2 t dt = d\left(\frac{\sin 2t}{4} - \frac{t}{2} + C\right).$$

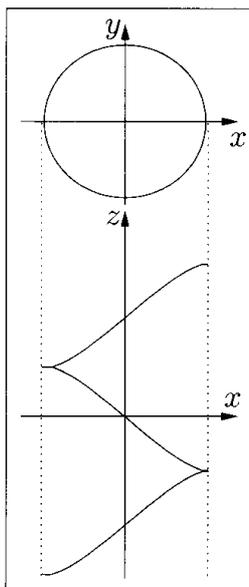


Figure I.6: Wave front of the circle

Nothing forbids us to represent the Lagrangian (non exact) immersion of the circle by a piece of the (non closed) wave front⁴ parametrized by

$$t \longmapsto \left(\cos t, \frac{\sin 2t}{4} - \frac{t}{2} + C \right)$$

and depicted on Figure I.6⁵.

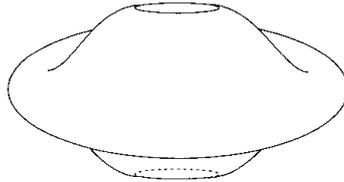


Figure I.7

If we rotate the (unbounded) wave front of Figure I.6 around a line parallel to the z -axis that does not intersect the wave front, we get the wave front of a Lagrangian torus, the one depicted on Figure I.7. One can then use the cylinder represented on Figure I.5 to perform connected sums of wave fronts. This way, Figure I.8 represents (the wave front of) a genus-2 Lagrangian surface. In the same way, one constructs Lagrangian immersions of all orientable surfaces in \mathbf{C}^2 . These figures are copied from Givental's paper [13], that contains many other examples.

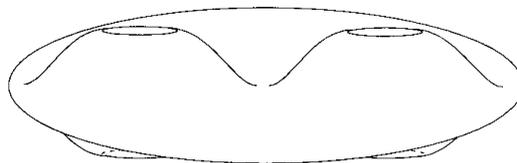


Figure I.8: A genus-2 surface

Remark I.4.6. Except for the torus, all the surfaces depicted here have double points, that show up in the wave fronts as points having the same projection on the horizontal plane and parallel tangent planes. It is rather easy to prove that the torus is the only orientable surface that can be *embedded* as a Lagrangian submanifold in \mathbf{C}^2 . As for non orientable surfaces, they can be embedded as Lagrangian surfaces when (and only when) their Euler characteristic is divisible by 4 (with the exception of the Klein bottle). See the pictures in [13]. As for the Klein bottle, it has long been unknown whether it had or had not a Lagrangian embedding. Mohnke [27] has recently proved that it has not.

⁴This is a place where one can really appreciate the difference between closed and exact 1-forms.

⁵Notice that wave fronts are defined only up to a "vertical" translation, the actual constant C used in Figure I.6 is $(\pi + 1)/4$.

Exact Lagrangian embeddings

Notice that, in all the examples of exact Lagrangian immersions we have given, there are double points. This is obviously necessary in dimension 2 ($n = 1$), due to Jordan theorem: an embedded curve cannot surround a zero area. This is also true in higher dimensions, due to a (hard) theorem of Gromov [16]: there is no exact Lagrangian submanifold in \mathbf{C}^n .

I.4.c Other examples

Here are a few other examples.

Grassmannians

Consider the map

$$\begin{array}{ccc} \mathrm{U}(n) & \longrightarrow & \mathrm{Sym}(n; \mathbf{C}) \\ A & \longmapsto & {}^tAA \end{array}$$

from the group $\mathrm{U}(n)$ to the complex vector space of symmetric matrices.

Proposition I.4.7. *The map $A \mapsto {}^tAA$ defines a Lagrangian immersion*

$$\Phi : \Lambda_n \longrightarrow \mathrm{Sym}(n; \mathbf{C}).$$

Proof. As ${}^tAA = \mathrm{Id}$ when $A \in \mathrm{O}(n)$, the map Φ is well defined. Call $[A]$ the class of a unitary matrix A in Λ_n . We have seen in §I.3.d that the tangent space to Λ_n at the point $[A]$ can be identified with

$$T_{[A]}\Lambda_n = \{AH \mid H \in i\mathrm{Sym}(n; \mathbf{R})\}.$$

It is mapped into $\mathrm{Sym}(n; \mathbf{C})$ par $T_{[A]}\Phi$ as follows

$$AH \longmapsto {}^tA(AH^t\bar{A} + \bar{A}^tH^tA)A.$$

The matrix $AH^t\bar{A} + \bar{A}^tH^tA$ has the form $K - \bar{K}$ for $K = AH^t\bar{A} = AHA^{-1}$ in $\mathfrak{u}(n)$ and this describes all the matrices in the vector space $i\mathrm{Sym}(n; \mathbf{R})$ when H varies in $i\mathrm{Sym}(n; \mathbf{R})$. The image of the tangent mapping $T_{[A]}\Phi$ is, thus, the subspace $\rho(A) \cdot i\mathrm{Sym}(n; \mathbf{R})$ where

$$\begin{array}{ccc} \rho : \mathrm{U}(n) & \longrightarrow & \mathrm{U}\left(\frac{n(n+1)}{2}\right) \\ A & \longmapsto & (B \mapsto {}^tABA) \end{array}$$

is the representation of $\mathrm{U}(n)$ operating on complex symmetric matrices. This image is, indeed, a Lagrangian subspace, being the image of the real part of the complex vector space $i\mathrm{Sym}(n; \mathbf{C})$ by a unitary matrix. \square

Tori, integrable systems

Integrable systems (mechanical systems with many conserved quantities) yield many Lagrangian tori. We use here a few standard symplectic notions: Hamiltonian vector fields, Poisson bracket, commuting functions. See if necessary Appendix I.6.c. Recall for instance that an integrable system on $\mathbf{C}^n = \mathbf{R}^{2n}$ is a map $f : \mathbf{R}^{2n} \rightarrow \mathbf{R}^n$ whose components f_1, \dots, f_n are functionally independent commuting functions.

This defines a local \mathbf{R}^n -action on \mathbf{R}^{2n} , which is locally free at the regular points of the system (the points at which the derivatives of the functions f_i are actually independent). Call X_1, \dots, X_n the Hamiltonian vector fields associated with the functions f_i . These vector fields commute:

$$[X_i, X_j] = X_{\{f_i, f_j\}} = 0.$$

The \mathbf{R}^n -action is given by integration:

$$t \cdot x = \varphi_n^{t_n} \circ \varphi_{n-1}^{t_{n-1}} \circ \dots \circ \varphi_1^{t_1}(x)$$

where φ_i denotes the flow of X_i and $t = (t_1, \dots, t_n) \in \mathbf{R}^n$ is close to 0 (in order that $\varphi_i^{t_i}$ be defined). This local action is indeed locally free on the open set of regular points because the vector fields X_i give independent tangent vectors at these points.

Assume moreover that the vector fields X_i are complete, namely that the flows $\varphi_i^{t_i}$ are defined for all values of t . We then have a locally free action of \mathbf{R}^n on the whole set of regular points. The vector fields X_i being tangent to the common level sets of the f_i 's, this action preserve the level sets. The connected components of the regular level sets of f are thus homogeneous spaces, quotients of \mathbf{R}^n by discrete subgroups. The discrete subgroups of \mathbf{R}^n are the lattices \mathbf{Z}^k in the linear subspaces of dimension k . The connected components of the regular level sets are thus diffeomorphic to $\mathbf{R}^{n-k} \times \mathbf{T}^k$ for some k such that $0 \leq k \leq n$. In particular, the compact connected components are tori \mathbf{T}^n and these tori are Lagrangian⁶, they are called the *Liouville tori*. The next proposition is the easiest part of the Arnold-Liouville theorem (see for instance [2, 6]).

Proposition I.4.8. *Compact connected components of the regular common level sets of an integrable system are Lagrangian tori.* \square

There are many examples of integrable systems and thus of Lagrangian tori, coming from mechanical systems (spinning top, pendulum...)⁷. The most classical example is that of the standard action of the torus

$$T^n = \{(t_1, \dots, t_n) \in \mathbf{C}^n \mid |t_j| = 1, \quad i = 1, \dots, n\}$$

⁶Notice that on a compact connected component, the flows are complete.

⁷See for instance [6].

on \mathbf{C}^n by

$$(t_1, \dots, t_n) \cdot (z_1, \dots, z_n) = (t_1 z_1, \dots, t_n z_n),$$

the orbits of which are the common level sets of the functions

$$g_1 = \frac{1}{2} |z_1|^2, \dots, g_n = \frac{1}{2} |z_n|^2,$$

tori $S^1 \times \dots \times S^1$ indeed, for the regular values of the g_i 's (namely every g_i non zero). We will come back to these examples in §I.5.e.

Normal bundles

Let now $f : V \rightarrow \mathbf{R}^n$ be any immersion of a k -dimensional manifold into \mathbf{R}^n . Consider the total space of its normal bundle

$$Nf = \{(x, v) \in V \times \mathbf{R}^n \mid x \in V, \quad v \in (T_x f(T_x V))^\perp\}.$$

It is naturally mapped into $\mathbf{R}^n \times \mathbf{R}^n$ by

$$\begin{aligned} \tilde{f} : Nf &\longrightarrow \mathbf{R}^n \times \mathbf{R}^n \\ (x, v) &\longmapsto (f(x), v). \end{aligned}$$

The manifold Nf has dimension $k + n - k = n$, and \tilde{f} is clearly an immersion. Moreover, it is Lagrangian. More precisely, we have:

Lemma I.4.9. *If λ is the Liouville form on $\mathbf{R}^n \times \mathbf{R}^n$, one has $\tilde{f}^* \lambda = 0$.*

Proof. Consider a vector $X \in T_{(x,v)} Nf$. Use the commutative diagram

$$\begin{array}{ccc} Nf & \xrightarrow{\quad} & \mathbf{R}^n \times \mathbf{R}^n \\ \pi \downarrow & \tilde{f} & \downarrow \pi \\ V & \xrightarrow{f} & \mathbf{R}^n \end{array}$$

to compute

$$\begin{aligned} (\tilde{f}^* \lambda)_{(x,v)}(X) &= \lambda_{(f(x), v)}(T_{(x,v)} \tilde{f}(X)) \\ &= v \cdot (T_{(f(x), v)} \pi \circ T_{(x,v)} \tilde{f}(X)) \\ &= v \cdot (T_x f \circ T_{(x,v)} \pi(X)) \\ &= 0 \text{ since } v \text{ is orthogonal to } T_x f(T_x V). \end{aligned}$$

□

This method allows to construct many (non compact) examples and can be generalized by replacing $\mathbf{R}^n \times \mathbf{R}^n = T^* \mathbf{R}^n$ by the cotangent bundle T^*M of a manifold and $V \rightarrow \mathbf{R}^n$ by an immersion into M . See §II.2.a.

I.4.d The Gauss map

Let $f : V \rightarrow \mathbf{C}^n$ be a Lagrangian immersion. Its tangent space at any point is a Lagrangian subspace of \mathbf{C}^n . One can globalize the data consisting of all these tangent spaces to define the “Gauss map”

$$\begin{array}{ccc} V & \xrightarrow{\gamma(f)} & \Lambda_n \\ x & \longmapsto & T_x f(T_x V). \end{array}$$

By definition of the tautological bundle (§I.3.c), one has

$$\gamma(f)^* E_n = TV.$$

In particular, the tangent bundle to V must have the same properties as E_n .

Proposition I.4.10. *For a manifold to have a Lagrangian immersion into \mathbf{C}^n , it is necessary that the complexification of its tangent bundle be trivializable. \square*

The converse is true, but less easy to prove. This is an application of Gromov’s h -principle [17], see also [23].

Examples I.4.11. (1) Spheres. We have seen examples of Lagrangian immersions of spheres in \mathbf{C}^n (in §I.4.a). One deduces that $TS^n \otimes_{\mathbf{R}} \mathbf{C}$ is a trivial complex bundle. Notice however that it is not true that the tangent bundle TS^n itself is trivial (except for $n = 0, 1, 3$ and 7).

(2) Surfaces. All orientable surfaces and half the non orientable surfaces have Lagrangian immersions in \mathbf{C}^2 (as we have seen it in §I.4.b). This is not the case, neither for the real projective plane nor for the connected sums of an odd number of copies of this plane.

(3) Normal bundles. This is a case where the tangent bundle itself is trivial (before complexification):

$$\begin{aligned} T_{(x,v)}(Nf) &= \{(\xi, U) \mid \xi \in T_x V, U \perp T_x f(T_x V)\} \\ &= T_x V \oplus N_x f \end{aligned}$$

and this is canonically isomorphic to the ambient space \mathbf{R}^n .

(4) Grassmannians. The Gauss map φ of the Lagrangian immersion Φ

$$\varphi : \Lambda_n \longrightarrow \Lambda_{\frac{n(n+1)}{2}}$$

satisfies of course

$$\varphi^* E_{\frac{n(n+1)}{2}} = \text{End Sym}(E_n).$$

The Maslov class

Every Lagrangian immersion has a Maslov class: use the Gauss map

$$\varphi(f) : V \longrightarrow \Lambda_n$$

to pull back $\mu \in H^1(\Lambda_n; \mathbf{Z})$ to a class

$$\mu(f) \in H^1(V; \mathbf{Z}).$$

One can also, with the notation of Remark I.3.3, define $\mu(f)$ as the cohomology class dual to $\gamma(f)^{-1}(\Sigma_n)$, see [24] for example.

I.5 Special Lagrangian submanifolds in \mathbf{C}^n

Lagrangian submanifolds are submanifolds of \mathbf{C}^n whose tangent space at each point is a Lagrangian subspace. They have a Gauss map into the Grassmannian Λ_n , namely into $U(n)/O(n)$. We look now at the submanifolds whose Gauss map takes values in $S\Lambda_n = SU(n)/SO(n)$. These are the special Lagrangian submanifolds, invented by Harvey and Lawson [18].

I.5.a Special Lagrangian subspaces

An oriented subspace P of \mathbf{C}^n is said to be *special Lagrangian* if it has a positive orthonormal basis that is a *special unitary* basis of \mathbf{C}^n .

For instance, if $n = 1$, as \mathbf{C} has a unique special unitary basis (the group $SU(1)$ is the trivial group), there is only one special Lagrangian subspace in \mathbf{C} , the line $\mathbf{R} \subset \mathbf{C}$... this will not be a very interesting notion in dimension 1. Fortunately, for $n \geq 2$, this is more exciting. Identify the space \mathbf{C}^2 with the skew-field \mathbf{H} of quaternions:

$$\begin{aligned} Z = (z_1, z_2) &= X + iY \\ &= (x_1 + iy_1, x_2 + iy_2) \\ &= (x_1 + iy_1) + j(x_2 + iy_2) \\ &= (x_1 + jx_2) + i(y_1 - jy_2). \end{aligned}$$

The 2×2 matrices that are in $SU(2)$ are the matrices of the form

$$\begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix} \text{ with } |z_1|^2 + |z_2|^2 = 1.$$

Thus the special Lagrangian planes are those who have an orthonormal basis (Z, Z') with Z and Z' of the form

$$\begin{cases} Z = (x_1 + iy_1) + j(x_2 + iy_2) \\ Z' = (-x_2 + iy_2) + j(x_1 - iy_1). \end{cases}$$

Notice that

$$Z' = [(x_1 + iy_1) + j(x_2 + iy_2)]j = Zj.$$

Thus a basis (Z, Z') of \mathbf{C}^2 is special unitary if and only if $Z' = Zj$. Now use multiplication by j to give \mathbf{H} the structure of a complex vector space. One has:

Proposition I.5.1. *The special Lagrangian subspaces of \mathbf{C}^2 are the complex lines with respect to the complex structure defined by the multiplication by j . The Grassmannian SA_2 is a complex projective line. \square*

Remark I.5.2. Notice also that $SA_2 = SU(2)/SO(2) = S^3/S^1$. This is indeed a dimension-2 sphere.

To distinguish the special Lagrangian subspaces among all the Lagrangian subspaces or the special unitary matrices among all the unitary matrices, one uses the (complex) determinant. To globalize the notion of special Lagrangian subspace and define special Lagrangian submanifolds, it will be practical (and natural) to describe the linear objects by differential forms. The form corresponding to the complex determinant is

$$\Omega = dz_1 \wedge \cdots \wedge dz_n.$$

Expressing the definition of the determinant, namely

$$(Ae_1) \wedge \cdots \wedge (Ae_n) = (\det A)e_1 \wedge \cdots \wedge e_n,$$

we see that, for $A \in GL(n; \mathbf{C})$, we have indeed

$$A^*\Omega = (\det A)\Omega.$$

Hence

$$\det A = 1 \iff A^*\Omega = \Omega.$$

In order to work with real subspaces, we need an additional notation: call α and β the two degree n real forms:

$$\alpha = \operatorname{Re} \Omega, \quad \beta = \operatorname{Im} \Omega.$$

For instance, in dimension 1, $\Omega = dz$, $\alpha = dx$ and $\beta = dy$. In dimension 2,

$$\begin{aligned} \Omega &= dz_1 \wedge dz_2 = (dx_1 + idy_1) \wedge (dx_2 + idy_2) \\ &= dx_1 \wedge dx_2 - dy_1 \wedge dy_2 + i(dy_1 \wedge dx_2 + dx_1 \wedge dy_2), \end{aligned}$$

that is

$$\begin{cases} \alpha = dx_1 \wedge dx_2 - dy_1 \wedge dy_2 \\ \beta = dy_1 \wedge dx_2 + dx_1 \wedge dy_2. \end{cases}$$

Proposition I.5.3. *Let P be an oriented (real) vector subspace of dimension n in \mathbf{C}^n . The number $\Omega(x_1 \wedge \cdots \wedge x_n)$ depends only on P and not on the positive orthonormal basis (x_1, \dots, x_n) of P used to express it.*

Proof. Consider the $2n$ vectors $(x_1, \dots, x_n, ix_1, \dots, ix_n)$ and the linear mapping $A : \mathbf{C}^n \rightarrow \mathbf{C}^n$ defined by the images of the vectors of the canonical basis:

$$A(e_j) = x_j, \quad A(ie_j) = ix_j$$

(so that A is complex linear). Then

$$\Omega(x_1 \wedge \cdots \wedge x_n) = \det_{\mathbf{C}} A.$$

If (gx_1, \dots, gx_n) is a positive orthonormal basis of P (that is, if $g \in \text{SO}(n)$), one gets

$$\begin{aligned} \Omega(gx_1 \wedge \cdots \wedge gx_n) &= \det_{\mathbf{C}}(gA) = \det_{\mathbf{C}} g \det_{\mathbf{C}} A = \det_{\mathbf{R}} g \det_{\mathbf{C}} A \\ &= \det_{\mathbf{C}} A = \Omega(x_1 \wedge \cdots \wedge x_n) \end{aligned}$$

(since $g \in \text{SO}(n) \subset \text{GL}(n; \mathbf{R}) \subset \text{GL}(n; \mathbf{C})$). □

We will thus denote $\Omega(P)$ the number $\Omega(x_1 \wedge \cdots \wedge x_n)$. Similarly, denote $\alpha(P)$ and $\beta(P)$ its real and imaginary parts.

Remark I.5.4. Notice that $\Omega(P)$ is non zero if and only if the $2n$ vectors

$$(x_1, \dots, x_n, ix_1, \dots, ix_n)$$

form a basis of \mathbf{C}^n over \mathbf{R} , that is, if and only if $P \cap iP = \{0\}$ or P does not contain any complex line. These subspaces are said to be *totally real*. This is in particular the case for Lagrangian subspaces.

Proposition I.5.5. *A real subspace P of \mathbf{C}^n has an orientation for which it is a special Lagrangian subspace if and only if P is Lagrangian and $\beta(P) = 0$.*

Proof. Let P be a Lagrangian subspace. Choose (x_1, \dots, x_n) , an orthonormal basis which is the image of the canonical basis of \mathbf{C}^n by a unitary matrix A . Thus

$$\Omega(P) = \det_{\mathbf{C}} A \in S^1.$$

For P to have a positive basis that is special unitary, it is necessary and sufficient that $\det_{\mathbf{C}} A$ be equal to ± 1 , that is, that

$$\text{Im } \Omega(P) = 0.$$

□

Here is a last elementary remark on linear subspaces:

Proposition I.5.6. *Let $Q \subset \mathbf{C}^n$ be an oriented isotropic linear subspace of dimension $n - 1$. There exists a unique special Lagrangian subspace that contains Q .*

Proof. Choose a positive orthonormal basis (x_1, \dots, x_{n-1}) of Q . In the complex line that is the orthogonal, with respect to the Hermitian form, of the complex subspace spanned by the x_i 's, there is a unique vector x_n such that the basis $(x_1, \dots, x_{n-1}, x_n)$ is a special unitary basis of \mathbf{C}^n . □

I.5.b Special Lagrangian submanifolds

A Lagrangian immersion

$$f : V \longrightarrow \mathbf{C}^n$$

of an oriented manifold into \mathbf{C}^n is *special* if $T_x f(T_x V)$ is a special Lagrangian subspace for every x . The Gauss map then takes values in $\mathrm{SL}_n \subset \widetilde{\Lambda}_n$.

Examples I.5.7. (1) In dimension 1, the tangent space must be the unique special Lagrangian $\mathbf{R} \subset \mathbf{C}$ for all x . If V is connected, f must thus be the immersion of an open subset of \mathbf{R} by $t \mapsto t + ia$. We have already noticed that this dimension will not be very exciting.

(2) In dimension 2, $T_x f(T_x V)$ must be a j -complex line for all x , f is thus the immersion of a j -complex curve into \mathbf{C}^2 . This gives quite a lot of examples.

Remark I.5.8. The Maslov class of a *special* Lagrangian immersion into \mathbf{C}^n is zero. Of course, as the examples above show it, there are much more Lagrangian immersions with zero Maslov class than there are special Lagrangian immersions.

In terms of forms, to say that the immersion

$$f : V \longrightarrow \mathbf{C}^n$$

is special Lagrangian is to say that it satisfies

- firstly $f^* \omega = 0$ (it is Lagrangian)
- secondly $f^* \beta = 0$ (it is special).

Proposition I.5.9. *If f is a special Lagrangian immersion, $f^* \Omega$ is a volume form on V .*

Proof. The complex form Ω has type $(n, 0)$ and defines an n -form $f^* \Omega$ on V , which is real since its imaginary part vanishes on V . Let x be a point in V and let (X_1, \dots, X_n) be a basis of $T_x V$. One has

$$(f^* \Omega)_x (X_1, \dots, X_n) = \Omega_{f(x)} (T_x f(X_1), \dots, T_x f(X_n)) \neq 0$$

because of Remark I.5.4 and since V is Lagrangian. Thus $f^* \Omega$ never vanishes. \square

In dimensions 1 and 2, the special Lagrangian submanifolds are non compact (in dimension 2, Liouville's theorem forbids complex curves in \mathbf{C}^2 to be compact). This is actually always the case, a straightforward application of Proposition I.5.9:

Corollary I.5.10. *There is no special Lagrangian immersion from a compact manifold into \mathbf{C}^n .*

Proof. If $f : V \rightarrow \mathbf{C}^n$ is a special Lagrangian immersion, $f^*\Omega$ is a volume form on V . But Ω is an exact complex form :

$$\Omega = dz_1 \wedge \cdots \wedge dz_n = d(z_1 dz_2 \wedge \cdots \wedge dz_n).$$

Decompose $z_1 dz_2 \wedge \cdots \wedge dz_n$ into its real and imaginary parts to get

$$\alpha = d\operatorname{Re}(z_1 dz_2 \wedge \cdots \wedge dz_n) = d\eta$$

and eventually

$$f^*\Omega = f^*\alpha = d(f^*\eta).$$

The manifold V thus has an exact volume form, and this prevents it of being compact. \square

Let us give now examples of special Lagrangian submanifolds in \mathbf{C}^n , starting from the examples of Lagrangians constructed in section I.4.

I.5.c Graphs of forms

Let us begin with Proposition I.4.1. Let $f : \mathbf{R}^n \rightarrow \mathbf{R}$ be a function. We require the graph of ∇f , a Lagrangian submanifold, to be a special Lagrangian submanifold. The $n = 1$ case is not interesting. For $n = 2$, the Lagrangian immersion associated with the function f is

$$F : (x, y) \longmapsto \left(x, y, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

and the form β is

$$\beta = dy_1 \wedge dx_2 + dx_1 \wedge dy_2.$$

Then

$$\begin{aligned} F^*\beta &= d\left(\frac{\partial f}{\partial x}\right) \wedge dy + dx \wedge d\left(\frac{\partial f}{\partial y}\right) \\ &= \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx \wedge dy. \end{aligned}$$

We thus have:

Proposition I.5.11. *Let U be an open subset of \mathbf{R}^2 and $f : U \rightarrow \mathbf{R}$ a function of class \mathcal{C}^2 . The graph of ∇f is a special Lagrangian submanifold of \mathbf{C}^2 if and only if f is a harmonic function. \square*

Notice that the condition is linear. Starting from dimension 3, this is no more the case. The function f must satisfy a complicated non linear partial differential

equation, expressed in Proposition I.5.12 below. Let us begin by a notation. Denote by $\text{Hess}(f)$ the Hessian matrix of f , namely the matrix

$$\text{Hess}(f)_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

and by $\sigma_k(\text{Hess}(f))$ the k -th elementary symmetric functions of its eigenvalues. More generally, for an $n \times n$ real matrix A , write

$$\det(A - X \text{Id}) = \sum_{k=0}^n (-1)^k \sigma_k(A) X^{n-k}.$$

For example, $\sigma_1(\text{Hess}(f))$ is the trace of the Hessian matrix, the Laplacian Δf of f .

Proposition I.5.12. *Let U be an open subset of \mathbf{R}^n and $f : U \rightarrow \mathbf{R}$ a function of class \mathcal{C}^2 . The graph of ∇f is a special Lagrangian submanifold of \mathbf{C}^n if and only if f satisfies the partial differential equation*

$$\sum_{k \geq 0} (-1)^k \sigma_{2k+1}(\text{Hess}(f)) = 0.$$

Examples I.5.13. For $n = 1$, the differential equation is $f''(t) = 0$ or $f'(t)$ constant, and this is precisely the differential equation of the special Lagrangian submanifolds. For $n = 2$, again, only σ_1 appears in the (linear) relation, which expresses the fact that the function f must be harmonic. For $n = 3$, the relation is

$$\sigma_1(\text{Hess}(f)) = \sigma_3(\text{Hess}(f))$$

or

$$\Delta f = \det(\text{Hess}(f)).$$

Remark I.5.14. The only order-1 term (in df) in this partial differential equation is Δf , so that the “linear part” of this equation is $\Delta f = 0$. This should be compared with McLean’s theorem (Theorem II.6.1 below).

Proof of the proposition. The tangent space to the graph of ∇f at the point $(x, \nabla f_x)$ is the image of the plane \mathbf{R}^n under the linear map $\text{Id} + i(d^2 f)_x$. This is a special Lagrangian subspace if and only if

$$\text{Im}(\det_{\mathbf{C}}(\text{Id} + i(d^2 f)_x)) = 0.$$

We still must check that, for any real symmetric matrix A , one has

$$\text{Im}(\det_{\mathbf{C}}(\text{Id} + iA)) = \sum_{k \geq 0} (-1)^k \sigma_{2k+1}(A).$$

Since A is real symmetric, it is diagonalizable in an orthonormal basis. It is clear that the two sides of the relation to be proved are invariant under conjugation by matrices in $O(n)$. One may thus assume that the matrix A is the diagonal $(\lambda_1, \dots, \lambda_n)$. The left hand side is then $\text{Im} \prod_j (1 + i\lambda_j)$ and it clearly coincides with the right hand side. \square

I.5.d Normal bundles of surfaces

Let $f : V \rightarrow \mathbf{R}^n$ be an immersion of a dimension- k manifold into \mathbf{R}^n . We know (see §I.4.c) that its normal bundle has a natural Lagrangian immersion into $\mathbf{R}^n \times \mathbf{R}^n$. Look now for the conditions under which this is a special Lagrangian immersion.

For the sake of simplicity, suppose here that $k = 2$ and $n = 3$ (case of surfaces in \mathbf{R}^3). There is a more general discussion in [18].

Fix a point x_0 in V , a unit normal vector field $n = n(x)$ on a neighbourhood of x_0 . The restriction of the normal bundle

$$Nf = \{(x, v) \mid x \in V, v \in \mathbf{R}^3, v \perp T_x f(T_x v)\}$$

to this neighborhood is isomorphic with $V \times \mathbf{R}$ by $(x, \mu) \mapsto (x, \mu n(x))$. We map Nf to \mathbf{C}^3 by

$$(x, \mu) \mapsto \mu n(x) + if(x)$$

(notice that, this time, the immersion f appears in the second copy of \mathbf{R}^n , that of purely imaginary vectors).

Let us now choose an orthonormal basis (e_1, e_2) of $T_{x_0}V$. Assume that this basis is orthogonal with respect to the second fundamental form, that is, to the symmetric bilinear form defined on $T_{x_0}V$ by

$$\Pi(X, Y) = -(T_{x_0}n(X), Y).$$

We have

$$T_{x_0}n(e_1) = -\lambda_1 e_1, \quad T_{x_0}n(e_2) = -\lambda_2 e_2$$

where λ_1 and λ_2 are the two ‘‘principal curvatures’’ of V at x_0 .

Consider now the tangent space to Nf at (x_0, v) where $v = \mu n(x_0) \in N_{x_0}f = \mathbf{R} \cdot n(x_0)$. The tangent mapping to our immersion is

$$\begin{aligned} P_0 = T_{(x_0, \mu)}(Nf) = T_{x_0}V \oplus N_{x_0}f &\longrightarrow \mathbf{R}^3 \times \mathbf{R}^3 \\ (\xi, \eta) &\longmapsto (\eta n(x_0) + \mu T_{x_0}n(\xi), T_{x_0}f(\xi)). \end{aligned}$$

The images of the basis vectors are

$$\begin{aligned} e_1 &\longmapsto (-\mu\lambda_1 e_1, e_1) \\ e_2 &\longmapsto (-\mu\lambda_2 e_2, e_2) \\ n &\longmapsto (n, 0). \end{aligned}$$

Thus

$$\begin{aligned} \Omega(P_0) &= (dz_1 \wedge dz_2 \wedge dz_3) (((i - \mu\lambda_1)e_1) \wedge ((i - \mu\lambda_2)e_2) \wedge n) \\ &= (i - \mu\lambda_1)(i - \mu\lambda_2), \end{aligned}$$

so that P_0 is a special Lagrangian if and only if $\mu(\lambda_1 + \lambda_2) = 0$. This is to say that the trace of $T_{x_0}n$ is zero. In other words, we have shown:

Proposition I.5.15. *The immersion of the normal bundle of*

$$f : V \longrightarrow \mathbf{R}^3$$

into \mathbf{C}^3 is a special Lagrangian immersion if and only if f is a minimal immersion. \square

For more information on minimal surfaces, see, for example, the beautiful surveys in [29] and the references quoted there.

Remark I.5.16. It is true that we have already mentioned Riemannian metrics in these notes, but up to now, they have had only an auxiliary role. The result presented here is a genuine Riemannian one.

I.5.e From integrable systems

Being compact, Lagrangian tori obtained as “Liouville tori” cannot be special Lagrangian submanifolds in \mathbf{C}^n . One can try to replace them by special Lagrangian submanifolds with the help of the remark included in Proposition I.5.6: the idea is to consider a (necessarily isotropic) subtorus in a Liouville torus T^n and to add a direction to construct another Lagrangian submanifold, which will be special.

Here is an example, coming from [18], of such a construction. Start from an orbit L of the standard action of T^n on \mathbf{C}^n (see §I.4.c), namely a common level set of the functions

$$g_1(z) = \frac{1}{2} |z_1|^2, \dots, g_n(z) = \frac{1}{2} |z_n|^2,$$

say $g_i = a_i$, none of the a_i 's being zero, so that L is a Lagrangian torus. Choose a subtorus of T^n :

$$T^{n-1} = \{(t_1, \dots, t_n) \in T^n \mid t_1 \cdots t_n = 1\}.$$

Let V be an orbit of this subtorus, an isotropic torus of dimension $n - 1$. Consider the Hamiltonian vector fields Y_1, \dots, Y_n associated to the functions g_i :

$$\begin{cases} Y_1(z_1, \dots, z_n) = (iz_1, 0, \dots, 0) \\ \vdots \\ Y_n(z_1, \dots, z_n) = (0, \dots, 0, iz_n). \end{cases}$$

Let $z = (z_1, \dots, z_n)$ be a point of V . The tangent space to L at z is spanned by the values of the Y_i 's, the tangent space to V is the hyperplane consisting of the vectors $\sum \lambda_i Y_i$ satisfying $\sum \lambda_i = 0$. It is spanned by the values at z of the vector fields

$$X_1 = Y_1 - Y_n, \dots, X_{n-1} = Y_{n-1} - Y_n,$$

that are the Hamiltonian vector fields of the functions

$$f_1 = g_1 - g_n, \dots, f_{n-1} = g_{n-1} - g_n.$$

We are looking now for an n -th function f such that the subspace spanned by the vectors X_1, \dots, X_{n-1} and X_f is a special Lagrangian at each point where the vectors are independent. The subspace $F = \langle X_1, \dots, X_{n-1} \rangle$ is isotropic and has dimension $n - 1$.

We look for X_f as a linear combination $X_f = \sum \lambda_j Y_j$ such that:

- The vector field X_f is in the subspace orthogonal to $\langle X_1, \dots, X_{n-1} \rangle$ (for the Hermitian form), that is, $\langle X_f, X_k \rangle = 0$ for $1 \leq k \leq n - 1$. This gives $\lambda_k |z_k|^2 - \lambda_n |z_n|^2 = 0$. Thus λ_k must have the form

$$\lambda_k = \frac{\mu(z_1, \dots, z_n)}{|z_k|^2}.$$

- The determinant

$$\begin{vmatrix} iz_1 & 0 & \lambda_1 iz_1 \\ 0 & \ddots & 0 \\ \vdots & iz_{n-1} & \vdots \\ -iz_n \dots & -iz_n & \lambda_n iz_n \end{vmatrix}$$

is real. This allows to determine the function μ .

Subtracting the linear combination $\lambda_1 X_1 + \dots + \lambda_{n-1} X_{n-1}$ from the last vector, this vector becomes $(\lambda_1 + \dots + \lambda_n) Y_n$, so that the determinant is $i^n (\lambda_1 + \dots + \lambda_n) z_1 \dots z_n$. We are thus looking for functions f and μ such that

$$X_f(z_1, \dots, z_n) = (\lambda_1 iz_1, \dots, \lambda_n iz_n) \quad \text{and} \quad i^n (\mu(z_1, \dots, z_n)) z_1 \dots z_n \text{ is real.}$$

For any index j , we must have:

$$2 \frac{\partial f}{\partial z_j} = \bar{\lambda}_j \bar{z}_j, \quad 2 \frac{\partial f}{\partial \bar{z}_j} = \lambda_j z_j, \quad \text{and} \quad i^n \mu(z_1, \dots, z_n) z_1 \dots z_n \in \mathbf{R}.$$

The functions

$$f(z_1, \dots, z_n) = z_1 \dots z_n + \overline{z_1 \dots z_n}, \quad \mu = 2 \overline{z_1 \dots z_n}$$

give a solution when $i^n \in \mathbf{R}$, namely when n is even. When n is odd, we rather take

$$f(z_1, \dots, z_n) = \frac{1}{i} (z_1 \dots z_n - \overline{z_1 \dots z_n}), \quad \mu = 2 \overline{z_1 \dots z_n}$$

Proposition I.5.17. *The functions f_1, \dots, f_n defined by*

$$f_1(z_1, \dots, z_n) = \frac{1}{2} (|z_1|^2 - |z_n|^2), \dots, f_{n-1}(z_1, \dots, z_n) = \frac{1}{2} (|z_{n-1}|^2 - |z_n|^2)$$

and

$$f_n(z_1, \dots, z_n) = \begin{cases} \operatorname{Re}(z_1 \dots z_n) & \text{if } n \text{ is even} \\ \operatorname{Im}(z_1 \dots z_n) & \text{if } n \text{ is odd} \end{cases}$$

form an integrable system on $\mathbf{C}^n = \mathbf{R}^n \times \mathbf{R}^n$, all the regular common level sets of which are special Lagrangian cylinders $T^{n-1} \times \mathbf{R}$.

Proof. The only thing that is left to prove is that the regular levels are “cylinders” $T^{n-1} \times \mathbf{R}$. As we are dealing with an integrable system, we know that the levels are endowed with an \mathbf{R}^n -action. Here the $n - 1$ first vector fields are periodic and in particular complete; the last one is complete too, because the level is a closed submanifold of \mathbf{C}^n . The action is thus an action of $T^{n-1} \times \mathbf{R}$ and this is a free action, as the level, being special Lagrangian, cannot be compact. \square

Exercise I.19 describes essentially the same construction.

I.5.f Special Lagrangian submanifolds invariant under $\mathrm{SO}(n)$

The next and sporadic examples also come from [18]. Start from a smooth curve Γ in

$$\mathbf{C} = \mathbf{C} \times \{0\} \subset \mathbf{C} \times \mathbf{C}^{n-1} = \mathbf{C}^n$$

and “rotate” it with the help of the diagonal $\mathrm{SO}(n)$ -action, namely

$$g \cdot (X + iY) = g \cdot X + ig \cdot Y \text{ for } g \in \mathrm{SO}(n) \text{ and } X, Y \in \mathbf{R}^n.$$

If we assume the curve does not pass through 0, we get a submanifold of \mathbf{C}^n :

$$V = \{(x + iy)u \mid x + iy \in \Gamma, u \in \mathbf{R}^n, u = g(e_1) \text{ for some } g \in \mathrm{SO}(n)\}$$

(notice that u describes a sphere $S^{n-1} \subset \mathbf{R}^n$). The tangent space to V at $(x + iy)u$ is spanned by the vectors $(x + iy)U$ with $U \in T_u S^{n-1}$ and the $(\xi + i\eta)u$ with $\xi + i\eta$ tangent to Γ at $x + iy$. The submanifold V is always Lagrangian, as is easily checked:

$$\begin{aligned} \omega((x + iy)U, (x + iy)U') &= xy(U \cdot U' - U' \cdot U) = 0, \\ \omega((x + iy)U, (\xi + i\eta)u) &= (x\eta - y\xi)U \cdot u = 0. \end{aligned}$$

It is special Lagrangian if and only if, denoting (U_1, \dots, U_{n-1}) a basis of $T_u S^{n-1}$,

$$\det_{\mathbf{C}}((x + iy)U_1, \dots, (x + iy)U_{n-1}, (\xi + i\eta)u) \in \mathbf{R}.$$

But this determinant is equal to $(x + iy)^{n-1}(\xi + i\eta) \det_{\mathbf{C}}(U_1, \dots, U_{n-1}, u)$, or to $(x + iy)^{n-1}(\xi + i\eta) \det_{\mathbf{R}}(U_1, \dots, U_{n-1}, u)$ since these vectors are in $\mathbf{R}^n \subset \mathbf{C}^n$. The condition is thus that

$$(x + iy)^{n-1}(\xi + i\eta) \in \mathbf{R} \text{ for any tangent vector } \xi + i\eta \text{ to } \Gamma.$$

We get eventually:

Proposition I.5.18. *The Lagrangian submanifold of \mathbf{C}^n*

$$V = \{(x + iy)u \mid (x + iy) \in \Gamma, u \in S^{n-1} \subset \mathbf{R}^n\}$$

is special Lagrangian if and only if, on Γ , the function $\mathrm{Im}((x + iy)^n)$ is constant. \square

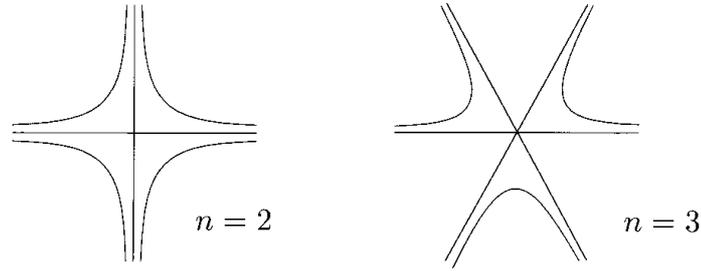


Figure I.9

Remark I.5.19. This method gives essentially one special Lagrangian submanifold in any dimension, which is not much!

Remark I.5.20. Any connected component of Γ is diffeomorphic to \mathbf{R} , the special Lagrangian submanifolds obtained are (unions of) copies of $S^{n-1} \times \mathbf{R}$.

To draw a picture of the special Lagrangian submanifold, one draws first the curve Γ (in the (x, y) plane), then its wave front (in the (x, z) plane). One then notices that the Liouville form $\lambda = Y \cdot dX$ is, on V :

$$\begin{aligned} \lambda = Y \cdot dX &= (yu) \cdot d(xu) \\ &= (yu) \cdot ((dx)u + xdu) \\ &= ydx \end{aligned}$$

(since $udu = \frac{1}{2} \|u\|^2 = 0$) so that the wave front of V is

$$\{(xu, z) \in \mathbf{R}^n \times \mathbf{R} \mid (x, z) \text{ is a point of the wave front of } \Gamma\}.$$

For example, for $n = 2$, the curve Γ is a hyperbola $xy = \text{constant}$, its wave front is the curve $z = \log x$ and the wave front of the special Lagrangian submanifold is the surface of revolution obtained by rotating the graph of the logarithm function about the z -axis (Figure I.10).

I.6 Appendices

I.6.a The topology of the symplectic group

Proposition I.6.1. *The manifold $\text{Sp}(2n)$ is diffeomorphic to the Cartesian product of the group $\text{U}(n)$ with a convex open cone of a vector space of dimension $n(n+1)$.*

Corollary I.6.2. *The symplectic group $\text{Sp}(2n)$ is path connected. The injection of $\text{U}(n)$ in $\text{Sp}(2n)$ induces an isomorphism*

$$\mathbf{Z} = \pi_1 \text{U}(n) \longrightarrow \pi_1 \text{Sp}(2n).$$

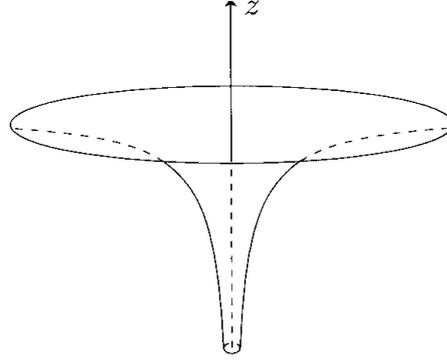


Figure I.10

Proof of the proposition. Let $A \in \text{Sp}(2n)$. As any invertible transformation of \mathbf{R}^{2n} , A can be written in a unique way as a product

$$A = S \cdot \Omega$$

where S is the positive definite symmetric matrix $S = \sqrt{A^t A}$ and Ω is the orthogonal matrix $\Omega = S^{-1}A$. As A is symplectic, the matrix S is also symplectic: ${}^t A$ and $A^t A$ are symplectic, the matrix $A^t A$ is symmetric, positive definite, thus it is diagonalizable in an orthonormal basis and S is the matrix that, in this basis, is the diagonal of the square roots of the eigenvalues of $A^t A$, so that S is indeed symplectic as is $A^t A$. One deduces that

$$\Omega = S^{-1}A \in \text{Sp}(2n) \cap \text{O}(2n) = \text{U}(n)$$

and thus that Ω is a unitary matrix. We have thus obtained a bijection

$$\begin{aligned} \text{Sp}(2n) &\longrightarrow \text{U}(n) \times \mathcal{S} \\ A &\longmapsto ((\sqrt{A^t A})^{-1}A, \sqrt{A^t A}) \end{aligned}$$

where \mathcal{S} denotes the set of positive definite symmetric matrices that are symplectic. We still have to prove that this space is an open convex cone in a vector space of dimension $n(n+1)$. Write the matrices as block matrices in a symplectic basis. Let $S \in \mathcal{S}$, we have

$$S = \begin{pmatrix} A & B \\ {}^t B & C \end{pmatrix} \text{ with } A \text{ and } C \text{ positive definite symmetric and } {}^t S J S = J.$$

The last condition, that expresses the fact that S is symplectic, is equivalent to

$$BA \text{ is symmetric and } C = A^{-1}(\text{Id} + B^2).$$

The mapping

$$\begin{aligned} \mathcal{S} &\longrightarrow \text{Sym}(n; \mathbf{R}) \times \text{Sym}^+(n; \mathbf{R}) \\ S &\longmapsto (BA, A) \end{aligned}$$

is the desired diffeomorphism. The open set $\text{Sym}^+(n; \mathbf{R})$ of all positive definite symmetric real matrices is obviously an open convex cone in the vector space $\text{Sym}(n; \mathbf{R})$ of all symmetric matrices, the product is an open convex cone of the product space, that has dimension $2\frac{n(n+1)}{2}$. \square

Proof of the corollary. The convex cone $\text{Sym}(n; \mathbf{R}) \times \text{Sym}^+(n; \mathbf{R})$ is contractible. \square

Remark I.6.3. There is another beautiful proof of this type of contractibility results, due to Sévenec, in [5].

I.6.b Complex structures

If E is a vector space endowed with a symplectic form ω , it is said that an endomorphism J of E is a complex structure *calibrated* by ω if $J^2 = -\text{Id}$ (J is a complex structure),

$$\omega(Jv, Jw) = \omega(v, w)$$

(J is symplectic) and

$$g(v, w) = \omega(v, Jw)$$

is a scalar product (namely a positive definite bilinear form) on E .

I.6.c Hamiltonian vector fields, integrable systems

In this appendix, denote for simplicity $\mathbf{C}^n = \mathbf{R}^{2n}$ by W . It can be replaced by any symplectic manifold W (see § II.1).

Hamiltonian vector fields

To any function $H : W \rightarrow \mathbf{R}$, the symplectic form allows to associate a vector field, a kind of gradient, the *Hamiltonian vector field* X_H (sometimes called the “symplectic gradient” H). This is the vector field defined by the relation

$$\omega_x(Y, X_H(x)) = (dH)_x(Y) \text{ for all } Y \in T_x W,$$

or by

$$\iota_{X_H} \omega = -dH.$$

In coordinates, one has

$$X_H(x_1, \dots, x_n, y_1, \dots, y_n) = \left(\frac{\partial H}{\partial y_1}, \dots, \frac{\partial H}{\partial y_n}, -\frac{\partial H}{\partial x_1}, \dots, -\frac{\partial H}{\partial x_n} \right).$$

Notice that the vector field X_H vanishes at x if and only if x is a critical point of the function H :

$$X_H(x) = 0 \iff (dH)_x = 0.$$

In particular, the singularities (or zeroes) of a Hamiltonian vector field are the critical points of a function.

Notice also that the function H is constant along the trajectories, or integral curves, of the vector field X_H : as ω_x is skew symmetric, we have $(dH)(X_H) = 0$ or $X_H \cdot H = 0$.

The Poisson bracket

Assume now that f and g are two functions on W . Define their “Poisson bracket” $\{f, g\}$ by the formula

$$\{f, g\} = X_f \cdot g = dg(X_f).$$

In coordinates, one has

$$\{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} - \frac{\partial g}{\partial y_i} \frac{\partial f}{\partial x_i} \right).$$

Notice that

$$X_f \cdot g = dg(X_f) = \omega(X_f, X_g) = -\omega(X_g, X_f) = -df(X_g) = -X_g \cdot f,$$

so that $\{f, g\} = -\{g, f\}$. This shows that the Poisson bracket is skew-symmetric in f and g . By definition, this is also a derivation (in both entries); in other words, the Poisson bracket satisfies the Leibniz identity

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

Using the general relation

$$\mathcal{L}_X \iota_Y - \iota_Y \mathcal{L}_X = \iota_{[X, Y]}$$

and Cartan formula

$$\mathcal{L}_X = d\iota_X + \iota_X d,$$

we get

$$\begin{aligned} \iota_{[X_f, X_g]} \omega &= \mathcal{L}_{X_f} \iota_{X_g} \omega - \iota_{X_g} \mathcal{L}_{X_f} \omega \\ &= d\iota_{X_f} \iota_{X_g} \omega + \iota_{X_f} d\iota_{X_g} \omega - \iota_{X_g} d\iota_{X_f} \omega - \iota_{X_g} \iota_{X_f} d\omega \\ &= d\iota_{X_f} \iota_{X_g} \omega = d(\omega(X_g, X_f)) = -d\{f, g\}, \end{aligned}$$

in other words

$$[X_f, X_g] = X_{\{f, g\}}.$$

We also have

$$[X_f, X_g] \cdot h = \{\{f, g\}, h\}.$$

From this, we deduce that the Poisson bracket satisfies the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

and thus defines a Lie algebra structure on $\mathcal{C}^\infty(W)$, the mapping

$$\begin{array}{ccc} \mathcal{C}^\infty(W) & \longrightarrow & \mathcal{X}(W) \\ f & \longmapsto & X_f \end{array}$$

being a morphism of Lie algebras from $\mathcal{C}^\infty(W)$ (with the Poisson bracket) into the Lie algebra of vector fields (with the Lie bracket of vector fields).

Proof of the Jacobi identity. Apply the definition of the bracket of vector fields:

$$[X_f, X_g] \cdot h = X_f \cdot (X_g \cdot h) - X_g \cdot (X_f \cdot h),$$

and the equality above to get

$$\begin{aligned} \{\{f, g\}, h\} &= [X_f, X_g] \cdot h \\ &= X_f \cdot (X_g \cdot h) - X_g \cdot (X_f \cdot h) \\ &= X_f \cdot \{g, h\} - X_g \cdot \{f, h\} \\ &= \{f, \{g, h\}\} - \{g, \{f, h\}\}. \end{aligned}$$

This, taking into account the skew-symmetry of the Poisson bracket, is equivalent to the Jacobi identity. \square

Integrable systems

As any vector field does it, the Hamiltonian vector field X_H defines a differential system on W , namely,

$$\dot{x}(t) = X_H(x(t)),$$

the *Hamiltonian system* associated with H . The function H is constant along the trajectories of this system, in other words

$$X_H \cdot H = 0 \text{ or } dH(X_H) = 0.$$

It is said that H is a first integral of the system. More generally, a function $f : W \rightarrow \mathbf{R}$ that is constant along the integral curves of a vector field X is called a *first integral* of X . In the case of a Hamiltonian vector field X_H , the equality $X_H \cdot f = 0$ is equivalent to $\{f, H\} = 0$, we say that the functions f and H *commute*.

It is said that a Hamiltonian system is integrable if it has “as many commuting first integrals as possible”. Let us explain this:

- Let f_1, \dots, f_k be commuting first integrals of the system X_H , so that $\{f_i, f_j\} = 0$ for all i and j . Each one is constant on the trajectories of the Hamiltonian system associated to each other one.

- The expression “as many as possible”: at any point x of W , the subspace of $T_x W$ spanned by the Hamiltonian vector fields of the functions f_i is isotropic:

$$\omega(X_{f_i}, X_{f_j}) = \pm \{f_i, f_j\} = 0.$$

Its dimension is thus at most $n = \frac{1}{2} \dim W$. It is required that, at least for x in an open dense subset of W , this subspace has maximal dimension n .

- Notice that the vectors X_{f_i} are independent at x if and only if the linear forms $(df_i)_x$ are independent.

Definition I.6.4. The function H or the Hamiltonian vector field X_H on W is qualified as *integrable* if it has n independent commuting first integrals.

Examples I.6.5. Every function depending only of the coordinates y_i ,

$$H = H(y_1, \dots, y_n)$$

is integrable: the functions y_i are independent commuting first integrals. Every Hamiltonian system on \mathbf{C} is integrable. Similarly, a Hamiltonian system on \mathbf{C}^2 is integrable if and only if it has a “second first integral”.

Exercises

Exercise I.1. Let V be a real vector space and V^* be its dual. Check that the form ω defined on $V \oplus V^*$ by

$$\omega((v, \alpha), (w, \beta)) = \alpha(w) - \beta(v)$$

is a symplectic form

Exercise I.2 (Relative linear Darboux theorem). Let F be a vector subspace of a symplectic vector space E . Assume that the restriction of the symplectic form to F has rank $2r$. Show that there exists a symplectic basis $(e_1, \dots, e_n, f_1, \dots, f_n)$ of E such that $(e_1, \dots, e_r, e_{r+1}, \dots, e_{r+k}, f_1, \dots, f_r)$ is a basis of F (k is the integer defined by $2r + k = \dim F$).

Exercise I.3. Show that the symplectic group of \mathbf{C} is isomorphic with the special linear group $\mathrm{SL}(2; \mathbf{R})$.

Exercise I.4. Prove directly that the symplectic group $\mathrm{Sp}(2)$ is diffeomorphic to the product of a circle by an open disk.

Exercise I.5. Let $A \in \mathrm{Sp}(2n)$. Check that the matrices ${}^t A$ and A^{-1} are similar⁸. Show that λ is an eigenvalue of A if and only if λ^{-1} is also an eigenvalue, and that both occur with the same multiplicity.

⁸Thus A and A^{-1} are similar too.

Exercise I.6. Check that a non zero vector of a symplectic space can be mapped to any other non zero vector by a symplectic transformation (in other words, the symplectic groups acts transitively on the set of non zero vectors).

Show that, for $n > 1$, the symplectic group does not act transitively on the set of real 2-dimensional subspaces of \mathbf{C}^n .

Exercise I.7. Let $n > 1$ be an integer. Let P be a real plane (dimension-2 subspace) in \mathbf{C}^n . Show that P is either isotropic or symplectic. What are the orbits of the action of the symplectic group on the set of planes in \mathbf{C}^n ?

Exercise I.8. Let V be a vector space and V^* be its dual. Endow $V \oplus V^*$ with the symplectic form defined in Exercise I.1. Let $A : V \rightarrow V^*$ be a linear map. Prove that the graph of A is a Lagrangian subspace if and only if the bilinear form defined by A on V is symmetric.

Exercise I.9. Let E be a vector space endowed with a symplectic form ω and let F be (any) subspace of E . Prove that ω induces a symplectic structure on the quotient $F/F \cap F^\circ$.

Exercise I.10. Let E be an even dimensional vector space and let ω, ω' be two symplectic forms on E . Prove that the symplectic groups $\text{Sp}(E, \omega)$ and $\text{Sp}(E, \omega')$ are conjugated subgroups of $\text{GL}(E)$.

Let $\Omega(E)$ be the space of all symplectic forms on the vector space E . Prove that the linear group of E acts on this space by

$$(g \cdot \omega)(X, Y) = \omega(gX, gY).$$

Deduce that $\Omega(E)$ is in one-to-one correspondence⁹ with the homogeneous space $\text{GL}(E)/\text{Sp}(E)$, where $\text{Sp}(E)$ is the symplectic group $\text{Sp}(E, \omega_0)$ for a given form ω_0 on E .

Exercise I.11. Prove that, on any symplectic vector space, there are complex structures. Prove that a complex structure is an isometry and that it is skew-symmetric for the scalar product it defines.

Exercise I.12. Let V be a real vector space. Using a scalar product on V , construct a complex structure calibrated by the standard symplectic form on $V \oplus V^*$ and such that

$$(J(v), w) = v \cdot w \text{ for all } v, w \in V.$$

Exercise I.13. Assume that the wave front

$$\begin{array}{ccc}] - \alpha, \alpha[& \longrightarrow & \mathbf{R}^2 \\ t & \longmapsto & (x(t), z(t)) \end{array}$$

has an ordinary cusp for $t = 0$ with a tangent line transversal to the z -axis. Prove that this is the wave front of a Lagrangian *immersion* of $] - \alpha, \alpha[$ into \mathbf{R}^2 .

⁹This is actually a homeomorphism.

Exercise I.14. Prove that the wave front of the Whitney immersion $S^n \rightarrow \mathbf{C}^n$ is the hypersurface in \mathbf{R}^{n+1} image of the sphere S^n by

$$(x, a) \longmapsto \left(x, a \|x\|^2 + \frac{a^3}{3} - a \right)$$

(using the notation of Example I.4.3). Find the singular points of this wave front and draw it in the cases $n = 1$ (this is the eye, Figure I.2) and $n = 2$ (this is the flying saucer, Figure I.4).

Exercise I.15 (The swallow tail). Determine... and draw the wave front of the Lagrangian immersion described in §I.4.5 and on Figure I.1.

Exercise I.16. Prove that the Maslov class of the standard (Lagrangian) embedding of the circle is ± 2 . What is that of the Whitney immersion? Of the immersion defined by the crossbow¹⁰?

Exercise I.17 (Lagrangian cobordisms [3]). The space \mathbf{C}^n is endowed with its Liouville form λ and its symplectic form $d\lambda$. It is said that a Lagrangian immersion $f : L \rightarrow \mathbf{C}^n$ is “cobordant to zero” if there exists an *oriented* manifold V of dimension $n + 1$, with boundary, whose boundary is L , and a Lagrangian immersion

$$\tilde{f} : V \longrightarrow \mathbf{C}^{n+1}$$

transversal to the co-isotropic subspace $F = \mathbf{C}^n \oplus i\mathbf{R} \subset \mathbf{C}^{n+1}$, such that

$$\tilde{f}^{-1}(F \cap V) = \partial V = L$$

and such that the composition

$$L \xrightarrow{\tilde{f}|_L} F \longrightarrow F/F^\circ = \mathbf{C}^n$$

is the immersion f .

- (1) Prove that the Whitney immersion $S^n \rightarrow \mathbf{C}^n$ (§I.4.3) is cobordant to zero.
- (2) Assume that $f : S^1 \rightarrow \mathbf{C}$ is cobordant to zero. What can be said of $\int_{S^1} f^* \lambda$? Prove that, if a Lagrangian immersion $S^1 \rightarrow \mathbf{C}$ is cobordant to zero, it is exact.
- (3) Consider an exact Lagrangian immersion

$$f : S^1 \longrightarrow \mathbf{C}$$

and its wave front in \mathbf{R}^2 . Assume the singularities of the wave front are ordinary cusps. The tangent line to the front at any point is transversal

¹⁰Hint: orient the circle and notice that the unit tangent vector to the Whitney immersion does not take all the values in the circle. For the crossbow, notice that this immersion of the circle into \mathbf{C} may be deformed, among immersion, into the standard embedding.

to the z -axis. The circle S^1 is oriented. Count the cusps of type (a) with a $+$ sign, those of type (b) with a $-$ sign (Figure I.11) and get a number $N(f) \in \mathbf{Z}$. What is the value of $N(f)$ for the Whitney immersion? For the crossbow (Figure I.3)?

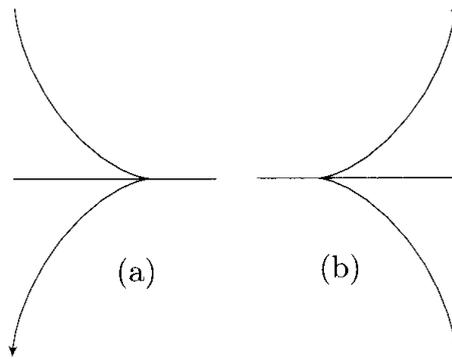


Figure I.11

- (4) The Lagrangian immersion $f : S^1 \rightarrow \mathbf{C}$ has a Gauss map $\gamma(f)$, taking its values in the Grassmannian $\tilde{\Lambda}_1$ of oriented Lagrangians in \mathbf{C} , that is a circle S^1 . Call σ the closed 1-form “ $d\theta$ ” on this circle. Prove that¹¹

$$N(f) = \frac{1}{2\pi} \int_{S^1} \gamma(f)^* \sigma.$$

- (5) Consider the mapping

$$\begin{aligned} j : \tilde{\Lambda}_1 &\longrightarrow \tilde{\Lambda}_2 \\ P &\longmapsto P \oplus \mathbf{R} \subset \mathbf{C} \oplus \mathbf{R} \subset \mathbf{C}^2. \end{aligned}$$

It can be shown (this is an additional question, use §I.3.f) that

$$j^* : H^1(\tilde{\Lambda}_2) \longrightarrow H^1(\tilde{\Lambda}_1)$$

is an isomorphism. Prove that if $f : S^1 \rightarrow \mathbf{C}$ is cobordant to zero, then $N(f) = 0$. Does there exist a Lagrangian immersion of a disk into \mathbf{C}^2 whose boundary is the crossbow?

¹¹This is to say that $N(f)$ is the Maslov class of the immersion f .

Exercise I.18 (From (x, y) to (z, \bar{z})). Writing

$$dz = dx + idy, \quad d\bar{z} = dx - idy$$

one gets a couple of relations between the expressions of the vector fields in coordinates (x, y) or (z, \bar{z}) . Prove for instance that

$$\begin{aligned} X_f &= \sum_{j=1}^n \left(\frac{\partial f}{\partial y_j} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial y_j} \right) \\ &= \frac{i}{2} \sum_{j=1}^n \left(\frac{\partial f}{\partial z_j} \frac{\partial}{\partial \bar{z}_j} - \frac{\partial f}{\partial \bar{z}_j} \frac{\partial}{\partial z_j} \right). \end{aligned}$$

Exercise I.19. Consider the vector field X given on \mathbf{C}^2 by

$$X(z_1, z_2) = (i\alpha_1 z_1, i\alpha_2 z_2)$$

(α_1 and α_2 being two real parameters).

(1) Check that

$$X(z_1, z_2) = \alpha_1 \left(iz_1 \frac{\partial}{\partial z_1} - i\bar{z}_1 \frac{\partial}{\partial \bar{z}_1} \right) + \alpha_2 \left(iz_2 \frac{\partial}{\partial z_2} - i\bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \right)$$

and show that the form $\iota_X \Omega$ is holomorphic.

- (2) Show that X preserves ω and find a function H such that $X = X_H$.
- (3) Under which condition does the vector field X preserve Ω ? Assume now that this condition holds. Find two functions g and h from \mathbf{C}^2 to \mathbf{R} such that

$$\iota_X \Omega = dg + idh.$$

Consider $H^{-1}(a) \cap h^{-1}(b)$. Show that, if a is a regular value of H , this is a special Lagrangian submanifold.

- (4) Describe the special Lagrangian submanifolds $H^{-1}(a) \cap h^{-1}(b)$ as complex j -curves, that is, by equations.
- (5) Check that they are diffeomorphic to $S^1 \times \mathbf{R}$. Hint: they are conics.

Chapter II

Lagrangian and special Lagrangian submanifolds in symplectic and Calabi-Yau manifolds

II.1 Symplectic manifolds

In order to deform a Lagrangian submanifold in \mathbf{C}^n , we must understand how a tubular neighbourhood looks like. We prove here that a Lagrangian submanifold has a neighbourhood which is diffeomorphic to a neighbourhood of the zero section in its cotangent bundle. To be precise and explicit, we need to define a symplectic structure on the cotangent bundles and more generally to say what a symplectic structure on a manifold is.

A *symplectic manifold* is a manifold W endowed with a non degenerate 2-form ω , namely, a non degenerate alternated bilinear form ω_x on each tangent space $T_x W$, which is required to be *closed*, ($d\omega = 0$). Notice that a symplectic manifold is even dimensional.

Examples II.1.1. (1) The first example is of course \mathbf{C}^n with the symplectic form we have used so far, considered as a differential form:

$$\omega = \sum_{j=1}^n dy_j \wedge dx_j$$

(where $(x_1 + iy_1, \dots, x_n + iy_n)$ stands for the complex coordinates in \mathbf{C}^n).

One also has:

$$\omega_z(Z, Z') = \omega(Z, Z') = \sum_{j=1}^n (X'_j Y_j - X_j Y'_j) = X' \cdot Y - X \cdot Y'.$$

And this is an exact, hence closed, form:

$$\omega = d\left(\sum_{j=1}^n y_j dx_j\right).$$

- (2) The next example is that of cotangent bundles. Think that $\mathbf{C}^n = \mathbf{R}^n \times \mathbf{R}^n$, then that $\mathbf{R}^n \times \mathbf{R}^n = T^*\mathbf{R}^n$ and simply replace \mathbf{R}^n by any manifold V . On $W = T^*V$, there is a *canonical* 1-form, the *Liouville form* λ , defined by the “compact” formula:

$$\lambda_{(x,\alpha)}(X) = \alpha(T_{(x,\alpha)}\pi(X))$$

... in which x denotes a point of V , α an element of T_x^*V (namely a linear form on the tangent space T_xV), and π the projection $T^*V \rightarrow V$ of the cotangent bundle. If (x_1, \dots, x_n) are local coordinates on V and (y_1, \dots, y_n) the cotangent coordinates, then

$$\lambda = \sum_{j=1}^n y_j dx_j.$$

The 2-form $d\lambda$ is both closed (!) and non degenerate.

- (3) Surfaces. On a surface W , any 2-form is closed. Moreover, in dimension 2, to say that a 2-form is non degenerate means that it nowhere vanishes, in other words that this is a volume form: all the orientable surfaces may be considered as symplectic manifolds.
- (4) The sphere. Consider, in particular, the unit sphere S^2 in \mathbf{R}^3 , whose tangent space at a point v is the plane orthogonal to the unit vector v . Put

$$\omega_v(X, Y) = v \cdot (X \wedge Y) = \det(v, X, Y).$$

This is a non degenerate 2-form and thus a symplectic form.

- (5) The projective space $\mathbf{P}^n(\mathbf{C})$ is a symplectic manifold. The nicest thing to do is to define its symplectic form starting from that of \mathbf{C}^{n+1} and using the symplectic reduction process. To define $\mathbf{P}^n(\mathbf{C})$, we factor out the unit sphere S^{2n+1} of \mathbf{C}^{n+1} by the S^1 -action (multiplication of coordinates):

$$t \cdot (z_1, \dots, z_{n+1}) = (tz_1, \dots, tz_{n+1})$$

At each point x of the sphere S^{2n+1} , the tangent space is the Euclidean orthogonal of x and the kernel of the restriction of the symplectic form is the line generated by ix . This line is also the tangent space to the circle through x on the sphere.

The symplectic form of \mathbf{C}^{n+1} defines a non degenerate alternated bilinear form ω on $\mathbf{P}^n(\mathbf{C})$. Its pull-back on the sphere is closed, so that ω is closed. It is actually a (the standard) Kähler form on $\mathbf{P}^n(\mathbf{C})$.

- (6) *Complex* submanifolds of the projective space are symplectic. The compatibility of ω with the complex structure gives that $\omega(X, iX) > 0$ for any vector X that is tangent to the submanifold, so that ω is indeed non degenerate on this submanifold.
- (7) More generally, all Kähler manifolds are symplectic. We will come back to this remark.

Notice that, on cotangent bundles, as on \mathbf{C}^n , the symplectic form is *exact*. This cannot be the case on a compact symplectic manifold.

Proposition II.1.2. *On a compact manifold, there exists no 2-form that is both non degenerate and exact.*

Proof. Let ω be a non degenerate 2-form on the $2n$ -dimensional manifold W . To say that ω is non degenerate is to say that $\omega^{\wedge n}$ is a *volume form*. But then, if $\omega = d\alpha$,

$$\omega^{\wedge n} = d(\alpha \wedge \omega^{\wedge(n-1)})$$

is also exact, thus W cannot be compact. □

Hamiltonian vector fields X_H for functions $H : W \rightarrow \mathbf{R}$ are defined exactly as in Appendix I.6.c and so is the Poisson bracket of two functions on W . Exercise II.3 explains why it is required that a symplectic form be closed.

II.2 Lagrangian submanifolds and immersions

An immersion $f : L \rightarrow W$ into a symplectic manifold is *Lagrangian* if $f^*\omega = 0$ and $\dim W = 2 \dim L$.

II.2.a In cotangent bundles

All what was done in \mathbf{C}^n in §I.4.a works as well in a cotangent bundle.

Graphs

Proposition I.4.1 generalizes as:

Proposition II.2.1. *Let $\alpha : L \rightarrow T^*L$ be a section of a cotangent bundle. Its image is a Lagrangian submanifold if and only if the 1-form α is closed.*

Proof. The most elegant thing to do is to state first a property of the Liouville form (which explains why it is called the “canonical” 1-form): for any form α , one has

$$\alpha^*\lambda = \alpha.$$

In this equality, α is considered as a section of the cotangent bundle in the left hand side and as a form in the right hand side. One has indeed:

$$\begin{aligned} (\alpha^*\lambda)_x(Y) &= \lambda_{(x, \alpha_x)}(T_x\alpha(Y)) && \text{by definition of } \alpha^* \\ &= \alpha_x(T_{(x, \alpha_x)}\pi \circ T_x\alpha(Y)) && \text{by definition of } \lambda \\ &= \alpha_x(Y) && \text{because } \alpha \text{ is a section.} \end{aligned}$$

Eventually, $\alpha^*\omega = 0$ if and only if $d(\alpha^*\lambda) = 0$, thus the graph of α is a Lagrangian submanifold if and only if α is closed. \square

Remark II.2.2. In particular, the zero section of $L \subset T^*L$ is a Lagrangian submanifold. What we plan to do next is to show that $L \subset T^*L$ is a model for *all* Lagrangian embeddings of L into a symplectic manifold (Theorem II.3.7).

Generating functions

A function

$$F : M \times \mathbf{R}^k \longrightarrow \mathbf{R}$$

allows to construct a Lagrangian submanifold (the graph of dF) into $T^*M \times \mathbf{C}^k$ and then, by reduction, a Lagrangian immersion into T^*M .

Wave fronts

Exact Lagrangian immersions into T^*M define wave fronts in $M \times \mathbf{R}$ and conversely.

Conormal bundles

Let

$$f : V \longrightarrow M$$

be any immersion. The *conormal* bundle is the subbundle of the pull back bundle

$$f^*T^*M = \left\{ (x, \varphi) \mid x \in V, \varphi \in T_{f(x)}^*M \right\} \longrightarrow V$$

defined by

$$\begin{aligned} N^*f &= \{(x, \varphi) \in f^*T^*M \mid \varphi|_{T_x f(T_x V)} = 0\} \\ &= \{(x, \varphi) \in f^*T^*M \mid \varphi \circ T_x f = 0\}. \end{aligned}$$

Map N^*f into T^*M by

$$F : (x, \varphi) \longmapsto (f(x), \varphi).$$

This is an immersion, since

$$T_{(x, \varphi)}F(\xi, \psi) = (T_x f(\xi), \psi).$$

It is Lagrangian, as we have $F^*\lambda = 0$. Indeed, calling π the two projections $T^*M \rightarrow M$ et $N^*f \rightarrow V$, we get

$$\begin{aligned} (F^*\lambda)_{(x, \varphi)}(X) &= \lambda_{(f(x), \varphi)}(T_{(x, \varphi)}F(X)) \\ &= \varphi(T_{(f(x), \varphi)}\pi \circ T_{(x, \varphi)}F(X)) \\ &= \varphi(T_x f(T_{(x, \varphi)}\pi(X))) \\ &= 0 \end{aligned}$$

as φ vanishes on the vectors that are tangent to V . □

One should check that the proof given for the normal bundle in §I.4.c for the case where $M = \mathbf{R}^n$ is identical to the one given here, the orthogonality used there being an *ersatz* of the duality used here.

II.3 Tubular neighborhoods of Lagrangian submanifolds

Let us now present a method, invented by Moser [28], which allows to describe a symplectic manifold in the neighbourhood of a point (they are all the same) or a neighbourhood of a Lagrangian submanifold in a symplectic manifold.

II.3.a Moser's method

The next “lemma” contains all these results.

Lemma II.3.1. *Let W be a $2n$ -dimensional manifold and let $Q \subset W$ be a compact submanifold. Assume that ω_0 and ω_1 are two closed 2-forms on W such that, at any point x of Q , ω_0 and ω_1 are equal and non degenerate on $T_x W$. Then there exists open neighborhoods \mathcal{V}_0 and \mathcal{V}_1 of Q and a diffeomorphism*

$$\psi : \mathcal{V}_0 \longrightarrow \mathcal{V}_1$$

such that $\psi|_Q = \text{Id}_Q$ and $\psi^*\omega_1 = \omega_0$.

Remark II.3.2. It is not easy to create a diffeomorphism “ex nihilo”. The remarkable idea of Moser is to construct *a whole path* of diffeomorphisms starting from the identity and ending at some diffeomorphism which has the desired property.

Let us write the proof of Moser lemma when $W = \mathbf{C}^n$ and explain then what should be done to get it in the general case (essentially to replace the Euclidean structure by a Riemannian metric). Consider the normal bundle to Q in \mathbf{C}^n ,

$$NQ = \{(x, v) \in Q \times \mathbf{C}^n \mid v \perp T_x Q\}$$

and the open subset

$$\mathcal{U}_\varepsilon = \{(x, v) \in NQ \mid \|v\| < \varepsilon\}.$$

Notice firstly that:

Lemma II.3.3. *Let Q be a compact submanifold of the Euclidean space \mathbf{R}^m . For ε small enough, the map*

$$\begin{aligned} E : NQ &\longrightarrow \mathbf{R}^m \\ (x, v) &\longmapsto x + v \end{aligned}$$

is a diffeomorphism from \mathcal{U}_ε onto its image.

Proof. In a neighbourhood of a point x_0 of Q , we describe Q by local coordinates $u = (u_1, \dots, u_k)$, namely by a mapping $x : U \rightarrow \mathbf{R}^m$ where U is open in \mathbf{R}^k and $x(0) = x_0$. One can choose vector fields $(v_1(u), \dots, v_{m-k}(u))$ of \mathbf{R}^m on U , that form, for all u , an orthonormal basis of the normal space of Q at $x(u)$. So we have local coordinates $(u_1, \dots, u_k, t_1, \dots, t_{m-k})$ on NQ in which the mapping E is

$$E(u, t) = x(u) + \sum_{i=1}^{m-k} t_i v_i(u).$$

The partial derivatives are

$$\begin{cases} \frac{\partial E}{\partial u_i} = \frac{\partial x}{\partial u_i} + \sum_j t_j \frac{\partial v_j}{\partial u_i} \\ \frac{\partial E}{\partial t_k} = v_k. \end{cases}$$

The matrix of partial derivatives is invertible for $t = 0$, thus it is invertible also for $\|t\|$ small enough¹. We conclude globally using the compactness of Q . \square

Call \mathcal{V}_0 the image of a suitable \mathcal{U}_ε . This is a neighbourhood of Q in \mathbf{C}^n .

Lemma II.3.4. *On \mathcal{V}_0 , the 2-form $\tau = \omega_1 - \omega_0$ is exact.*

¹It is interesting to see “how far” we can go. This leads to the notion of *focal point*, see for example [26].

First proof. The vector bundle NQ retracts on its zero section. The inclusion $j : Q \rightarrow \mathcal{V}_0$ thus induces an isomorphism $j^* : H_{DR}^2(\mathcal{V}_0) \rightarrow H_{DR}^2(Q)$. As $j^*[\omega_1] = j^*[\omega_0]$, the cohomology classes of ω_1 and ω_0 are equal in $H_{DR}^2(\mathcal{V}_0)$, which means that their difference is an exact form. \square

Second proof. We explicitly construct a 1-form σ that is a primitive of τ . Consider the dilatation of factor t in the fibers

$$\begin{array}{ccc} \varphi_t : \mathcal{V}_0 & \longrightarrow & \mathcal{V}_0 \\ x + v & \longmapsto & x + tv \end{array} \quad t \in [0, 1].$$

This is a diffeomorphism (onto its image) for $t > 0$ and we have $\varphi_0(\mathcal{V}_0) = Q$, $\varphi_1 = \text{Id}_{\mathcal{V}_0}$ and $\varphi_t|_Q = \text{Id}_Q$. The form $\tau = \omega_1 - \omega_0$ is a 2-form on \mathbf{C}^n . Consider its restriction to \mathcal{V}_0 . It is identically zero along Q by assumption. We have

$$\varphi_0^* \tau = 0, \quad \varphi_1^* \tau = \tau.$$

Consider now the (time depending) radial vector field X_t (tangent to the dilatation) on \mathcal{V}_0 . This is the vector field defined by

$$X_t(y) = \left(\frac{d}{ds} \varphi_s \right) (\varphi_t^{-1}(y)) |_{s=t}.$$

It is defined only for $t > 0$, in the same way that φ_t is a diffeomorphism only for $t > 0$. In a very concrete way, the vector field is

$$X_t(x + v) = \frac{1}{t}v.$$

For all t , consider also the 1-form σ^t defined by

$$\sigma_{x+v}^t(Y) = \tau_{x+tv}(v, T_{x+v}(\varphi_t)(Y)).$$

Notice that, if y is in Q , one has

$$\varphi_t(y) = y \text{ and } \frac{d}{dt} \varphi_t(y) = 0$$

thus σ^t is zero along Q . For $t > 0$, one has

$$\begin{aligned} (\varphi_t^* \iota_{X_t} \tau)_{x+v}(Y) &= (\iota_{X_t} \tau)_{x+tv}(X_t(x + tv), T_{x+v}(\varphi_t)(Y)) \\ &= \tau_{x+tv}(v, T_{x+v}(\varphi_t)(Y)) \\ &= \sigma_{x+v}^t(Y). \end{aligned}$$

Hence, for $t > 0$,

$$\sigma^t = \varphi_t^* \iota_{X_t} \tau$$

and consequently

$$\begin{aligned}
d\sigma^t &= d(\varphi_t^* \iota_{X_t} \tau) \\
&= \varphi_t^* (d\iota_{X_t} \tau + \iota_{X_t} d\tau) \\
&= \varphi_t^* (\mathcal{L}_{X_t} \tau) \\
&= \frac{d}{dt} (\varphi_t^* \tau).
\end{aligned}$$

Eventually, we get

$$d\sigma^t = \frac{d}{dt} (\varphi_t^* \tau)$$

for $t > 0$ and thus also for all $t \in [0, 1]$. Now

$$\tau = \tau - 0 = \varphi_1^* \tau - \varphi_0^* \tau = \int_0^1 \frac{d}{dt} (\varphi_t^* \tau) dt = \int_0^1 (d\sigma^t) dt = d\sigma$$

writing $\sigma = \int_0^1 \sigma^t dt$. We have thus proved that, in a neighbourhood of Q , $\omega_1 - \omega_0 = d\sigma$ is an exact form (with σ identically zero on Q). \square

To finish the proof of Lemma II.3.1, we use the actual method of Moser. We consider the path of symplectic forms

$$\omega_t = \omega_0 + t(\omega_1 - \omega_0) = \omega_0 + t d\sigma.$$

For $t = 0$, this is the non degenerate form ω_0 . Also, along Q , this is the very same form ω_0 . Restricting again \mathcal{V}_0 if necessary (using compactness again) one can assume that ω_t is non degenerate on \mathcal{V}_0 for all $t \in [0, 1]$. Let Y_t be the vector field defined by

$$\iota_{Y_t} \omega_t = -\sigma$$

(the existence and uniqueness of Y_t are consequences of the fact that ω_t is non degenerate). Let ψ_t be its flow:

$$\frac{d}{dt} \psi_t = Y_t \circ \psi_t.$$

We have

$$\begin{aligned}
\frac{d}{dt} (\psi_t^* \omega_t) &= \psi_t^* \left(\frac{d}{dt} \omega_t + \mathcal{L}_{Y_t} \omega_t \right) \\
&= \psi_t^* (d(\sigma) + d\iota_{Y_t} \omega_t) \\
&= d(\psi_t^* (\sigma + \iota_{Y_t} \omega_t)) \\
&= 0
\end{aligned}$$

by definition of Y_t . Hence $\psi_t^* \omega_t = \psi_0^* \omega_0 = \omega_0$ and eventually

$$\psi_1^* \omega_1 = \omega_0.$$

\square

Remark II.3.5. In a general symplectic manifold W , the proof is identical to the one given here; what we need is the notion of a normal bundle, that is, of orthogonality in TW , and a way to replace the mapping $(x, v) \mapsto x + v$. One uses a Riemannian metric on W and its exponential mapping: the point $\exp_v(x)$ that replaces $x + v$ is the point reached at time 1 by a geodesic² starting from x (at time 0) with tangent vector v .

The most direct application of Lemma II.3.1 is the Darboux theorem. This is the case where Q is a point x_0 , ω_1 is a symplectic form on W and ω_0 is the symplectic form induced on $T_{x_0}W$.

Theorem II.3.6 (Darboux theorem). *Let x be a point of a manifold W endowed with a symplectic form ω . There exists local coordinates*

$$(x_1, \dots, x_n, y_1, \dots, y_n)$$

centered at x in which $\omega = \sum dy_i \wedge dx_i$.

Proof. The form induced by ω_1 on $T_{x_0}W$ defines, using a diffeomorphism from a neighbourhood of 0 in $T_{x_0}W$ onto a neighbourhood of x_0 in W , a symplectic form ω_0 on a neighbourhood of x_0 . Lemma II.3.1 gives a diffeomorphism ψ from a neighbourhood of x_0 into itself, that fixes x_0 and satisfies $\psi^*\omega_1 = \omega_0$. By definition of ω_0 , there exists local coordinates centered at x_0 in which it can be written $\sum dy_i \wedge dx_i$. \square

II.3.b Tubular neighborhoods

The next application is a theorem of Weinstein that describes the tubular neighborhoods of the Lagrangian submanifolds.

Theorem II.3.7 (Weinstein [34]). *Let (W, ω) be a symplectic manifold and let $L \subset W$ be a compact Lagrangian submanifold. There exists a neighbourhood \mathcal{N}_0 of the zero section in T^*L , a neighbourhood \mathcal{V}_0 of L in W and a diffeomorphism $\varphi : \mathcal{N}_0 \rightarrow \mathcal{V}_0$ such that*

$$\varphi^*\omega = -d\lambda \text{ and } \varphi|_L = \text{Id}.$$

Proof. Let us check that we can apply Lemma II.3.1. The submanifold Q is the Lagrangian submanifold L and the form ω_0 is the restriction of ω . The form ω_1 is the symplectic form of T^*L . We are going to compare them in T^*L . As in the previous proof, let us assume firstly that $W = \mathbf{C}^n$. Let φ be the composed mapping

$$\begin{array}{ccccc} T^*L & \longrightarrow & N_L & \longrightarrow & \mathbf{C}^n \\ (x, \alpha) & \longmapsto & (x, Jv_\alpha) & \longmapsto & x + Jv_\alpha \end{array}$$

²To extend the geodesics, we also need an assumption on the completeness of the metric, or on the manifold W .

where

- $\alpha \mapsto v_\alpha$ is the isomorphism between cotangent and tangent spaces given by the Euclidean structure of \mathbf{C}^n restricted to L :

$$\alpha(u) = (u, v_\alpha),$$

- J is the multiplication by i . Recall (see Lemma I.2.1) that L is Lagrangian if and only if $TL^\perp = JTL$.

Call \mathcal{N}_0 a neighbourhood of the zero section in T^*L , mapped onto a suitable \mathcal{U}_ε , so that $\varphi : \mathcal{N}_0 \rightarrow \mathbf{C}^n$ is a diffeomorphism onto its image. We want to compare, in $\mathcal{N}_0 \subset T^*L$, the two forms $\omega_1 = -d\lambda$ and $\omega_0 = \varphi^*\omega$. To apply Lemma II.3.1, we have to check that they coincide along the zero section.

Let $(x, 0) \in L \subset \mathcal{N}_0$. We have

$$T_{(x,0)}\mathcal{N}_0 = T_{(x,0)}(T^*L) = T_xL \oplus T_x^*L.$$

Recall that there is an exact sequence

$$0 \longrightarrow \text{Ker } T_{(x,\alpha)}\pi \longrightarrow T_{(x,\alpha)}(T^*L) \xrightarrow{T_{(x,\alpha)}\pi} T_xL \longrightarrow 0$$

which splits along the zero section s , using

$$T_xL \xrightarrow{T_x s} T_{(x,0)}(T^*L),$$

and that the kernel $\text{Ker } T_{(x,\alpha)}\pi$ is canonically identified with T_x^*L . Compute then $\varphi^*\omega$ along the zero section. For $v, w \in T_xL$ and $\alpha, \beta \in T_x^*L$, we have

$$\begin{aligned} (\varphi^*\omega)_{(x,0)}((v, \alpha), (w, \beta)) &= \omega_{\varphi(x,0)}(v + Jv_\alpha, w + Jv_\beta) \\ &= \omega_x(v + Jv_\alpha, w + Jv_\beta) \\ &= (v, v_\beta) - (w, v_\alpha) \\ &= \beta(v) - \alpha(w). \end{aligned}$$

But we have seen (in Exercise II.1) that

$$(d\lambda)_{(x,0)}((v, \alpha), (w, \beta)) = \left(\sum dy_j \wedge dx_j \right) ((v, \alpha), (w, \beta)) = \alpha(w) - \beta(v).$$

The forms $\varphi^*\omega$ and $-d\lambda$ coincide along the zero section, therefore we can apply the lemma. \square

In the general situation where W is a symplectic manifold, we need a Riemannian metric and an analogue of J . We use an “almost complex structure” J calibrated by ω , namely an endomorphism J of the tangent bundle TW such that $J^2 = -\text{Id}$ and

$$(X, Y) \longmapsto \omega(X, JY)$$

is a Riemannian metric. Such structures exist and form a contractible set. See for instance [5, 24]. Notice that this notion is a globalization of the linear notion, mentioned in §I.6.b.

II.3.c “Moduli space” of Lagrangian submanifolds

We consider now, for a given manifold L , the *space* of Lagrangian immersions

$$f : L \longrightarrow W.$$

We call it a “space” because this set is actually a topological space, a fact which allows to consider immersions that are “close” to a given immersion. We use the Whitney \mathcal{C}^1 -topology.

The \mathcal{C}^1 -topology

Let V and W be two manifolds. The \mathcal{C}^1 -topology is a topology on the space of \mathcal{C}^1 -maps from V to W . Consider the vector bundle $\mathcal{L}(TV, TW)$ over $V \times W$, the fiber at (x, y) of which is the vector space $\mathcal{L}(T_x V, T_y W)$. The total space is usually called $J^1(V, W)$ rather than $\mathcal{L}(TV, TW)$. Every map $f \in \mathcal{C}^1(V, W)$ defines a mapping

$$\begin{aligned} j^1 f : V &\longrightarrow J^1(V, W) \\ x &\longmapsto (x, f(x), T_x f). \end{aligned}$$

If U is an open subset of $J^1(V, W)$, denote

$$\mathcal{V}(U) = \{f \in \mathcal{C}^1(V, W) \mid j^1 f \in U\}.$$

The \mathcal{C}^1 -topology is the topology for which the $\mathcal{V}(U)$ are a basis. It is said that a map f is “ \mathcal{C}^1 -close” to f_0 if it is close to f_0 for the \mathcal{C}^1 -topology.

Diffeomorphism group

The group of diffeomorphisms of L acts on this space by $\varphi \cdot f = f \circ \varphi^{-1}$. We want to consider Lagrangian immersions only up to this action: we do not want to take into account the way the manifold L is “parametrized”.

Moduli space

We consider the space of Lagrangian \mathcal{C}^1 -immersions from L to W up to the action of the diffeomorphism group. The quotient space is called the “moduli space” of Lagrangian immersions from L to W and denoted $\mathcal{L}(L)$. The next theorem describes the Lagrangian immersions that are close to a fixed Lagrangian embedding of L into W .

Theorem II.3.8. *Let L be a compact and connected manifold. A neighbourhood of a Lagrangian embedding*

$$L \longrightarrow W$$

in the space $\mathcal{L}(L)$ can be identified with a neighbourhood of 0 in the vector space of closed 1-forms of class \mathcal{C}^1 on L .

Proof. Let $f_0 : L \rightarrow W$ be a Lagrangian embedding and $f : L \rightarrow W$ be a Lagrangian immersion close to f_0 . In particular, f is close to f_0 for the “ \mathcal{C}^0 -topology³”, we can consider that everything lies in a neighbourhood of L . Thanks to the tubular neighbourhood theorem (here Theorem II.3.7) we can assume that everything takes place in a neighbourhood of the zero section in T^*L . The map f is \mathcal{C}^1 -close to the inclusion of the zero section $L \rightarrow T^*L$ (this is what f_0 has become when we have identified the neighbourhood of $f_0(L)$ in W with a neighbourhood of the zero section in T^*L). Thus the composition of f with the projection of the cotangent is a \mathcal{C}^1 -mapping $L \rightarrow L$, close to the identity. Recall the next lemma, which is a consequence of the inverse function theorem.

Lemma II.3.9. *Let L be a compact and connected manifold. Let f be a \mathcal{C}^1 -map $L \rightarrow L$ that is \mathcal{C}^1 -close to the identity. Then f is a diffeomorphism. \square*

According to this lemma, the composition is a diffeomorphism g of L . Composing with g^{-1} , we get an embedding

$$\alpha : L \longrightarrow T^*L$$

which is still \mathcal{C}^1 -close to the zero section... but now the composition

$$L \longrightarrow T^*L \longrightarrow L$$

is the identity. Thus α is a section, that is, a 1-form on L , and α is closed because the embedding is Lagrangian. Conversely, all the closed 1-forms that are close to the zero section define Lagrangian embeddings close to f_0 . \square

Remark II.3.10. One should have noticed that the section $L \rightarrow T^*L$ defined by a 1-form is a \mathcal{C}^1 -mapping if and only if the form is a \mathcal{C}^1 -form. The \mathcal{C}^1 -topology thus defines the structure of a topological vector space on the space of 1-forms. In §II.6 below, we will need a Banach space structure.

Remark II.3.11. The vector space we have obtained is infinite dimensional. It can be considered as a neighbourhood of f_0 in the “manifold” of deformations of f_0 , or as its tangent space at f_0 .

II.4 Calabi-Yau manifolds

We want now to describe, in a way analogous to what we have done in §II.3.b, the moduli space of special Lagrangian submanifolds. In order to apply Theorem II.3.7 (special Lagrangian submanifolds are, firstly, Lagrangian submanifolds) we need a *compactness* assumption on the Lagrangian submanifold. Unfortunately, as we have seen it in §I.5.b, the special Lagrangian submanifolds of \mathbf{C}^n are never compact. We thus need to consider more general manifolds, in which it is possible to define special Lagrangian submanifolds. These are the “Calabi-Yau” manifolds.

³The \mathcal{C}^0 -topology, defined similarly to the \mathcal{C}^1 -topology, is simply the compact open topology.

The point is to define a structure that globalizes the structures on \mathbf{C}^n which have allowed us to speak of special Lagrangian submanifolds. Recall that, in addition to the \mathbf{R} -bilinear alternated form ω , we have used the form $\Omega = dz_1 \wedge \cdots \wedge dz_n$ of the complex determinant.

We will use here the best adapted definition of a Calabi-Yau manifold, the point is not to spend time on the Calabi-Yau manifold itself but rather on its special Lagrangian submanifolds. For more information on Calabi-Yau manifolds, see [33, 8] and the references they contain.

II.4.a Definition of the Calabi-Yau manifolds

Our manifolds should be complex and endowed with a symplectic form ω and a type- $(n, 0)$ holomorphic form Ω that is nowhere zero (this is sometimes called a holomorphic volume form). Consider thus a manifold M , on which are given

- a complex structure J (multiplication by i),
- a closed non degenerate type $(1, 1)$ -form ω (the Kähler form)
- a Riemannian metric

$$g(X, Y) = \omega(X, iY),$$

- a Hermitian metric

$$h(X, Y) = g(X, Y) - i\omega(X, Y),$$

- a trivialization of the “canonical” bundle $\Lambda^n T^*M$, namely a type- $(n, 0)$ holomorphic form Ω which is nowhere zero.

We still need a relation between the forms ω and Ω . Notice that both forms $\omega^{\wedge n}$ and $\Omega \wedge \bar{\Omega}$ are of type (n, n) and both do not vanish on M , in particular, both are volume forms. We thus have

$$\Omega \wedge \bar{\Omega} = f\omega^{\wedge n}$$

for some function f on M . The additional compatibility condition is that f should be constant. Let us look at the case of \mathbf{C}^n . We have

$$\omega^n = \left(\sum_{j=1}^n dy_j \wedge dx_j \right)^{\wedge n} = n!(dy_1 \wedge dx_1) \wedge \cdots \wedge (dy_n \wedge dx_n).$$

Writing

$$dy = \frac{1}{2i}(dz - d\bar{z}) \text{ and } dx = \frac{1}{2}(dz + d\bar{z})$$

and noticing that

$$dy \wedge dx = \frac{1}{4i}(dz - d\bar{z}) \wedge (dz + d\bar{z}) = \frac{1}{2i}dz \wedge d\bar{z},$$

we can also write

$$\omega^{\wedge n} = \frac{n!}{2^n i^n} (dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n).$$

The computation of $\Omega \wedge \bar{\Omega}$ gives

$$\Omega \wedge \bar{\Omega} = (dz_1 \wedge \cdots \wedge dz_n) \wedge (d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n).$$

We thus have

$$\omega^{\wedge n} = \frac{(-1)^{\frac{n(n-1)}{2}} n!}{2^n i^n} \Omega \wedge \bar{\Omega}.$$

We will use the same normalization formula to define a Calabi-Yau manifold in general.

Definition II.4.1. A complex manifold M is said to be a *Calabi-Yau manifold* if it is Kähler, has a trivialized canonical bundle, and if the Kähler form ω and the type- $(n, 0)$ form Ω trivializing the bundle $\Lambda^n T^*M$ are related by

$$\omega^{\wedge n} = \frac{(-1)^{\frac{n(n-1)}{2}} n!}{2^n i^n} \Omega \wedge \bar{\Omega}.$$

Remark II.4.2. Recall that it is possible to express the fact that the form ω is Kähler by saying that the complex structure is “parallel” with respect to the Levi-Civita connection associated with the metric it defines with ω . Similarly, it is possible to express the compatibility condition for Ω by saying that it is parallel with respect to the same connection.

Remark II.4.3. In general, it is required that the Kähler metric be *complete*, in other words that it is possible to extend geodesics. This is equivalent to requiring that the manifold be complete (in the sense of metric spaces).

II.4.b Yau’s theorem

Consider a (complex algebraic) projective smooth manifold M of complex dimension n . Assume that all the $H^{p,0}(M)$ are zero for $1 \leq p \leq n-1$ and that the canonical bundle $\Lambda^n T^*M = K_M$ is trivialized by a type- $(n, 0)$ form Ω . Notice that M is Kähler, call the Kähler form ω . Rescaling ω if necessary, we get

$$\int_M \omega^{\wedge n} = \frac{(-1)^{\frac{n(n-1)}{2}} n!}{2^n i^n} \int_M \Omega \wedge \bar{\Omega}.$$

A hard theorem of Yau [35] asserts that there exists a unique Kähler form $\tilde{\omega}$ on M such that $[\tilde{\omega}] = [\omega] \in H_{DR}^2(M)$ and which, together with Ω , gives M the structure of a Calabi-Yau manifold.

II.4.c Examples of Calabi-Yau manifolds

Of course \mathbf{C}^n is a Calabi-Yau manifold.

Affine quadrics

We have defined in §II.1 a symplectic form on the unit sphere $S^2 \subset \mathbf{R}^3$ by the formula

$$\omega_x(X, X') = \det(x, X, X').$$

Similarly, the formula

$$\Omega_z(Z, Z') = \det_{\mathbf{C}}(z, Z, Z')$$

defines a “holomorphic symplectic” form of the complex quadric

$$Q = \{(z_1, z_2, z_3) \mid z_1^2 + z_2^2 + z_3^2 = 1\}.$$

In “differential” terms,

$$\Omega = z_1 dz_2 \wedge dz_3 + z_2 dz_3 \wedge dz_1 + z_3 dz_1 \wedge dz_2.$$

On the open subset of Q where $z_3 \neq 0$, z_1 and z_2 are coordinates and, using the relation

$$z_1 dz_1 + z_2 dz_2 + z_3 dz_3 = 0,$$

we can write

$$\Omega = \frac{1}{z_3} dz_1 \wedge dz_2,$$

so that

$$\Omega \wedge \bar{\Omega} = \frac{1}{|z_3|^2} dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2.$$

Modifying the restriction ω_0 to Q of the standard Kähler form of \mathbf{C}^3 , let us construct a Kähler form ω on Q such that

$$\omega \wedge \omega = \frac{1}{4} \Omega \wedge \bar{\Omega}.$$

Call h the restriction to Q of the function $|z|^2$. We look for ω of the form

$$\omega = \frac{i}{2} \partial \bar{\partial} (f \circ h)$$

for some function f . A straightforward computation (see also [31]) shows that $f(h) = \sqrt{h+1}$ works.

The quadric Q , equipped with Ω and ω is (thus) a Calabi-Yau manifold. Recall that Q is diffeomorphic to the tangent bundle TS^2 by

$$\begin{aligned} Q &\longrightarrow TS^2 \\ X + iY &\longmapsto \left(\frac{X}{\sqrt{1 + \|Y\|^2}}, Y \right). \end{aligned}$$

In this way, what we have got is the structure of a Calabi-Yau manifold on the tangent (or cotangent) bundle of the sphere S^2 . It is possible (but a little more complicated) to do the same for the cotangent bundles of all the spheres S^n and more generally for those of all “rank-1 symmetric spaces” (see [31]).

Remark II.4.4. Recall that we have identified \mathbf{C}^2 with the skew field \mathbf{H} of quaternions (in §I.5.a). Similarly, the surface Q has the structure of a “quaternionic” or “hyperkähler” manifold.

Call I the complex structure defined on Q by that of \mathbf{C}^3 (this is the multiplication by i) and notice that the symmetric bilinear form that is an equation for Q is still non degenerate when restricted to $z^\perp = T_z Q$. Define an operator J_z on the tangent space $T_z Q$ by the fact that $J_z(Z)$ is the unique vector in $T_z Q$ that is orthogonal to Z for the complex bilinear form and such that

$$\det_{\mathbf{C}}(z, Z, J_z(Z)) = \|Z\|^2.$$

This is an almost complex structure since

$$\det_{\mathbf{C}}(z, J_z Z, -Z) = \|Z\|^2$$

thus $J_z^2 = -\text{Id}$. This is an isometry since

$$\|JZ\|^2 = \det_{\mathbf{C}}(z, JZ, J^2(Z)) = \det_{\mathbf{C}}(z, JZ, -Z) = \|Z\|^2.$$

Moreover, J “anti-commutes” with I :

$$\begin{aligned} \det_{\mathbf{C}}(z, IZ, JIZ) &= \|IZ\|^2 = \|Z\|^2 \text{ on the one hand} \\ &= i \det_{\mathbf{C}}(z, Z, JIZ) \text{ by linearity.} \end{aligned}$$

We thus have

$$\det_{\mathbf{C}}(z, Z, JIZ) = -i \|Z\|^2 = -\det_{\mathbf{C}}(z, Z, IJZ)$$

so that $JI = -IJ$. Hence I , J and IJ form a quaternionic structure on Q . On Q , we thus have

- the Kähler form ω ,
- the complex structure I defined by multiplication by i in \mathbf{C}^3 ,
- the associated Riemannian metric g , so that $\omega(X, IY) = g(X, Y)$,
- the “holomorphic symplectic form” Ω ,
- the complex structure J defined in such a way that Ω be a J -Kähler form, associated with the same metric g .

It is said that Q is hyperkähler. See Exercise II.7 for a kind of converse statement.

Let us give now a few examples of compact Calabi-Yau manifolds.

Elliptic curves

The quotient M of \mathbf{C} by a lattice Λ is an *elliptic curve*. The two forms

$$\omega = \frac{1}{2i} dz \wedge d\bar{z} \text{ and } \Omega = dz$$

give it the structure of a dimension-1 Calabi-Yau manifold. One can, more generally, perform the quotient of \mathbf{C}^n by a lattice. It is time for a remark: no other “explicit” example of compact Calabi-Yau manifold is known. In all the known examples, the existence of the Kähler metric with all the desired properties is obtained as a consequence of the Yau theorem (§ II.4.b).

Hypersurfaces

Recall that complex elliptic curves can be considered as degree-3 curves in $\mathbf{P}^2(\mathbf{C})$, thanks to the Weierstrass \wp -function. They are thus the $n = 1$ case in the next theorem.

Theorem II.4.5. *A degree- d hypersurface in $\mathbf{P}^{n+1}(\mathbf{C})$ is a dimension- n Calabi-Yau manifold if and only if $d = n + 2$.*

Proof. The condition on the degree is necessary, as we show it now by the computation of the first Chern classes. We want that the bundle $\Lambda^n T^*M$ be trivializable, we must thus have $c_1(T^*M) = -c_1(TM) = 0$. Calling j the inclusion of M in $\mathbf{P}^{n+1}(\mathbf{C})$, we have

$$c_1(TM) + j^*c_1(\mathcal{O}(d)) = j^*c_1(T\mathbf{P}^{n+1}(\mathbf{C}))$$

since the normal bundle of M in $\mathbf{P}^{n+1}(\mathbf{C})$ is $\mathcal{O}(d)$. Denoting by t the dual class to the hyperplane section in $H^2(\mathbf{P}^{n+1}(\mathbf{C}))$, we have

$$(n + 2 - d)j^*t = 0$$

so that $d = n + 2$.

Assume conversely that $d = n + 2$. Let us construct explicitly a holomorphic n -form on M . Let F be a degree- $(n + 2)$ homogeneous polynomial that describes the hypersurface M . Every point of M lies in an affine chart $Z_i \neq 0$ of $\mathbf{P}^{n+1}(\mathbf{C})$. In affine coordinates $z_k = Z_k/Z_i$, there is an index j such that

$$\frac{\partial}{\partial z_j} F(z_0, \dots, 1, \dots, z_{n+1}) \neq 0$$

since M is smooth. The formula

$$\Omega = (-1)^{i+j-1} \frac{dz_0 \wedge \dots \wedge \widehat{dz}_i \wedge \dots \wedge \widehat{dz}_j \wedge \dots \wedge dz_{n+1}}{\frac{\partial F}{\partial z_j}(z_0, \dots, 1, \dots, z_{n+1})}$$

defines a homogeneous holomorphic n -form on M that is nowhere zero. This is a consequence of the theorem of Yau (§II.4.b) that there is, indeed, in the same cohomology class as the standard Kähler form ω , another Kähler form $\omega + i\partial\bar{\partial}\varphi$ giving a Calabi-Yau structure on M . \square

Remark II.4.6. The form Ω above is defined as “Poincaré residue⁴” starting from the $n + 1$ -form on $\mathbf{P}^{n+1}(\mathbf{C})$ with poles along M defined by

$$\sigma_i = (-1)^i \frac{dz_0 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_{n+1}}{F(z_0, \dots, 1, \dots, z_{n+1})}$$

in the affine chart $Z_i \neq 0$.

Remark II.4.7. Calabi-Yau manifolds of dimension 2 are hyperkähler. The proof of this fact is the subject of Exercises II.6 and II.7.

II.4.d Special Lagrangian submanifolds

An immersion $f : V \rightarrow M$ from a manifold of real dimension n into a Calabi-Yau manifold M of complex dimension n is said *special Lagrangian* if it satisfies $f^*\omega = 0$ and $f^*\beta = 0$. As in the case of \mathbf{C}^n , the form $f^*\Omega = f^*\alpha$ is then a volume form.

II.5 Special Lagrangians in real Calabi-Yau manifolds

II.5.a Real manifolds

A complex analytic manifold is *real* if it is endowed with a “real structure”, that is, with an anti-holomorphic involution S : an involution such that, for any holomorphic function f over an open subset U of M , $\overline{f \circ S}$ is a holomorphic function. For example, on the algebraic submanifolds of $\mathbf{P}^N(\mathbf{C})$ described by real polynomial equations, the complex conjugation is an anti-holomorphic involution. These manifolds are thus real manifolds. In particular, the projective space $\mathbf{P}^N(\mathbf{C})$ itself is a real manifold.

The real part, or set of real points of a real manifold is, by definition, the set of fixed points of S . For example, the real part of the real manifold $\mathbf{P}^N(\mathbf{C})$ is $\mathbf{P}^N(\mathbf{R})$. Notice that there exists respectable real manifolds that have no real point at all, as is, for example, the “Euclidean quadric”

$$\sum_{i=1}^{N+1} X_i^2 = 0$$

in $\mathbf{P}^N(\mathbf{C})$.

⁴See [15] p. 147.

Proposition II.5.1. *The real part of a real manifold of complex dimension n , if it is non empty, is a submanifold all connected components of which have dimension n .*

Proof. The connected components of the set of fixed points of the action of a finite group (here the order-2 group generated by S) are always submanifolds. The tangent space at x to such a component is the subspace of fixed points of the \mathbf{R} -linear involution $\sigma = T_x S$.

The fact that S is a real structure implies that $\overline{f \circ \sigma}$ is a complex linear form for any complex linear form f on the tangent space at x . We have to check that the fixed subspace of σ has dimension n . To do this, we simply verify that the eigensubspaces associated with the eigenvalues 1 and -1 are isomorphic. Indeed, if $\sigma(X) = X$, then for any complex linear form f , we have

$$\overline{f \circ \sigma}(iX) = i\overline{f \circ \sigma}(X) = i\bar{f}(X) = \bar{f}(-iX).$$

For any complex linear form f , we thus have

$$f(\sigma(iX)) = f(-iX)$$

so that $\sigma(iX) = -iX$. Hence, there are “as many” eigenvectors for the eigenvalue -1 than there are for the eigenvalue 1. \square

II.5.b Real Calabi-Yau manifolds

A Calabi-Yau manifold is *real* if it is both a Calabi-Yau manifold and a real manifold, with a couple of compatibility conditions

$$S^* \omega = -\omega \text{ and } S^* \Omega = \bar{\Omega}$$

(similarly to what happens in \mathbf{C}^n with the complex conjugation and the two usual forms Ω and ω).

Examples II.5.2. • The affine quadric $\sum z_i^2 = 1$ of \mathbf{C}^3 , endowed with the complex conjugation of coordinates is a real manifold. It is also clear that this is a real Calabi-Yau manifold. Its real part is simply the unit sphere $S^2 \subset \mathbf{R}^3$. If we consider Q as the tangent bundle to S^2 , notice that the complex conjugation is the multiplication by -1 on the fibers and that the real part is the zero section.

- A real hypersurface of degree $n+2$ in $\mathbf{P}^{n+1}(\mathbf{C})$ is a real Calabi-Yau manifold. This is checked by computing $S^* \Omega$ and $S^*(\omega + i\partial\bar{\partial}\varphi)$, for S the involution induced by the real structure (complex conjugation) of $\mathbf{P}^{n+1}(\mathbf{C})$ and Ω, ω as in the proof of Theorem II.4.5.

II.5.c The example of elliptic curves

Let us come back to the example of $\Gamma = \mathbf{C}/\Lambda$ where Λ is a lattice that we assume here to have the form

$$\Lambda = \{m + n\tau \mid m, n \in \mathbf{Z}\}$$

for some fixed τ such that $0 \leq \operatorname{Re}(\tau) < 1$ et $\operatorname{Im}(\tau) > 0$. To define a real structure on \mathbf{C}/Λ from the complex conjugation in \mathbf{C} , it is necessary that Λ be invariant, that is, that

$$\bar{\tau} = m + n\tau$$

for some $m, n \in \mathbf{Z}$. Considering the real and imaginary parts of τ , it is seen that $m = 2\operatorname{Re}(\tau)$, thus $\operatorname{Re}(\tau) = \frac{1}{2}$ or 0.

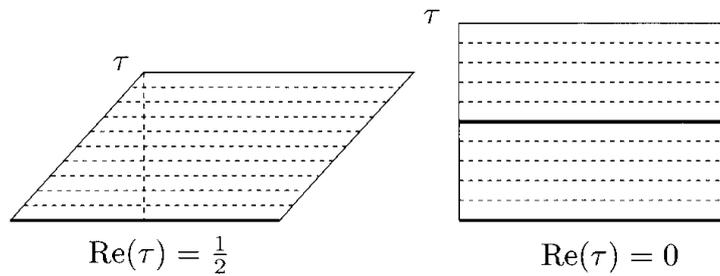


Figure II.1: Real elliptic curves

In the second case, the real part of Γ has two connected components, but in the first case, it has only one, as can be seen solving the equation

$$\bar{z} = z + m + n\tau$$

in both cases. These components are depicted in bold on Figure II.1.

Notice (although this is a trivial remark) that the lines that are parallel to the x axis constitute a real foliation of \mathbf{C}/Λ by circles (dimension-1 tori) that are special Lagrangian submanifolds of Γ , represented by dotted lines on Figure II.1. The space of these special Lagrangian submanifolds is parametrized by the axis generated by τ or rather by its image in Γ , a circle.

We shall see more generally in §II.6 that the moduli space of special Lagrangian submanifolds in a Calabi-Yau manifold is, in the neighbourhood of a submanifold V , a manifold whose dimension is the first Betti number of V (here V is a circle and its first Betti number is 1).

II.5.d Special Lagrangians in real Calabi-Yau manifolds

Assume now that M is a *real* Calabi-Yau manifold. We know that

$$S^*\omega = -\omega \text{ and } S^*\Omega = \bar{\Omega}.$$

Assume now that the real part $M_{\mathbf{R}}$ is not empty. Call j the inclusion of $M_{\mathbf{R}}$ into M . We have $S \circ j = j$ and in particular

$$j^*\omega = (S \circ j)^*\omega = j^*(S^*\omega) = j^*(-\omega) = -j^*\omega$$

hence $j^*\omega = 0$. Similarly

$$j^*\Omega = (S \circ j)^*\Omega = j^*(S^*\Omega) = j^*\bar{\Omega} = \overline{j^*\Omega}$$

thus $j^*\beta = 0$. We have proved:

Proposition II.5.3. *Let M be a real Calabi-Yau manifold. The real part of M , if it is not empty, is a special Lagrangian submanifold of M . \square*

Let us describe now a few examples of this situation.

The affine quadric

The sphere S^2 is a special Lagrangian submanifold of the affine quadric $Q \in \mathbf{C}^3$. In other words, with the Calabi-Yau structure on TS^2 defined in §II.4.c, the zero section is a special Lagrangian submanifold.

In the next examples, we consider a smooth hypersurface defined by a real homogeneous polynomial of degree $n + 2$ in $\mathbf{P}^{n+1}(\mathbf{C})$, with its real Calabi-Yau structure.

Elliptic curves

The $n = 1$ case, that of plane cubics, is isomorphic to the example of quotients of \mathbf{C} by lattices (§II.5.c). The real part of a plane cubic has zero, one or two connected components (see Figure II.1). All components are (topologically) circles. Cubics are foliated by special Lagrangian circles, drawn in dotted lines on Figure II.1.

Degree-4 surfaces

Consider now real algebraic surfaces of degree 4 in $\mathbf{P}^3(\mathbf{C})$ (the real part of this subject has been investigated and explained in [21]). Here is an example from [9]. Consider the real polynomial

$$P(z_0, z_1, z_2, z_3) = z_0^4 + z_1^4 - z_2^4 - z_3^4$$

that describes a smooth surface M which has a non empty real part $M_{\mathbf{R}}$. This real part is

$$M_{\mathbf{R}} = \{(x_0, x_1, x_2, x_3) \in \mathbf{R}^4 - \{0\} \mid x_0^4 + x_1^4 = x_2^4 + x_3^4\} / (x \sim \lambda x).$$

Normalize the non zero vectors of \mathbf{R}^4 by the choice, in each real line through zero, of one of the two vectors such that

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 2.$$

Then $M_{\mathbf{R}}$ is the quotient

$$\{(x_0, x_1), (x_2, x_3) \in \mathbf{R}^2 \times \mathbf{R}^2 \mid x_0^4 + x_1^4 = x_2^4 + x_3^4 = 1\} / (u, v) \sim (-u, -v).$$

It is clear that the curve C described in \mathbf{R}^2 by the equation $x^4 + y^4 = 1$ is diffeomorphic to a circle (radially). Eventually

$$M_{\mathbf{R}} = (C \times C) / ((u, v) \sim (-u - v))$$

is diffeomorphic to a torus. We have thus found a special Lagrangian torus in the Calabi-Yau surface M .

II.6 Moduli space of special Lagrangian submanifolds

We want now an analogue of Theorem II.3.8, more precisely a description of a neighbourhood of a given special Lagrangian submanifold in the space of all special Lagrangian submanifolds.

Theorem II.6.1 (McLean [25]). *Let V be a compact manifold. The moduli space of the special Lagrangian embeddings of V in the Calabi-Yau manifold M is a manifold of finite dimension $b_1(V) = \dim H^1(V; \mathbf{R})$. Its tangent space at a given point is isomorphic to the vector space of harmonic 1-forms on V .*

As usual, it is understood that the empty set is a manifold of any dimension — this is not an existence theorem.

Remark II.6.2. There are two main differences between this statement and Theorem II.3.8. The first one is that the moduli space here has finite dimension. The second one is that the condition “to be special Lagrangian” is no longer linear, so that this is indeed the tangent space that is identified to a space of differential forms.

Example II.6.3. Let us come back to the example of the Calabi-Yau structure on TS^2 described in §II.4.c. We have said in §II.5.d that the zero section is a special Lagrangian submanifold. As there are no non zero harmonic 1-forms on S^2 , the theorem of McLean asserts that the zero section is “rigid”, that is, it cannot be deformed. In the moduli space of special Lagrangian submanifolds, this is an isolated point.

Proof of Theorem II.6.1.

Using the tubular neighbourhood theorem (here in §II.3.b), replace M by a tubular neighbourhood of the submanifold V that is isomorphic to a neighbourhood of the zero section in the normal bundle of V (as are all tubular neighborhoods) and to a neighbourhood of the zero section in the cotangent T^*V . We will use the structures induced by those of M on this neighbourhood, keeping their names, for example $\Omega = \alpha + i\beta$.

We have said in §II.3.c that the space of Lagrangian submanifolds can be identified with a neighbourhood of 0 in the space $Z^1(V)$ of the closed 1-forms on V . The special Lagrangian submanifolds are described, in this space, by the equation $F(\eta) = 0$ where

$$F : Z^1(V) \longrightarrow \Omega^n(V)$$

is the mapping defined by $F(\eta) = \eta^*\beta$. Although these spaces are infinite dimensional, the strategy of the proof is to show that F is submersive at 0. It is thus better to restrict, as much as possible, its target space. Notice first:

Lemma II.6.4. *The image of F is contained in the subspace $d\Omega^{n-1}(V)$ of exact n -forms on V .*

Proof of Lemma II.6.4. If η is the zero form, the mapping $\eta : V \rightarrow T^*V$ is the inclusion of the zero section, a special Lagrangian, thus $F(0) = 0$. Given a form η , it is possible to consider the path (segment) $(t\eta)_{t \in [0,1]}$ joining it to the zero form... and giving a homotopy from the section η to the zero section. The cohomology class of the closed form $(t\eta)^*\beta$ does not depend on t , thus it is identically zero, and that means, indeed, that $\eta^*\beta$ is an exact form. \square

We thus consider F as a mapping

$$F : Z^1(V) \longrightarrow d\Omega^{n-1}(V)$$

and compute its differential at 0.

Lemma II.6.5. *The differential of F at 0 is the mapping*

$$(dF)_0(\eta) = d(\star\eta)$$

where \star denotes the Hodge star operator⁵ associated with the metric defined by the Calabi-Yau structure on the special Lagrangian V .

Proof of Lemma II.6.5. To compute $(dF)_0(\eta)$, one chooses a path of forms $\tilde{\eta}_t$ whose tangent vector at 0 is the form η . Let η be a 1-form on V and X be the vector field that corresponds to it via the metric on V , that is, the vector field such that $g(X, \cdot) = \eta$. Let $Y = JX$ be the vector field normal to V . This is the vector corresponding to η under the isomorphism $NV \simeq T^*V$. The vector field Y is only defined along V , we extend it (arbitrarily) in a vector field \tilde{Y} on the tubular neighbourhood under consideration. Call $\tilde{\varphi}_t$ the flow of \tilde{Y} , so that $\tilde{\varphi}_t$ is a diffeomorphism defined for t small enough. The restriction φ_t of $\tilde{\varphi}_t$ to V is an embedding of V into NV (one pushes V using φ_t). For $t = 0$, this is the zero section. Hence for t small enough, this is still a section of NV . We have, for all x in V ,

$$\frac{d}{dt}\varphi_t(x)|_{t=0} = Y(x).$$

⁵See Exercise II.5.

Under the identification $NV \simeq T^*V$, the section φ_t of NV corresponds to a section $\tilde{\eta}_t$ of T^*V which is a path of forms, whose tangent vector at 0 is the form η .

Consider now the $(n, 0)$ -form Ω , still on our neighbourhood of the zero section in NV . We have

$$\frac{d}{dt} \tilde{\varphi}_t^* \Omega|_{t=0} = \mathcal{L}_{\tilde{Y}} \Omega = d\iota_{\tilde{Y}} \Omega$$

applying Cartan formula together with the fact that Ω is closed. For the embedding $\varphi_t : V \rightarrow NV$, we thus have

$$\frac{d}{dt} \varphi_t^* \Omega|_{t=0} = d\iota_{JX} \Omega = id(\iota_X \Omega)$$

since Ω is \mathbf{C} -linear. We then have

$$\begin{aligned} (dF)_0(\eta) &= \frac{d}{dt} \tilde{\eta}_t^* \beta|_{t=0} \\ &= \text{Im}(id(\iota_X \Omega)) \\ &= \text{Re } d(\iota_X \Omega) \\ &= d(\iota_X \alpha). \end{aligned}$$

We still have to convince ourselves that $\iota_X \alpha = \star \eta$. The $(n-1)$ -form $\star \eta$ is the unique form satisfying

$$\psi \wedge (\star \eta) = g(\psi, \eta) \alpha$$

for any 1-form ψ . But, as the $(n+1)$ -form $\psi \wedge \alpha$ is zero, its interior product by X is also zero and we have

$$\psi \wedge (\iota_X \alpha) = (\iota_X \psi) \alpha = g(\psi, \eta) \alpha$$

by definition of X and of the metric g on the space of 1-forms. \square

Notice that this implies in particular that the differential dF_0 is onto: if σ is an $(n-1)$ -form,

$$d\sigma = (dF_0)(\pm \star \sigma).$$

To end the proof of the theorem, we need to precise what kind of implicit function theorem we use to go from “differential is surjective” to “inverse image is a submanifold”. The simplest here is to use the standard implicit function theorem for Banach spaces (see, for example, [10]). We need to endow the spaces of forms $Z^1(V)$ and $\Omega^n(V)$ with structures of Banach spaces. Let us precise the regularity of the forms we use. We consider forms of class $\mathcal{C}^{1,\varepsilon}$ in $Z^1(V)$ and of class $\mathcal{C}^{0,\varepsilon}$ in $\Omega^n(V)$. The Hölder norm used here on forms is deduced from the usual Hölder norm on functions: recall that $\mathcal{C}^{k,\varepsilon}(U)$ is the space of functions of class \mathcal{C}^k on the open set U of \mathbf{R}^n all the derivatives (of order $\leq k$) of which have a finite Hölder norm $\|u\|_\varepsilon$ (for $\varepsilon \in]0, 1]$), with

$$\|u\|_\varepsilon = \sup_{x,y \in U} \frac{|u(x) - u(y)|}{\|x - y\|^\varepsilon} + \sup_{x \in U} |u(x)|.$$

The implicit function theorem gives the fact that $F^{-1}(0)$ is a submanifold in a neighbourhood of 0, whose tangent space at 0 is the kernel $\mathcal{H}^1(V)$ of $(dF)_0$. It is important here that this kernel has finite dimension. The isomorphism between $H_{DR}^1(V)$ and the space $\mathcal{H}^1(V)$ of harmonic 1-forms is the contents in degree 1 of the Hodge theorem, see [15]. \square

Remark II.6.6. The vector space $H_{DR}^{n-1}(V)$ is isomorphic to the vector space dual to $H_{DR}^1(V)$, so that

$$H_{DR}^1(V) \oplus H_{DR}^{n-1}(V)$$

has a natural symplectic structure (see Exercise I.1), here

$$\omega((\alpha, \eta), (\alpha', \eta')) = \int_V (\alpha \wedge \eta' - \alpha' \wedge \eta).$$

The space of harmonic 1-forms is a Lagrangian subspace, by

$$\begin{array}{ccc} \mathcal{H}^1(V) & \longrightarrow & H_{DR}^1(V) \oplus H_{DR}^{n-1}(V) \\ \alpha & \longmapsto & (\alpha, \star\alpha) \end{array}$$

(this is the graph of the mapping \star , which is symmetric with respect to the metric... see Exercise I.8). If $j_0 : V \rightarrow W$ is a special Lagrangian submanifold, call \mathcal{B} the moduli space in a neighbourhood of j_0 . We thus have a Lagrangian subspace

$$T_{j_0} \mathcal{B} \longrightarrow H_{DR}^1(V) \oplus H_{DR}^{n-1}(V)$$

and it is possible to “integrate” it in a Lagrangian embedding (see [19])

$$F : \mathcal{B} \longrightarrow H_{DR}^1(V) \oplus H_{DR}^{n-1}(V).$$

See also [11] for a description of all these structures by symplectic reduction.

II.7 Towards mirror symmetry?

The “mirror conjecture” asserts the existence, for any Calabi-Yau manifold M , of another Calabi-Yau manifold M^* of the same dimension, related with M in a way we briefly describe now, sending the readers to [33] for missing detail.

Call \mathcal{M}_M the space of isomorphism classes of

- a complex structure J_t deforming the complex structure J of M
- a “complexified Kähler class” on (M, J_t) , namely a cohomology class of the form $\alpha + i\beta$, for some Kähler class (for J_t) $\alpha \in H_{DR}^2(M)$ and some element $\beta \in H_{DR}^2(M)/2\pi H^2(M; \mathbf{Z})$.

Notice that, locally, $\alpha + i\beta$ varies in an open subset of $H^2(M; \mathbf{C})$, so that the space \mathcal{M}_M is, locally, a product. The manifold M and its “mirror” partner M^* should be related by an isomorphism of the moduli spaces

$$\mathcal{M}_M \longrightarrow \mathcal{M}_{M^*}$$

that exchanges the factors of this local decomposition as a product.

Using in an essential way the symplectic structure of the loop space of M and techniques that go far beyond the level of these notes, Givental has proved the conjecture in [14], following a series of previous papers, the references of which can be found in [14] and [33].

Special Lagrangian submanifolds have been a few years ago the central object of another approach to mirror symmetry, more speculative and having given so far very few results — but a very beautiful approach indeed, that I intend to describe very briefly here.

II.7.a Fibrations in special Lagrangian submanifolds

We are no more interested in a single special Lagrangian submanifold but in a whole family. More precisely, we consider a compact Calabi-Yau manifold M and a differential mapping

$$p : M \longrightarrow B$$

to a manifold B , whose general fibers are special Lagrangian submanifolds. The dimension of B , as that of the fibers of p , must be n . It is not required that p be everywhere regular. Some of the fibers may be singular. The other ones, who correspond to regular values of p , are called *general* fibers.

We know (see §I.6.c and [4]) that in any proper Lagrangian fibration, the general fibers are unions of tori, so this must be the case here. The first Betti number of a torus of dimension n is precisely n , so that it can be expected that B “looks like” the moduli space of special Lagrangian submanifolds.

So, let $b \in B$ be a regular value of p and let $V \subset p^{-1}(b)$ be a connected component of the fiber $p^{-1}(b)$. If $X \in T_b B$ is a tangent vector, there exists a unique vector field Y normal to V in M and such that, for all x in V ,

$$T_x p(Y_x) = X.$$

To this field Y corresponds a harmonic 1-form η on V , as in the proof of the theorem of McLean (here Theorem II.6.1). As B has dimension n , starting from n independent vectors X_1, \dots, X_n in $T_b B$, one constructs n fields Y_1, \dots, Y_n , that are normal to V and linearly independent at each point of V . Dually, we thus have n harmonic 1-forms η_1, \dots, η_n that form a basis of $\mathcal{H}^1(V)$ and are linearly independent at each point of V .

In order that such a fibration $p : M \rightarrow B$ exists in a neighbourhood of a special Lagrangian tors $V \subset M$, it is necessary that, for the metric induced by the

Calabi-Yau structure on V , there exists a basis of $\mathcal{H}^1(V)$ consisting of forms that are independent at each point of V .

It is time to mention that (except in dimension 1) there is no known example having all the properties mentioned here.

- Notice first that, abstractly, a basis of harmonic 1-forms that are independent at each point exists on the flat torus, the basis dx_1, \dots, dx_n having this property. The metrics that are close enough to the flat metric thus have the same property.
- We have seen in §II.5.c that the situation of a Calabi-Yau manifold foliated by special Lagrangian submanifolds occurs in dimension 1.
- In dimension 2, on a special Lagrangian torus, one always has a basis of harmonic 1-forms as expected. We have seen that a special Lagrangian submanifold in dimension 2 is simply a complex curve (for a different complex structure). Assuming the submanifold is a torus, it must be an elliptic curve and it has a nowhere vanishing holomorphic form. Actually, the real and imaginary part of this form are harmonic forms on V and they are independent at every point.

II.7.b Mirror symmetry

The Strominger, Yau and Zaslow approach to mirror symmetry [32] is to associate, to a Calabi-Yau manifold M endowed with a fibration in special Lagrangian tori (assuming it exists), another Calabi-Yau manifold M^* . The latter should be the “extended” moduli space of special Lagrangian submanifolds of M equipped with a flat unitary line bundle. Call, as above, \mathcal{B} the moduli space of special Lagrangian submanifolds in the neighbourhood of V . Locally, the extended moduli space is

$$M^* = \mathcal{B} \times H^1(V; \mathbf{R}/\mathbf{Z}).$$

Its tangent space at a point m is

$$T_m M^* = H^1(V; \mathbf{R}) \oplus H^1(V; \mathbf{R}) \simeq H^1(V; \mathbf{R}) \otimes \mathbf{C}.$$

Thus, M^* has a natural almost complex structure, it is even Kähler:

Theorem II.7.1 (Hitchin [19]). *The complex structure on M^* is integrable, the metric of $H^1(V; \mathbf{R})$ defines a Kähler metric on M^* .*

We have seen (Remark II.6.6 above) that \mathcal{B} is a Lagrangian submanifold of $H_{DR}^1(V) \oplus H_{DR}^{n-1}(V)$, a symplectic vector space endowed by the metric of an almost complex structure (see Exercise I.12). It can be shown (see [19]) that M^* is a Calabi-Yau manifold if \mathcal{B} is... a special Lagrangian submanifold in this complex vector space. See [11, 20].

Exercises

Exercise II.1. Check that the Liouville form λ of the cotangent T^*V satisfies

$$(d\lambda)_{(x,0)}((v, \alpha), (w, \beta)) = \alpha(w) - \beta(v)$$

(see Exercise I.1).

Exercise II.2. Let $\varphi : L \rightarrow L$ be a diffeomorphism. Prove that the formula

$$\Phi(x, \alpha) = (\varphi(x), ((d\varphi)_x^{-1})^* \alpha)$$

defines a diffeomorphism of T^*L into itself. Determine $\Phi^*\lambda$ and prove that Φ preserves the symplectic form.

Exercise II.3. Let ω be a non degenerate 2-form on a manifold W . Define the Hamiltonian vector fields and Poisson brackets as above (this does not use the fact that ω is closed). Express

$$(d\omega)_x(X, Y, Z)$$

when X, Y et Z are tangent vectors to W at x that are the values at x of the Hamiltonian vector fields of three functions f, g and h . Prove that ω is a closed form if and only if the Poisson bracket it defines satisfies the Jacobi identity.

Exercise II.4. Assume X and Y are two “locally Hamiltonian” vector fields on a symplectic manifold, namely that $\iota_X\omega$ et $\iota_Y\omega$ are closed forms. Prove that their Lie bracket $[X, Y]$ is a globally Hamiltonian vector field, namely that $\iota_{[X, Y]}\omega$ is an *exact* form.

Exercise II.5 (The Hodge star operator). Let V be an n -dimensional oriented manifold endowed with a Riemannian metric g and let α be the Riemannian volume form. Check that the formula

$$g(u_1 \wedge \cdots \wedge u_p, v_1 \wedge \cdots \wedge v_p) = \det(g(u_i, v_j)_{1 \leq i, j \leq p})$$

defines an metric on $\Lambda^p T^*V \dots$ and that the map

$$\star : \Lambda^p T^*V \longrightarrow \Lambda^{n-p} T^*V$$

defined by

$$u \wedge (\star v) = g(u, v)\alpha$$

for all $u \in \Lambda^p T^*V$ defines, indeed, an operator, the *Hodge star operator*, which is an isometry. Check that

$$\star\star = (-1)^{p(n-p)} \text{Id}_{\Lambda^p T^*V}.$$

Exercise II.6 (Multilinear algebra in \mathbf{R}^4). Consider the vector space \mathbf{R}^4 , with its Euclidean structure $g(X, Y) = (X, Y)$ and canonical basis (e_1, e_2, e_3, e_4) , and the vector space

$$\Lambda = \Lambda^2(\mathbf{R}^4)^*$$

of alternated bilinear forms on \mathbf{R}^4 .

- (1) What is the dimension of Λ ? Check that Λ is isomorphic to the vector space of skew-symmetric endomorphisms of \mathbf{R}^4 .
- (2) Endow Λ with the Euclidean structure (\cdot, \cdot) induced by that of \mathbf{R}^4 , namely such that the basis $(e_i^* \wedge e_j^*)/\sqrt{2}$ (for $1 \leq i < j \leq 4$) is orthonormal. Define the (Hodge) star operator \star on Λ by the formula

$$(\star\alpha) \wedge \eta = (\alpha, \eta) \det \text{ for all } \eta \in \Lambda$$

(where \det , namely the determinant, is the generator of $\Lambda^4(\mathbf{R}^4)^*$ such that $\det(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = 1$).

Check that \star is an involution. Determine the $\star(e_i^* \wedge e_j^*)$ and the eigenspaces of \star .

- (3) Call Λ_+ the subspace of forms that are invariant by \star (they are called “self-dual” forms). To any α in Λ_+ , associate as in (1) a skew-symmetric endomorphism

$$J_\alpha : \mathbf{R}^4 \longrightarrow \mathbf{R}^4.$$

Prove that $J_\alpha^2 = -\text{Id}$ if and only if $(\alpha, \alpha) = 1$.

Exercise II.7 (Calabi-Yau surfaces). Let M be a Calabi-Yau surface with Kähler form ω and holomorphic 2-form Ω .

- (1) There exists a local basis (φ_1, φ_2) of the vector space of holomorphic forms on M in which

$$\omega = \frac{1}{2i} (\varphi_1 \wedge \bar{\varphi}_1 + \varphi_2 \wedge \bar{\varphi}_2)$$

(see [15]). Prove that, on the open set where φ_1 and φ_2 are defined, one has

$$\Omega = \lambda \varphi_1 \wedge \varphi_2$$

for some constant λ .

- (2) Check that $\star\Omega = \bar{\Omega}$. Deduce that the real α and imaginary β parts of Ω are self-dual in the sense that $\star\alpha = \alpha$ and $\star\beta = \beta$.
- (3) Prove that the formula

$$\alpha(X, JY) = g(X, Y)$$

defines a skew-symmetric endomorphism J of the tangent bundle TM and that

$$JX \in \langle X, IX \rangle^\perp = (\mathbf{C} \cdot X)^\perp.$$

- (4) Prove that $J^2 = -\text{Id}$, so that J is an almost⁶ complex structure on M , and that J is an isometry for g .
- (5) Prove that M is endowed with a hyperkähler structure, namely with three isometries I, J and K that are almost complex structures and anti-commute and with three non degenerate 2-forms that are Kähler for the metric g and respectively for each of the complex structures I, J and K .

Exercise II.8. Using the notation of Exercise II.7, prove that the special Lagrangian submanifolds of the Calabi-Yau manifold M are the complex curves for the complex structure J .

Exercise II.9. In this exercise, W denotes a complex analytic manifold⁷ of complex dimension 2, endowed with the structure of a Calabi-Yau manifold, with the Kähler form ω , the holomorphic volume form Ω and the metric γ . Assume moreover that $H_{DR}^1(W) = 0$.

Consider a vector field X on W , assume that it is not identically zero, and that it preserves ω and Ω , namely that it satisfies the relations $\mathcal{L}_X\omega = 0$ and $\mathcal{L}_X\Omega = 0$. Assume moreover that the 1-form $\iota_X\Omega$ is *holomorphic*.

The metric γ and the vector field X are assumed to be complete.

- (1) Prove that X is the Hamiltonian vector field of a function $H : W \rightarrow \mathbf{R}$.
- (2) Prove that $\iota_X\Omega$ is preserved by X and that there exists a holomorphic function $f : W \rightarrow \mathbf{C}$ such that $\iota_X\Omega = df$. Let $x \in W$ be a point such that $X_x \neq 0$. Prove that the kernel of $(\iota_X\Omega)_x$ is the complex line in T_xW spanned by X_x .
- (3) Assume that L is a Lagrangian submanifold of W that is preserved by X (this means that $X_x \in T_xL$ for all $x \in L$). Check that the connected components of L are contained in the level sets $H^{-1}(a)$ of the Hamiltonian H .
- (4) Call g and h respectively the real and imaginary part of f . Assume now that L is a *special* Lagrangian submanifold. Prove that h is locally constant on L .
- (5) Let $a \in \mathbf{R}$ be a *regular* value of H and let $Q = H^{-1}(a)$ be the corresponding level set in W . Fix a point x in Q . Prove that the orthogonal of X_x for the metric γ in T_xQ is a complex line D_x and that the complex linear form $(\iota_X\Omega)_x$ is non zero on D_x . Deduce that the two real linear forms $dh(x)$ and $dH(x)$ are independent. Prove that, for all $b \in \mathbf{R}$, $L = Q \cap h^{-1}(b)$ is a dimension-2 submanifold of W and that it is special Lagrangian.
- (6) Prove that $g|_L$ has no critical point.

⁶It is not very hard to prove that this is a *genuine* complex structure, namely that M is a complex manifold for some structure such that the multiplication by i is J .

⁷This exercise (slightly) generalizes the construction given in §1.5.e and in particular that of Exercise I.19.

- (7) Assume that the Hamiltonian vector field X is periodic. Prove that the connected components of L are diffeomorphic to $S^1 \times \mathbf{R}$.

Bibliography

- [1] V. I. ARNOLD – “A characteristic class entering in quantization conditions”, *Funct. Anal. Appl.* **1** (1965).
- [2] — , *Mathematical methods in classical mechanics*, Springer, 1978.
- [3] — , “Lagrange and Legendre cobordisms I and II”, *Funct. Anal. Appl.* **14** (1980), p. 167–177 et 252–260.
- [4] V. I. ARNOLD et A. B. GIVENTAL – “Symplectic geometry”, *Dynamical systems, Encyclopædia of Math. Sci.*, Springer (1985).
- [5] M. AUDIN – “Symplectic and almost complex manifolds”, in [7], p. 41–74.
- [6] — , *Les systèmes hamiltoniens et leur intégrabilité*, Cours Spécialisés, 8, Société Mathématique de France & EDP Sciences, 2001.
- [7] M. AUDIN et J. LAFONTAINE (éds.) – *Holomorphic curves in symplectic geometry*, Progress in Math., Birkhäuser, 1994.
- [8] J. BERTIN, J.-P. DEMAILLY, L. ILLUSIE et C. PETERS – *Introduction à la théorie de Hodge*, Panoramas et Synthèses, 3, Société Mathématique de France, 1996.
- [9] R. BRYANT – “Some examples of special Lagrangian tori”, *preprint* (1998).
- [10] H. CARTAN – *Calcul différentiel*, Hermann, Paris, 1967.
- [11] S. DONALDSON – “Moment maps and diffeomorphisms”, *Asian Journal of Math.* (2000), à paraître.
- [12] D. B. FUKS – “Maslov-Arnold characteristic classes”, *Soviet Math. Dokl.* **9** (1968), p. 96–99.
- [13] A. B. GIVENTAL – “Lagrangian imbeddings of surfaces and the open Whitney umbrella”, *Funktsional. Anal. i Prilozhen.* **20** (1986), no. 3, p. 35–41, 96.
- [14] A. B. GIVENTAL – “Equivariant Gromov-Witten invariants”, *Internat. Math. Res. Notices* **13** (1996), p. 613–663.

- [15] P. A. GRIFFITHS et J. HARRIS – *Principles of algebraic geometry*, Wiley, 1978.
- [16] M. GROMOV – “Pseudo-holomorphic curves in symplectic manifolds”, *Invent. Math.* **82** (1985), p. 307–347.
- [17] M. GROMOV – *Partial differential relations*, Springer, Berlin, 1986.
- [18] R. HARVEY et H. B. LAWSON, JR. – “Calibrated geometries”, *Acta Math.* **148** (1982), p. 47–157.
- [19] N. J. HITCHIN – “The moduli space of special Lagrangian submanifolds”, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **25** (1997), no. 3-4, p. 503–515 (1998), Dedicated to Ennio De Giorgi.
- [20] — , “Lectures on special Lagrangian submanifolds”, *preprint* (1999).
- [21] V. M. KHARLAMOV – “On the classification of nonsingular surfaces of degree 4 in \mathbf{RP}^3 with respect to rigid isotopies”, *Funktsional. Anal. i Prilozhen.* **18** (1984), no. 1, p. 49–56.
- [22] J. LAFONTAINE – *Introduction aux variétés différentielles*, Presses universitaires de Grenoble, 1996.
- [23] J. A. LEES – “On the classification of Lagrange immersions”, *Duke Math. J.* **43** (1976), no. 2, p. 217–224.
- [24] D. MCDUFF et D. SALAMON – *Introduction to symplectic topology*, The Clarendon Press Oxford University Press, New York, 1995, Oxford Science Publications.
- [25] R. C. MCLEAN – “Deformations of calibrated submanifolds”, *Comm. Anal. Geom.* **6** (1998), no. 4, p. 705–747.
- [26] J. MILNOR – *Morse theory*, Princeton University Press, 1963.
- [27] K. MOHNKE – *preprint* (2001).
- [28] J. MOSER – “On the volume elements on a manifold”, *Trans. Amer. Math. Soc.* **120** (1965), p. 286–294.
- [29] H. ROSENBERG et D. HOFFMAN – *Surfaces minimales et solutions de problèmes variationnels*, Société Mathématique de France, Paris, 1993.
- [30] A. CANNAS DA SILVA – *Lectures on symplectic geometry*, Lecture Notes in Mathematics, Springer, 2001.
- [31] M. STENZEL – “Ricci-flat metrics on the complexification of a compact rank one symmetric space”, *Manuscripta Math.* **80** (1993), p. 151–163.

- [32] A. STROMINGER, S. T. YAU et E. ZASLOW – “Mirror symmetry is T -duality”, *Nuclear Phys.* **B 479** (1996), p. 243–259.
- [33] C. VOISIN – *Symétrie miroir*, Panoramas et Synthèses, 2, Société Mathématique de France, 1996, English translation: *Mirror symmetry*, SMF/AMS Texts and Monographs 1, 1999.
- [34] A. WEINSTEIN – *Lectures on symplectic manifolds*, CBMS Regional Conference Series in Mathematics, 29, Amer. Math. Soc., 1977.
- [35] S. T. YAU – “On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I”, *Comm. Pure and Appl. Math.* **31** (1978), p. 339–411.