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EIGENVECTORS OF LAX MATRICES, SPACES OF HYPERELLIPTIC CURVES AND ACTION COORDINATES FOR MOSER SYSTEMS

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We describe a Poisson structure on some open subset of the relative Jacobian of hyperelliptic curves and use eigenvectors of Lax matrices to derive action variables for Moser systems.

Special Kovalevskaya Edition

Introduction

In this work, we carry on with the idea that eigenvectors of Lax matrices describe the geometry of the differential (Lax) equation. In previous work (see [8, 3, 4, 6]) eigenvectors were used to describe the Liouville tori for some systems of classical mechanics. We try now to use them to describe the *transverse* affine structure. In this paper we consider some rather simple systems, related both to quadrics and to hyperelliptic curves, the Moser systems.

According to a general idea of Novikov and Veselov ([19], see also [11]) we describe a Poisson structure on some (big) open subset of the relative Jacobian over a Hurwitz space of hyperelliptic curves.

In the second part of the paper, we relate it (through eigenvectors of Lax matrices) to the Moser systems and derive action coordinates for these systems.

1. A Poisson structure on the relative Jacobian of hyperelliptic curves

Let M be a manifold endowed with a foliation $\mathscr{S}$ and with a closed 2-form ω. Assume that for any leaf S of $\mathscr{S}$

$$S \xhookrightarrow{i_S} M\,,$$

the induced form $\omega_S = i_S^\star\omega$ is a symplectic form on S. Then there exists a unique Poisson structure on M such that the foliation by the symplectic manifolds (S, ω_S) is the symplectic foliation (and the first integrals of $\mathscr{S}$ are Casimir functions): simply define the Poisson bracket of two functions f and g by the formula

$$\{f, g\}(x) = \{f|_S, g|_S\}_S(x) \quad \text{when} \quad x \in S\,.$$

We will use this construction in the case where M is an open subset of the relative Jacobian over a moduli space of hyperelliptic curves and the foliation is induced by a foliation of this moduli space.

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1.1. A space of hyperelliptic curves

In this paper, a genus-n *hyperelliptic curve* is a smooth complete complex genus-n curve C endowed with a degree-2 map

$$z\colon C \longrightarrow \mathbb{P}^1$$

(I insist that the map z is part of the data). Two such curves (C, z) and (C', z') are *isomorphic* if there exists an isomorphism $\varphi\colon C \to C'$ such that $z' \circ \varphi = z$.

The $2n+2$ branch points of z in $\mathbb{P}^1$, a configuration of $2n+2$ distinct points on the line are associated, the space of isomorphism classes of data (C, z) is thus a complex manifold of dimension $2n+2$ with the curve (C, z).

It is convenient to consider z as a meromorphic function on C, that is, as an affine coordinate on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. A configuration of $2n+2$ distinct points in $\mathbb{P}^1$ can then be described by the monic polynomial whose roots are the given points — it has degree $2n+1$ if ∞ is one of the points, $2n+2$ otherwise.

A covering and a foliation. We will consider an n-dimensional foliation of this "moduli space" obtained by fixing some of the branch points. It is natural to do so

- because this is the simplest (or the more naïve) way to lower the dimension of the moduli space,
- but also because the dimension-n families of hyperelliptic curves obtained this way do appear naturally in the theory of integrable systems (in particular in the Jacobi system giving the geodesics on an ellipsoid, see Section 2.1 below).

We will thus consider a covering $\mathscr{H}_n$ of the space of isomorphism classes of data (C, z). The branch points of z are divided into two subsets

$$\{x_1, \ldots, x_n\} \cup \{\alpha_0, \ldots, \alpha_{n+1}\}$$

the x_i's being genuine points of $\mathbb{C}$ (that is, $x_i \neq \infty$ for all i).

The manifold $\mathscr{H}_n$ is foliated by n-dimensional leaves obtained by fixing the second subset. The functions on $\mathscr{H}_n$ which depend only on the α_j's form the algebra $\mathscr{A}$ of first integrals of this foliation.

Equations. Any curve (C, z) is the nonsingular completion of an affine curve of equation

$$y^2 = P(z)\,,$$

where P is a polynomial whose roots are the branch points (different from ∞) of z. The covering space $\mathscr{H}_n$ is the space of polynomials (with simple roots)

$$P(z) = Q(z)R(z)\,,$$

where $\deg Q = n$ and $\deg R = n+1$ if ∞ is a branch point, $n+2$ otherwise.

Remark 1. As we will deal with real problems, we must be careful with sign conventions. To agree with the notation that is usual in the mechanical problems we want to consider, we will not assume that Q and R are monic but rather than

$$Q(z) = \prod (x_i - z) \quad \text{and} \quad R(z) = \prod (\alpha_j - z)\,.$$

The leaf denoted $\mathscr{H}_n^R$ corresponds to fixing R. To emphasize the fact that we have now two sorts of branch points, we will use the function $h = \dfrac{y}{R(z)}$ and write an affine equation for our curves in the form

$$h^2 = \frac{Q(z)}{R(z)}\,.$$

We will use the functions z, y and h with this signification throughout the paper. Notice that each $\mathscr{H}_n^R$ is just an open subset of the space of degree-n polynomials with leading coefficient $(-1)^n$.

1.2. The relative Jacobian

On $\mathscr{H}_n$, there is a "tautological" curve $(\mathscr{C}, z)$

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{z} & \mathbb{C} \\ \downarrow & & \\ \mathscr{H}_n & & \end{array}$$

that can be described in a very down-to-earth way as a (partial) completion of

$$\mathscr{C} = \left\{ (Q, R, h, z) \in \mathscr{H}_n \times \mathbb{C}^2 \mid h^2 = \frac{Q(z)}{R(z)} \right\}.$$

Similarly, there is a k-fold symmetric (fibered) product of $\mathscr{C}$:

$$\mathscr{C}^{(k)} = \left\{ (Q, R, h_1, z_1, \ldots, h_k, z_k) \mid h_i^2 = \frac{Q(z_i)}{R(z_i)} \right\} / \mathfrak{S}_k$$

and a degree-k Picard $\mathrm{Pic}^k(\mathscr{C})$.

What I will call the *relative Jacobian* is the degree-n Picard $\mathscr{J}_n = \mathrm{Pic}^n\mathscr{C} \to \mathscr{H}_n$ (the space of linear equivalence classes of degree-n divisors). Recall that n is the genus of the curves in the family. We will use the natural onto map

$$\begin{aligned} u\colon \mathscr{C}^{(n)} &\longrightarrow \mathrm{Pic}^n\mathscr{C}, \\ (C, p_1 + \cdots + p_n) &\longmapsto (C, [p_1 + \cdots + p_n]) \end{aligned}$$

denoting by square brackets the linear equivalence class of the divisor.

The relative Jacobian is foliated by the inverse images $\mathscr{J}_n^R$ of the leaves $\mathscr{H}_n^R$. We will also need the degree-$(n+1)$ version $\mathscr{J}_{n+1}$ (linear equivalence classes of divisors of degree $n+1$) and its variant $\mathscr{J}_{n+1}^R$.

1.3. The Liouville form α

On a curve C of (affine) equation

$$h^2 = \frac{Q(z)}{R(z)}$$

we will consider the meromorphic 1-form

$$\sigma = h dz = \frac{Q(z) dz}{y}.$$

It has poles only over $z = \infty$. This is a double pole if ∞ is a branch point; there are two simple poles otherwise.

Considering $\mathscr{C}$ as a subset of $\mathscr{H}_n \times \mathbb{C}^2$, the same formulae give a form, also called σ, on $\mathscr{C}$ and a form σ_k on $\mathscr{C}^{(k)}$.

Forms on the relative degree-n Jacobian. Let C be a hyperelliptic curve of genus n and let D be an element of $\mathrm{Pic}^n C$, considered as the class of an effective divisor

$$D = [p_1 + \cdots + p_n].$$

Look at the map

$$T_D\mathrm{Pic}^n C \xrightarrow{-D} T_0\mathrm{Pic}^0 C = H^0(\Omega^1_C)^\star$$

and at the transposed isomorphism

$$H^0(\Omega^1_C) \longrightarrow T^\star_D \mathrm{Pic}^n C\,.$$

To give a holomorphic 1-form on $\mathrm{Pic}^n C$, it is sufficient to give, for any D, a holomorphic 1-form α_D on C (depending holomorphically on D).

We are going to construct a holomorphic 1-form α on an open subset of $\mathrm{Pic}^n\mathscr{C}$ satisfying, for the natural map $u\colon \mathscr{C}^{(n)} \to \mathrm{Pic}^n\mathscr{C}$, the equation

$$u^\star\alpha = \sigma_n\,.$$

Let f be any polynomial. Let $p_i = (\eta_i, \zeta_i)$ (for $1 \leqslant i \leqslant n$) be n points on the curve $\mathscr{C}$, so that

$$\eta_i^2 = \frac{Q(\zeta_i)}{R(\zeta_i)}\,.$$

Define a polynomial $f_{p_1+\cdots+p_n}$ of degree at most $n-1$ by the conditions

$$f_{p_1+\cdots+p_n}(\zeta_i) = f(\zeta_i) \quad \text{for} \quad 1 \leqslant i \leqslant n$$

and a holomorphic 1-form on $\mathscr{C}$ by

$$\beta^f_{p_1+\cdots+p_n} = \frac{f_{p_1+\cdots+p_n}(z)}{y}dz\,.$$

Let $\mathscr{V}$ be the open subset of $\mathscr{C}^{(n)}$ defined by

$$\mathscr{V} = \{p_1 + \cdots + p_n \mid p_i \neq \tau p_j \quad \text{and} \quad z(p_i) \neq \infty\}$$

(τ is the hyperelliptic involution $h \mapsto -h$, we do not require that the p_i's be distinct).

Proposition 1. *If the effective divisor $p_1 + \cdots + p_n$ belongs to the open subset $\mathscr{V}$, the 1-form $\beta^f_{p_1+\cdots+p_n}$ depends only on the linear equivalence class of $p_1 + \cdots + p_n$. The form α defined on $\mathscr{V}$ by*

$$\alpha_{[p_1+\cdots+p_n]} = \beta^Q_{p_1+\cdots+p_n}$$

is such that $u^\star\alpha = \sigma_n$.

Proof.

The first assertion is a consequence of the following well-known lemma:

Lemma 1. *Let V be the open subset of $C^{(n)}$ defined by*

$$V = \{p_1 + \cdots + p_n \mid p_i \neq \tau p_j \quad \text{and} \quad z(p_i) \neq \infty\}\,.$$

If $p_1 + \cdots + p_n$ belongs to V, then $u^{-1}([p_1 + \cdots + p_n]) = p_1 + \cdots + p_n$.

Let us check the second assertion of Proposition 1. For $X_i \in T_{p_i}C$ and with obvious notation, we have

$$(u^\star\alpha)_{p_1+\cdots+p_n}(X_1 + \cdots + X_n) = \alpha_{[p_1+\cdots+p_n]}(T_{p_1+\cdots+p_n}u(X_1 + \cdots + X_n))\,.$$

Now, u is the Abel–Jacobi map, so that $T_{p_1+\cdots+p_n}u(X_1 + \cdots + X_n)$ is the linear form on $H^0(\Omega^1_C)$

$$\omega \longmapsto \sum_{i=1}^n \omega_{p_i}(X_i)\,.$$

Hence, we have

$$(u^\star\alpha)_{p_1+\cdots+p_n}(X_1+\cdots+X_n)=\sum_{i=1}^{n}\left(\alpha_{[p_1+\cdots+p_n]}\right)_{p_i}(X_i)\,.$$

Now, by the very definition of α_D,

$$\left(\alpha_{[p_1+\cdots+p_n]}\right)_{p_i}(X_i)=\left(\beta^Q_{p_1+\cdots+p_n}\right)_{p_i}(X_i)=\left(\frac{Q_{p_1+\cdots+p_n}(z)}{y}dz\right)_{p_i}(X_i)=$$
$$=\frac{Q_{p_1+\cdots+p_n}(\zeta_i)}{y_i}dz(X_i)=\frac{Q(\zeta_i)}{y_i}dz(X_i)=\eta_i dz(X_i)$$

so that $u^\star\alpha=\sigma_n$ as announced. ■

Proof of Lemma 1.

We include a proof of Lemma 1 for completeness. It says that the only functions on C having at most simple poles at $p_1,\dots,p_n$ are constants, a straightforward application of the Riemann–Roch theorem.

The dimension of the space of such functions is

$$\deg(p_1+\cdots+p_n)-\mathrm{genus}(C)+1+i=1+i\,,$$

where i is the dimension of the space of holomorphic forms on C with a zero at each p_k. We know all the holomorphic 1-forms on C: these are the

$$\frac{g(z)}{y}dz\quad\text{for } g\qquad\text{a polynomial of degree }\leqslant n-1\,.$$

If $\ell=\deg g$, the divisor of the holomorphic form is

$$\left(\frac{g(z)}{y}dz\right)=Z_1+\tau Z_1+\cdots+Z_\ell+\tau Z_\ell+(n-\ell-1)\infty\,,$$

where ∞ stands for the (degree-2) pole divisor of z and Z_i is a point of C above a root of g. Of course, to have

$$Z_1+\tau Z_1+\cdots+Z_\ell+\tau Z_\ell+(n-\ell-1)\infty\geqslant p_1+\cdots+p_n\,,$$

it is necessary that $p_i=\tau p_j$ or $z(p_i)=\infty$ at least once. Thus, if $p_1+\cdots+p_n\in\mathscr{V}$, we have $i=0$. ■

We will call the 1-form α on $\mathscr{V}\subset\mathrm{Pic}^n\mathscr{C}$ the *Liouville form*. Note that some of the p_i's may coincide in $\mathscr{V}$. We will need to use the smaller open set $\mathscr{U}\subset\mathscr{V}$ defined by the additional conditions $p_i\neq p_j$:

$$\mathscr{U}=\{D=[p_1+\cdots+p_n]\mid p_i\neq p_j\quad\text{for}\quad i\neq j, z(p_i)\neq\infty\quad\text{and}\quad p_i\neq\tau p_j\}\,.$$

1.4. The Poisson structure on $\mathscr{U}\subset\mathrm{Pic}^n\mathscr{C}$

The idea that a form like α defines a Poisson structure on an open subset of the relative Jacobian comes from [19]. Let us determine, in our case, where $d\alpha$ is nondegenerate.

Proposition 2. *The restriction of the* 2*-form* $d\alpha$ *to the leaf* $\mathscr{J}^R_n\cap\mathscr{U}$ *in* $\mathscr{U}$ *is a nondegenerate* 2*-form.*

Proof.

The proof is a direct computation. Let S be the rational fraction $\frac{Q}{R}$:

$$S(z)=S(x_1,\dots,x_n,\alpha_1,\dots,\alpha_{n+1};z)=\frac{\prod_{i=1}^{n}(x_i-z)}{\prod_{j=1}^{n+1}(\alpha_j-z)}\,.$$

Using the equation $h^2 = S(z)$, we get

$$2hdh = \sum_{i=1}^{n} \frac{\partial S}{\partial x_i} dx_i + \sum_{j=1}^{n+1} \frac{\partial S}{\partial \alpha_j} d\alpha_j + S'(z)dz\,,$$

so that

$$d(hdz) = dh \wedge dz = \frac{1}{2h}\left[\sum_{i=1}^{n} \frac{\partial S}{\partial x_i} dx_i \wedge dz + \sum_{j=1}^{n+1} \frac{\partial S}{\partial \alpha_j} d\alpha_j \wedge dz\right].$$

Recall that we look only at the leaves defined by $\alpha_j =$ const so that we need only to consider the terms $dx_i \wedge dz$. As

$$\frac{\partial S}{\partial x_i}(z) = \frac{S(z)}{x_i - z} \quad \text{and} \quad \frac{S(z)}{h} = h\,,$$

we get that, in restriction to $\mathscr{H}_n^R$,

$$dh \wedge dz = \frac{h}{2} \sum_{i=1}^{n} \frac{dx_i \wedge dz}{x_i - z}\,.$$

Hence, over $\mathscr{C}^{(n)} \to \mathscr{H}_n^R$, we have

$$d\sigma_n = \frac{1}{2} \sum_{i,k=1}^{n} h_k \frac{dx_i \wedge dz_k}{x_i - z_k}\,.$$

This 2-form is given by a matrix

$$\begin{pmatrix} 0 & A \\ -{}^tA & 0 \end{pmatrix} \quad \text{with} \quad A_{i,j} = \frac{1}{2} h_j \frac{dx_i \wedge dz_j}{x_i - z_j}\,.$$

It is nondegenerate if and only if A is invertible. The determinant of A is, up to a $\left(-\frac{1}{2}\right)^n$ factor

$$h_1 \dots h_n \begin{vmatrix} \dfrac{1}{z_1 - x_1} & \cdots & \dfrac{1}{z_n - x_1} \\ \vdots & & \vdots \\ \dfrac{1}{z_1 - x_n} & \cdots & \dfrac{1}{z_n - x_n} \end{vmatrix} = h_1 \dots h_n \frac{\mathscr{P}(x_1, \dots, x_n, z_1, \dots, z_n)}{\prod_{i,j=1}^{n} (z_i - x_j)}\,,$$

where $\mathscr{P}$ is a polynomial divisible by the $x_k - x_\ell$ and the $z_k - z_\ell$. Comparing degrees, one gets

$$\det A = C h_1 \dots h_n \frac{\prod_{k<\ell} (x_k - x_\ell) \prod_{k<\ell} (z_k - z_\ell)}{\prod_{i,j} (z_i - x_j)}$$

for some nonzero constant C. Eventually, the form $d\sigma_n$ is nondegenerate on the open subset $\mathscr{U}'_R$ of $\mathscr{C}_R^{(n)}$ defined by

$$\mathscr{U}'_R = \{p_1 + \dots + p_n \mid p_i \neq p_j \quad \text{for} \quad i \neq j \quad \text{and} \quad p_i \neq \tau p_j\}\,.$$

Using Proposition 1, we get that $u^\star d\alpha$ is a nondegenerate 2-form on $\mathscr{U}'_R$, implying that $d\alpha$ is nondegenerate on $\mathscr{J}_n^R \cap \mathscr{U}$. ■

Corollary 1. *The derivative $d\alpha$ of the Liouville form α defines a Poisson structure on the open subset $\mathscr{U}$ in the relative Jacobian $\operatorname{Pic}^{n}\mathscr{C} \to \mathscr{H}_n$. The symplectic leaves are the intersections of $\mathscr{U}$ with the relative Jacobians $\mathscr{J}_n^R \to \mathscr{H}_n^R$. The fibrations $\mathscr{J}_n^R \cap \mathscr{U} \to \mathscr{H}_n^R$ have Lagrangian fibers.* ■

1.5. A remark on real curves

The case of real hyperelliptic curves is of special interest in the applications to integrable systems. To be very specific, let us consider the subspace of real curves in $\mathscr{H}_n$:

$$\mathscr{H}_n(\mathbb{R}) = \left\{ (C, z) \in \mathscr{H}_n \quad \text{with equation} \quad h^2 = \frac{Q(z)}{R(z)} \text{ for } Q, R \in \mathbb{R}[z] \right\}$$

and its (open and closed) subspace $\mathscr{H}_n(\mathbb{R})_0$ defined by the following conditions:

- all the roots $x_1, \ldots, x_n, (\alpha_0,)\alpha_1, \ldots, \alpha_{n+1}$ of Q and R are real (recall they are distinct),
- if they are ordered[1] $b_0 < b_1 < \cdots < b_{2n+2}$, then $x_i = b_{2i-1}$ or b_{2i}.

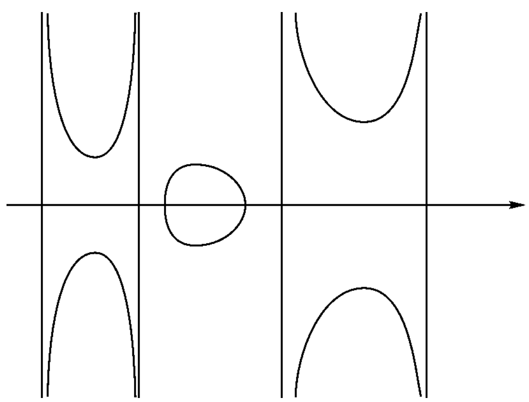
Fix $\alpha_0, \alpha_1, \ldots, \alpha_{n+1}$ in increasing order in $\{-\infty\} \cup \mathbb{R}$ and let

$$R(z) = \prod_{j=1}^{n+1} (\alpha_j - z)\,.$$

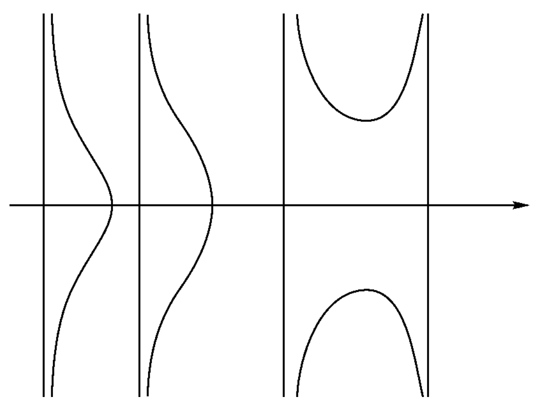
The curves (C, z) in $\mathscr{H}_n^R(\mathbb{R})_0$ are characterized by the following properties:

- the real part $C(\mathbb{R})$ has $n+1$ connected components $C_0, \ldots, C_n$,
- and $z(C_i) \subset]\alpha_i, \alpha_{i+1}[\subset \mathbb{R}$ for $0 \leqslant i \leqslant n$.

Figure 1 shows the affine parts (in the (z, h)-plane) of two possible[2] configurations, both in the case $n = 2$ and $\alpha_0 \neq \infty$. In the first one, $x_1 = b_2$ and $x_2 = b_3$ while in the second one $x_1 = b_1$ and $x_2 = b_3$.



(1) $x_1 = b_2$ and $x_2 = b_3$



(2) $x_1 = b_1$ and $x_2 = b_3$

Fig. 1

Look now at the real relative Pic^n over $\mathscr{H}_n(\mathbb{R})_0$. It also has several components. We will be specially interested in the (union of) component(s) $\mathscr{J}_n(\mathbb{R})_0 \to \mathscr{H}_n(\mathbb{R})_0$:

$$\mathscr{J}_n(\mathbb{R})_0 = \{D = [p_1 + \cdots + p_n] \mid p_i \in C_i \subset C(\mathbb{R})\}\,.$$

Let $\mathscr{U}_0 \subset \mathscr{J}_n(\mathbb{R})_0$ be the open subset defined by the condition "p_i is not a ramification point of z".

[1] The good convention to use here is that $\infty = -\infty$ in the real case, namely that if $\alpha_0 = \infty$, α_0 is less than all the roots of Q and R.

[2] Notice that there are 2^n configurations allowed (see also Figure 4 and Section 2.6).

Proposition 3. *The projection $\mathscr{J}_n(\mathbb{R})_0 \to \mathscr{H}_n(\mathbb{R})_0$ is a fibration by n-dimensional real tori. The Liouville form α defines a Poisson structure on the open subset $\mathscr{U}_0$ of $\mathscr{J}_n(\mathbb{R})_0$, the symplectic leaves of which are the $\mathscr{J}_n^R(\mathbb{R})_0 \cap \mathscr{U}_0$. The fibers of the map $\mathscr{J}_n(\mathbb{R}) \cap \mathscr{U}_0 \to \mathscr{H}_n(\mathbb{R})_0$ are open subsets of Lagrangian tori.* ■

Remark 2. The tori here are the products $C_1 \times \cdots \times C_n$. On each C_i, there are two forbidden points, their complement giving the open subset in question.

2. Action variables for Moser systems

In this part, I consider some Moser systems and show that the previous constructions can be used to construct action coordinates for these systems.

2.1. The Moser systems

The systems I will consider are

- geodesics on quadrics,
- geodesic flow (Euler–Arnold equations) on small coadjoint orbits of the rotation group $SO(N)$.

Geodesics on quadrics. We are given a family of confocal quadrics Q_z of equations

$$\frac{x_1^2}{\alpha_1 - z} + \cdots + \frac{x_{n+1}^2}{\alpha_{n+1} - z} - 1 = 0$$

in $\mathbb{R}^{n+1}$. The α_i's are distinct real numbers. Since Jacobi [14], it is known that to look at a geodesic on Q_z amounts to looking at a common tangent line to Q_z and $n-1$ other quadrics of the family (see also e. g. [22, 15, 17, 3]).

This is why we look at the space $\mathscr{D}_n$ of (affine) oriented straight lines in $\mathbb{R}^{n+1}$. As such a line is well defined by a unit vector and the projection of the origin, this space is diffeomorphic to the tangent bundle of the unit sphere:

$$\mathscr{D}_n \cong \left\{(p, u) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|p\| = 1 \quad \text{and} \quad p \cdot u = 0\right\} = TS^n \,.$$

This will be our phase space.

Euler–Arnold equations on small orbits of *SO*(*N*). The motion of a free rigid body is described by the differential equation $\dot{M} = M \wedge \Omega$, where M and Ω are the angular momentum and velocity, related by $M = \mathscr{J}\,\Omega$, for some symmetric matrix $\mathscr{J}$ (the "inertia" matrix). This is easily generalized in dimension N in the system

$$\dot{M} = [M, \Omega]\,, \qquad M = J\Omega + \Omega J\,.$$

Now M is in the Lie algebra $\mathfrak{so}(N)$ and J is a diagonal matrix. This system has been put in Lax form by Manakov [16], the Lax equation being

$$\frac{d}{dt}\left(M + hJ^2\right) = \left[M + hJ^2, \Omega + hJ\right]\,.$$

It also describes the geodesic flow on $SO(N)$ under a right-invariant metrics as studied by Arnold (see Appendix 2 in [2]), this is why the system is called Euler–Arnold equations.

It is clear that the motion takes place on (co)adjoint orbits in $\mathfrak{so}(N)$. In this paper, we will concentrate on the smallest orbits. We will thus look at the space of $N \times N$ skew-symmetric matrices of rank 2. Let M be such a matrix. It defines a skew-symmetric bilinear form on $\mathbb{R}^N$ with a kernel of codimension 2. Let P be the plane orthogonal to the kernel (orthogonality is of course with respect to the canonical Euclidean structure). Choose an orthonormal basis (x, y) in this 2-plane to see that the matrix M is conjugate to

$$M_\theta = \begin{pmatrix} \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\theta = {}^t x M y$ is nonzero and is the only orbital invariant. Notice that if $N \geqslant 3$,

$$\begin{pmatrix} \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{pmatrix} \text{ modulo the } SO(N)\text{-action}$$

so that one can assume that $\theta > 0$.

Call W_θ this set of matrices. This is an adjoint orbit in $\mathfrak{so}(N)$. Notice that the matrix M can be recovered from x and y: obviously

$$M = \theta x \wedge y\,,$$

where $x \wedge y$ denotes the matrix $(x \wedge y)_{i,j} = x_i y_j - x_j y_i$. Thus W_θ is diffeomorphic to the Grassmannian of oriented 2-planes in $\mathbb{R}^N$. Notice that W_θ is a *compact* manifold of dimension $2(N-2) = 2n$.

Note that, in order to describe W_θ, we can

- either choose an *orthonormal* basis (x, y) of P and get the orbit invariant $\theta = {}^t x M y$,
- or choose an orthogonal symplectic basis (x, y') with x unitary and get the (same) orbit invariant $\theta = \|y'\|$.

Symplectic structures. The manifold $\mathscr{D}_n$ of straight lines has a natural symplectic structure, either as TS^n or as the space of characteristics for the geodesic flow on $\mathbb{R}^{n+1}$ (up to a sign, this is the same structure).

The manifold W_θ also has a natural symplectic structure as it is a (co)adjoint orbit. It can also be described by symplectic reduction. Look at the complement of the zero section in TS^N, namely at

$$V = \left\{(x, y') \in \mathbb{R}^N \times \mathbb{R}^N \mid \|x\| = 1, x \cdot y' = 0 \quad \text{and} \quad y' \neq 0\right\}$$

endowed with the $SO(2)$-action

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \cdot (x, y') = (\cos\varphi x + \sin\varphi \frac{y'}{\|y'\|}, -\sin\varphi \left\|y'\right\| x + \cos\varphi y')\,.$$

This is a Hamiltonian action with Hamiltonian $H(x, y') = \|y'\|$ and the quotient of the θ-level V_θ is our W_θ. Note that $V_\theta \to W_\theta$ is, by definition, a principal $SO(2)$-bundle.

Lax equations. Consider $N \times N$ matrices of the form

$$A_h = h\alpha + a \wedge b + h^{-1} B_{a,b}\,.$$

In this equality,

- the two vectors a, b are in $\mathbb{R}^N$ and $a \wedge b$ is a short name for the matrix $(a \wedge b)_{i,j} = (a_i b_j - a_j b_i)$,
- the matrix $B_{a,b}$ is symmetric, depends on a and b and has rank $\leqslant 2$,
- the matrix α is diagonal, invertible and real,
- the "spectral parameter" h is an independent variable.

We can be more specific in the examples of interest:

- In the "geodesics on quadrics" case, $N = n+1$, $a = p$, $b = u$ and $B_{a,b} = -(p \otimes p) = -(p_i p_j)$ is a rank-1 matrix,
- In the Euler–Arnold case, $N = n+2$, $a = x$, $b = y'$ and $B_{a,b} = 0$, we recover a matrix A_h of the form $M + hJ^2$.

The Moser systems under consideration are (according to [17]) the differential systems on the phase spaces W (either $\mathscr{D}_n$ or W_θ):

$$\frac{d}{dt} A_h = [A_h, \Gamma + h\beta]$$

(the square bracket is, of course, the commutator of matrices) for some skew-symmetric matrix Γ depending on a and b (the systems are nonlinear!) and some diagonal matrix β depending on α (e.g. this is Manakov's equation in the Euler–Arnold case).

This differential system is a Lax form expression of a Hamiltonian system on the symplectic manifold W.

2.2. The algebro-geometric situation

To the differential system above it is associated an affine plane curve, the "spectral curve" defined by the vanishing of the characteristic polynomial of the matrix A_h. As A_h is a perturbation of the $N \times N$ diagonal matrix α by terms of rank $\leqslant 2$, this is easily computed (see [17]). We write

$$A_h - hz\,\mathrm{Id} = h(\alpha - z\,\mathrm{Id})\left[\mathrm{Id} + h^{-1}\underbrace{(\alpha - z\,\mathrm{Id})^{-1}(a \wedge b + h^{-1}B_{a,b})}_{C}\right].$$

Using any basis $e_1, \ldots, e_N$, we get

$$\begin{aligned}\det(\mathrm{Id} + h^{-1}C) e_1 \wedge \cdots \wedge e_N &= (\mathrm{Id} + h^{-1}C)^{\wedge N}(e_1 \wedge \cdots \wedge e_N) = \\ &= e_1 \wedge \cdots \wedge e_N + h^{-1}(\operatorname{tr} C) e_1 \wedge \cdots \wedge e_N + h^{-2}(\operatorname{tr} C^2) e_1 \wedge \cdots \wedge e_N\end{aligned}$$

(as $\operatorname{rk} C \leqslant 2$, $C^{\wedge 3} = 0$ and there are no additional terms). Now, as $\alpha - z\,\mathrm{Id}$ is diagonal and the diagonal terms of $a \wedge b$ are zero, the trace of the matrix $(\alpha - z\,\mathrm{Id})^{-1} a \wedge b$ is zero. Thus

$$\det(\mathrm{Id} + h^{-1}C) = 1 + h^{-2}\left[\operatorname{tr}((\alpha - z\,\mathrm{Id})^{-1}B_{a,b}) + \operatorname{tr}\left((\alpha - z\,\mathrm{Id})^{-1} a \wedge b\right)^2\right]$$

and eventually

$$\det(A_h - hz\,\mathrm{Id}) = h^N \det(\alpha - z\,\mathrm{Id})^N \left\{1 - h^{-2}\frac{Q_{a,b}(z)}{\prod(\alpha_i - z)}\right\}.$$

In this expression, $Q_{a,b}$ is a polynomial of degree $N-1$ in the variable z whose coefficients are functions of a and b. We will also decompose

$$\frac{Q_{a,b}(z)}{\prod(\alpha_i - z)} = \sum_{i=1}^{N} \frac{F_i(a,b)}{\alpha_i - z}$$

with

$$F_i(p,u) = p_i^2 + \sum_{j \neq i} \frac{(p_i u_j - p_j u_i)^2}{\alpha_i - \alpha_j} \quad \text{for} \quad 1 \leqslant i \leqslant n+1$$

in the case of geodesics on quadrics and

$$F_i(x,y)=\sum_{j\neq i}\frac{(x_iy_j-x_jy_i)^2}{\alpha_i-\alpha_j}\quad \text{for}\quad 0\leqslant i\leqslant n+1$$

in the Euler–Arnold case.

The spectral curve is the affine curve of equation

$$h^2=\frac{Q_{a,b}(z)}{\prod(\alpha_i-z)}\,.$$

A differential system in Lax form describes isospectral deformations of the matrix A_h. Thus the characteristic polynomial of a solution $A_h(t)$ does not depend on t. This means here that the F_i's are first integrals. It is also true that they are in involution (see [17] and Remark 6).

Note that our spectral curve is a hyperelliptic genus-n curve, a double cover of $\mathbb{P}^1$ branched at the α_i's, at the roots of $Q_{a,b}$ (and, if necessary, that is, if $N=n+1$, at ∞).

We are thus in the situation depicted in Section 1: a solution $A_h(t)$ of our differential system stays on an isospectral subset of the spaces of matrices, this subset being described by a curve which is precisely a curve C as in Section 1.1.

Remark 3. Notice also that $Q(\alpha_i)=F_i$, so that, if $F_i(a,b)=0$, α_i is a root of $Q_{a,b}$ and the corresponding curve is not smooth.

Real curves. It is not hard to show (for a proof, see e. g. [15, 18, 3] and Remark 4 below) that, for real $\alpha_0,\ldots,\alpha_{n+1}$, the real curve C satisfies the properties listed in Section 1.5, namely that (writing the α_i's in increasing order), the real part has $n+1$ components $C_0,\ldots,C_n$ with $z(C_i)\subset]\alpha_i,\alpha_{i+1}[$. See Figures 1, 2 and 3.

2.3. The eigenvector mapping

The Lax equation gives us a precise way to relate the dynamical system to the algebro-geometric situation, namely it gives us a *mapping*

$$\varphi_C\colon \mathcal{T}_C\longrightarrow \operatorname{Jac}(C)\,,$$

where $\mathcal{T}_C$ denotes the level set of the first integrals corresponding to the given curve C. To a point of $\mathcal{T}_C$, φ_C associates a line bundle L (or a divisor) on the spectral curve C. This is the *eigenvector bundle*. It has the property that its fiber at the point (h,z) of C is the dual of the eigenline of the actual matrix A_h defined by the given point of $\mathcal{T}_C$ corresponding to the eigenvalue hz.

Degrees. Consider the spectral parameter h as a meromorphic function on C, that is, as a degree-N map $C\to\mathbb{P}^1$. The direct image $h_\star L$ is the trivial rank-N bundle over $\mathbb{P}^1$ (this is to say that A_h is diagonalizable for almost all values of h). This allows to compute the degree d of L, using the Riemann–Roch formula:

$$\operatorname{ch}(h_\star L)\operatorname{td}(\mathbb{P}^1)=h_\star(\operatorname{ch}(L)\operatorname{td}(C))\,,$$

namely, writing t and u for the generators of $H^2(\mathbb{P}^1;\mathbb{Z})$, $H^2(C;\mathbb{Z})$ respectively,

$$N(1+t)=h_\star((1+du)(1+(1-n)u))\,.$$

With $h_\star 1=N$ and $h_\star u=t$, this gives $d=N+n-1$, specifically $d=2n$ in the case of geodesics on quadrics and $d=2n+1$ for the Euler–Arnold system.

In the two examples considered here, it is easy to determine explicitly the eigenvectors of the Lax matrix. Let f_z denote the symmetric bilinear form

$$f_z(a,b) = \sum \frac{a_j b_j}{\alpha_j - z}$$

and call z_i the zeroes of $f_z(a,a)$ (they depend on a). In [3], we noticed that

Proposition 4 (Eigenvectors, geodesics on quadrics). *Assume that the spectral curve C is smooth. The eigenvector bundle has a meromorphic section (eigenvector) with no zeroes and whose pole divisor is of the form $R_1 + \cdots + R_n + S_1 + \cdots + S_n$ where*

- *the point $R_i = (f_{z_i}(p,u), z_i)$ is a real point of the spectral curve C, belonging to the component C_i of $C(\mathbb{R})$,*
- *and the point $S_i = (0, x_i)$ is one of the branch points x_i of z on C.* ■

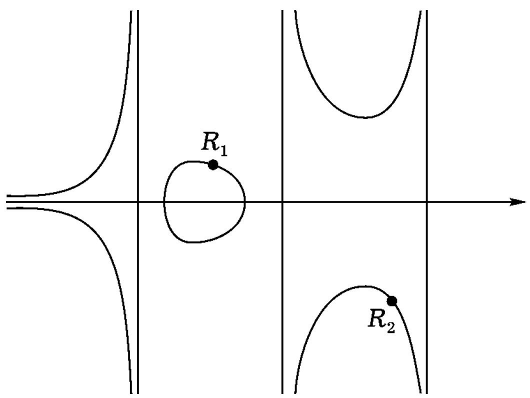


Fig. 2

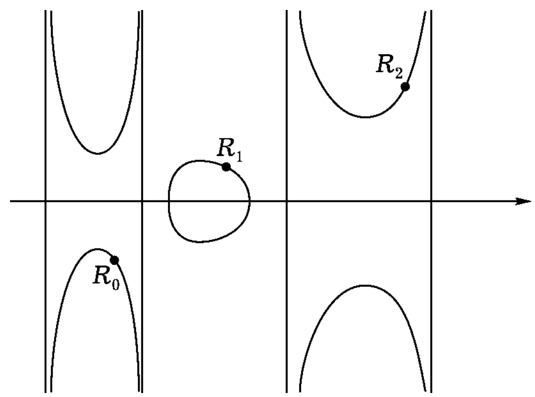


Fig. 3

See Figure 2. The situation in the Euler–Arnold case considered here is quite similar, except that there are $n+1$ points R_i in this case (Figure 3).

Proposition 5 (Eigenvectors, Euler–Arnold case). *Let (x,y) be a point in V. Assume the corresponding spectral curve C is smooth. The eigenvector bundle has a meromorphic section (eigenvector) with no zeroes and pole divisor of the form $R_0 + R_1 + \cdots + R_n + S_1 + \cdots + S_n$ where*

- *the point $R_i = (f_{z_i}(x,y), z_i)$ is a real point of the spectral curve C, belonging to the component C_i of $C(\mathbb{R})$,*
- *and the point $S_i = (0, x_i)$ is one of the branch points x_i of z on C.*

Proof.
The proof is similar to that of Proposition 4 (see [3]). Let us use the same notation as above (except that we use y instead of y'). We are looking for a (nonzero) eigenvector of $A_h = h\alpha + x \wedge y$. Write it

$$x + \lambda y + v \text{ with } v \in \langle x, y \rangle^{\perp}.$$

As $\|x\| = 1$, we are sure it will not vanish. To say that this is an eigenvector is to say that

$$(A_h - hz\,\mathrm{Id})(x + \lambda y + v) = 0\,.$$

We do not really want to determine these eigenvectors, we rather want to know for which points (h, z) on C the eigenvector can become infinite. Thus we look for the points (h, z) for which

$$\operatorname{Ker}(A_h - hz\,\mathrm{Id}) \cap x^\perp \neq \{0\}\ .$$

This is why we will look at $(A_h - hz\,\mathrm{Id})|_{x^\perp}$, noticing that

$$\begin{aligned}(A_h - zh\,\mathrm{Id})(\lambda y + v) &= (h(\alpha - z\,\mathrm{Id}) + x \wedge y)\,(\lambda y + v) = \\ &= h(\alpha - z\,\mathrm{Id})(\lambda y + v) + (x \wedge y)(\lambda y + v) = \\ &= h(\alpha - z\,\mathrm{Id})(\lambda y + v) + \lambda \|y\|^2 x\,.\end{aligned}$$

- If $h \neq 0$, we look for λ and v such that

$$\lambda \|y\|^2 x + h(\alpha - z\,\mathrm{Id})(\lambda y + v) = 0\,.$$

  - Notice that for $\lambda = 0$, this gives $(\alpha - z\,\mathrm{Id}) \cdot v = 0$. Recall that α is the diagonal matrix $(\alpha_0, \ldots, \alpha_{n+1})$. Thus $z = \alpha_j$ for some j (and $h \neq \infty$) and $v = \mu_i e_i$ is a multiple of a basis vector. It must also be orthogonal to x and y, thus the j-th coordinate of these two vectors should vanish. But this implies that $F_i(x, y) = 0$, a contradiction with the assumption that the spectral curve is smooth (see Remark 3 above).
  - We thus assume that $\lambda \neq 0$. Our equation is equivalent to the following:

$$v = \lambda \|y\|^2 h^{-1}(\alpha - z\,\mathrm{Id})^{-1}(x) - \lambda y\,.$$

  Writing that v is orthogonal both to x and y, we get

$$\begin{cases}\lambda h^{-1} \|y\|^2 (\alpha - z\,\mathrm{Id})^{-1}(x) \cdot x = 0\,,\\ \lambda \|y\|^2 \left[h^{-1}(\alpha - z\,\mathrm{Id})^{-1}(y) \cdot x - 1\right] = 0\,.\end{cases}$$

  Recall that we have denoted $f_z(u, v)$ the symmetric bilinear form

$$(\alpha - z\,\mathrm{Id})^{-1}(u) \cdot v$$

  and that $y \neq 0$ in V_θ. We thus obtain

$$f_z(x, x) = 0 \quad \text{and} \quad h = f_z(x, y)\,.$$

  Writing

$$f_z(x, x) = \frac{\mathcal{Q}(z)}{\prod(\alpha_j - z)}\,,$$

  we define a degree-$(N-1)$ (or $n+1$) polynomial $\mathcal{Q}$, the roots of which being what we denoted $z_0, \ldots, z_n$. The points $R_i = (f_{z_i}(x, y), z_i)$ are indeed the desired points.

- If $h = 0$,

  - with $z \neq \infty$, z is one of the roots of Q, the eigenvalue hz is zero, the kernel of $A_h = x \wedge y$ is the subspace $\langle x, y\rangle^\perp$. This gives us the points $S_1, \ldots, S_n$.
  - with $z = \infty$, $h^2 \sim \dfrac{\varepsilon^2}{z^2}$ for some module-1 complex number ε and thus the eigenvalue $hz \sim \varepsilon$. Then $h(\alpha - z\,\mathrm{Id}) + x \wedge y = -\varepsilon\,\mathrm{Id} + x \wedge y$ and

$$\begin{aligned}(A_h - hz\,\mathrm{Id})(x + \lambda y + v) &= -\varepsilon(x + \lambda y + v) + (-y + \lambda \|y\|^2 x) = \\ &= (\lambda \|y\|^2 - \varepsilon)x - (\lambda\varepsilon + 1)y - \varepsilon v\end{aligned}$$

  vanishes for $v = 0$ and $\|y\|^2 = -\varepsilon^2$. Thus no eigenvector of the form $x + \lambda y + v$ can become infinite at these points.

We still need to prove the "reality" claims. Notice first that

$$\mathcal{Q}(z)=\sum_{i=0}^{n+1} x_i^2 \prod_{j\neq i}(\alpha_j - z)$$

so that $\mathcal{Q}(\alpha_i) = x_i^2 \prod_{j\neq i}(\alpha_j-\alpha_i)$. Thus we have $\mathcal{Q}(\alpha_0) \geqslant 0$ (equality only if $x_0 = 0$) and the signs of the $\mathcal{Q}(\alpha_i)$'s alternate. This gives a real root of $\mathcal{Q}$ in every interval $[\alpha_j, \alpha_{j+1}]$ ... and thus all its roots are real. ∎

REMARK 4. Notice that the last argument ensures that the real part of the spectral curve has at least one point over each interval $[\alpha_j, \alpha_{j+1}]$, thus giving a short proof of the assertions on this real part stated in Section 1.5.

2.4. The Liouville form

Now we want to let the curve vary and to define the eigenvector mapping φ globally as a map from the whole phase space to the relative Jacobian. Recall that, in order to define φ_C, we need the curve C to be smooth. Using the differential of the map φ_C, we proved in [3] that the level $\mathcal{T}_C$ is regular if and only if the curve C is smooth[3].

Let us thus call W° the open set of regular points in W. We have a map

$$\varphi_R\colon W^\circ \longrightarrow \mathcal{J}_k^R \qquad k=n \quad \text{or} \quad n+1$$

from the phase space to the relative Jacobian. More globally, let $\mathcal{A}$ be the open set

$$\mathcal{A} = \begin{cases} \text{either} & \left\{(\alpha_1,\ldots,\alpha_{n+1}) \in \mathbb{R}^{n+1} \mid \alpha_i < \alpha_{i+1} \quad 1 \leqslant i \leqslant n\right\}, \\ \text{or} & \left\{(\alpha_0,\ldots,\alpha_{n+1}) \in \mathbb{R}^{N} \mid \alpha_i < \alpha_{i+1} \quad 0 \leqslant i \leqslant n\right\}. \end{cases}$$

Recall that the first integrals F_i's depend on the α_j's and let $P \subset W \times \mathcal{A}$ be the open subset of regular points:

$$P = \{(w,a) \in W \times \mathcal{A} \mid w \quad \text{is a regular point of the} \quad F_i\text{'s}\}$$

(it is fibered over $\mathcal{A}$ with fiber the corresponding W°). We even have a map

$$\varphi\colon P \longrightarrow \mathcal{J}_k \qquad k = n \text{ or } n+1\,.$$

REMARK 5. According to the construction of Poisson structures at the beginning of Section 1, $W \times \mathcal{A}$, endowed with the symplectic form ω of W, is a Poisson manifold, symplectically foliated by the $W \times \{a\}$.

Geodesics on quadrics. In this case, $W = \mathcal{D}_n = TS^n$ is endowed with a Liouville form

$$\lambda = \sum_{j=1}^{n+1} u_j dp_j\,.$$

It turns out that it is very simply related with the algebraic Liouville form α of Section 1.3.

Theorem 1. *The eigenvector mapping φ satisfies*

$$\varphi^\star \alpha = 2(-1)^n \lambda\,.$$

The proof is a computation and quite analogous to that of the corresponding assertion (Theorem 2) in the Euler–Arnold case (see below). ∎

[3] In [3], this result is proved only in the case of the geodesics on quadrics but the proof in the Euler–Arnold case is quite the same.

Corollary 2. *The eigenvector mapping is a Poisson map*

$$P \longrightarrow \mathscr{J}_n .$$

Proof.

The 2-form $\varphi^\star d\alpha$ is, up to a constant, the form $\omega = d\lambda$ on P. Both define the Poisson structures. ■

Remark 6. On the one hand, we know that $d\lambda$ is a nondegenerate 2-form, thus Theorem 1 implies that $\varphi^\star d\alpha$ is nondegenerate on the leaves $W^\circ \times \{a\}$, giving another (still computational) proof of Proposition 2. On the other hand, we know that the fibers of $\mathscr{J}_n \to \mathscr{H}_n$ are isotropic, thus Theorem 1 shows that the Hamiltonian vector fields must commute and thus that the first integrals Poisson commute — notice that we have *not* used this fact so far.

Euler–Arnold equations. We would like to do something analogous in the Euler–Arnold case. However, the symplectic manifold is compact (a Grassmannian) and there is no hope of finding a global primitive of the symplectic form. But W_θ was obtained as a reduced symplectic manifold from TS^{n+1} which has a Liouville form

$$\lambda = \sum_{j=0}^{n+1} y_j dx_j .$$

Fortunately, the degree-$(n+1)$ divisor exhibited in Proposition 5 depends on the point (x, y) in $V_\theta \subset TS^{n+1}$ and not only on its class in W_θ. In other words, Proposition 5 defines a lifted map $\widetilde{\varphi}$ fitting in a commutative diagram

$$\begin{array}{ccc} V_\theta \times \mathscr{A} & \xrightarrow{\widetilde{\varphi}} & \mathscr{C}^{(n+1)} \\ \big\downarrow & & \big\downarrow \\ W_\theta \times \mathscr{A} & \xrightarrow{\varphi} & \mathscr{J}_{n+1} \end{array}$$

(φ and $\widetilde{\varphi}$ are defined on open subsets that I will not mention anymore to save on notation).

Theorem 2. *The lifted map* $\widetilde{\varphi}\colon V_\theta \to \mathscr{C}^{(n+1)}$ *satisfies*

$$\widetilde{\varphi}^\star \sigma_{n+1} = 2(-1)^{n+1} \lambda .$$

Proof.

I am not very happy with this proof, which is a direct computation: we check that, for the roots z_i of $\mathscr{Q}$, we have

$$\sum_{i=0}^{n} f_{z_i}(x, y) dz_i = 2(-1)^{n+1} \sum_{j=0}^{n+1} y_j dx_j .$$

We change slightly notation and write

$$\Phi(z, x) = (-1)^{n+1} f_z(x, x) = \frac{\prod_{i=0}^{n}(z - z_i)}{\prod_{j=0}^{n+1}(\alpha_j - z)} = \frac{(-1)^{n+1}\mathscr{Q}(z)}{\prod_{j=0}^{n+1}(\alpha_j - z)} .$$

Let ζ be one of the roots z_i of the numerator, so that $\Phi(\zeta, x) = 0$. Differentiating this relation, we get

$$\frac{\partial \Phi}{\partial z}(\zeta, x) d\zeta + (-1)^{n+1} 2 f_\zeta(x, dx) = 0 , \qquad d\zeta = 2(-1)^n \frac{f_\zeta(x, dx)}{\frac{\partial \Phi}{\partial z}(\zeta, x)}$$

and

$$f_\zeta(x,y)d\zeta = 2(-1)^n \frac{f_\zeta(x,y)f_\zeta(x,dx)}{\frac{\partial\Phi}{\partial z}(\zeta,x)}.$$

Before summing over all the roots $\zeta = z_i$ of $\mathcal{Q}$, notice that

$$\frac{\partial\Phi}{\partial z}(\zeta,x) = (-1)^{n+1}\frac{\mathcal{Q}'(\zeta)}{\prod_{j=0}^{n+1}(\alpha_j-\zeta)}$$

as $\mathcal{Q}(\zeta)=0$, so that $\dfrac{\partial\Phi}{\partial z}(z_i,x) = \dfrac{\prod_{k\neq i}(z_i-z_k)}{\prod_{j=0}^{n+1}(\alpha_j-z_i)}$. Then

$$\sum_{i=0}^n f_{z_i}(x,y)dz_i = 2(-1)^n\sum_{i=0}^n \frac{f_{z_i}(x,y)f_{z_i}(x,dx)}{\frac{\partial\Phi}{\partial z}(z_i,x)} = 2(-1)^n\sum_{j=0}^{n+1}A_j dx_j$$

with $A_j = \sum_i \dfrac{f_{z_i}(x,y)x_i}{\frac{\partial\Phi}{\partial z}(z_i,x)(\alpha_j-z_i)}$. All the indices j behave in the same way. Let us concentrate on A_0 (for notational simplicity):

$$A_0 = \sum_{i=0}^n \frac{f_{z_i}(x,y)x_0}{\frac{\partial\Phi}{\partial x}(z_i,x)(\alpha_0-z_i)} = x_0\sum_{i=0}^n \frac{f_{z_i}(x,y)}{\prod_{k\neq i}(z_i-z_k)}\frac{\prod_{j=0}^{n+1}(\alpha_j-z_i)}{\alpha_0-z_i} =$$

$$= x_0\left\{x_0y_0\underbrace{\sum_{i=0}^n\frac{\prod_{j\neq0}(\alpha_j-z_i)}{(\alpha_0-z_i)\prod_{k\neq i}(z_i-z_k)}}_{B} + \sum_{\ell=1}^{n+1}x_\ell y_\ell\underbrace{\sum_{i=0}^n\frac{\prod_{j\neq0,\ell}(\alpha_j-z_i)}{\prod_{k\neq i}(z_i-z_k)}}_{(-1)^nC_\ell}\right\}.$$

Let us compute the term C_ℓ:

$$C_\ell = \sum_{i=0}^n\frac{\prod_{j\neq0,\ell}(z_i-\alpha_j)}{\prod_{k\neq i}(z_i-z_k)} = \frac{\prod_{j\neq0,\ell}(z_0-\alpha_j)}{\prod_{k\neq0}(z_0-z_k)} + \sum_{i=1}^n\frac{\prod_{j\neq0,\ell}(z_i-\alpha_j)/\prod_{k\neq0,i}(z_i-z_k)}{z_i-z_0}.$$

Expanding the first term (as a rational function of z_0), we get

$$\frac{\prod_{j\neq0,\ell}(z_0-\alpha_j)}{\prod_{k\neq0}(z_0-z_k)} = 1+\sum_{i=1}^n\frac{\prod_{j\neq0,\ell}(z_i-\alpha_j)/\prod_{k\neq0,i}(z_i-z_k)}{z_0-z_i}$$

so that $C_\ell = 1$. Moreover, as x and y are orthogonal vectors,

$$\sum_{k=1}^{n+1}x_ky_k = -x_0y_0,$$

hence

$$A_0 = x_0\left\{x_0y_0B+(-1)^n\sum_{k=1}^{n+1}x_ky_k\right\} = x_0^2y_0(B+(-1)^{n+1}).$$

Let us thus compute the term B:

$$B = \sum_{i=0}^n\frac{\prod_{j\neq0}(\alpha_j-z_i)}{(\alpha_0-z_i)\prod_{k\neq i}(z_i-z_k)} = \sum_{i=0}^n\frac{\prod_{j\neq0}(\alpha_j-z_i)/\prod_{k\neq i}(z_i-z_k)}{\alpha_0-z_i} =$$

$$= \frac{1}{\prod_{i=0}^n(\alpha_0-z_i)}\sum_{i=0}^n\frac{\prod_{\ell\neq i}(\alpha_0-z_\ell)\prod_{j\neq0}(\alpha_j-z_i)}{\prod_{k\neq i}(z_i-z_k)} = \frac{1}{\mathcal{Q}(\alpha_0)}\left\{\sum_{i=0}^n\prod_{\ell\neq i}\frac{\alpha_0-z_\ell}{z_i-z_\ell}\prod_{j\neq0}(\alpha_j-z_i)\right\}.$$

The term in the braces is a polynomial in α_0, has degree n and takes the value $\prod_{j\neq 0}(\alpha_j - z_i)$ at $\alpha_0 = z_i$ for $0 \leqslant i \leqslant n$. This polynomial is thus equal to

$$\prod_{j\neq 0}(\alpha_j - \alpha_0) - (-1)^{n+1}\prod_{i=0}^{n}(\alpha_0 - z_i)$$

(which has exactly the same properties), that is, to $\prod_{j\neq 0}(\alpha_j - \alpha_0) - (-1)^{n+1}\mathcal{Q}(\alpha_0)$. We thus get

$$B = \frac{\prod_{j\neq 0}(\alpha_j - \alpha_0)}{\mathcal{Q}(\alpha_0)} - (-1)^{n+1}.$$

Recall now that, by definition, $\mathcal{Q}(z) = \sum_{k=0}^{n+1} x_k^2 \prod_{j\neq k}(\alpha_j - z)$, so that $\mathcal{Q}(\alpha_0) = x_0^2\prod_{j\neq 0}(\alpha_j - \alpha_0)$. Eventually,

$$B = \frac{1}{x_0^2} - (-1)^{n+1}$$

so that $A_0 = y_0$. Similarly, $A_j = y_j$, hence

$$\widetilde{\varphi}^{\star}\sigma_{n+1} = 2(-1)^{n+1}\sum_{j=0}^{n+1} y_j dx_j\,.$$

■

2.5. Action variables

We expect to construct a map

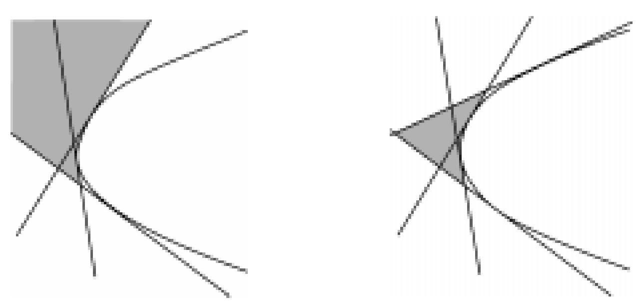
$$s = (s_1, \ldots, s_n)\colon W^{\circ} \longrightarrow \mathbb{R}^n$$

such that

$$s_i(a,b) = \int_{\gamma_i}\eta$$

where η is a primitive of the symplectic form on the component of W° containing (a,b) and $(\gamma_1, \ldots, \gamma_n)$ is a basis of the local system defined by the groups $H_1(\mathcal{T}_C;\mathbb{Z})$.

Remark 7. No monodromy problem should be expected in these examples: the connected components of the set of regular values of the momentum mapping are n-dimensional open simplices as shown in Figure 4 (see Section 2.6). This is why we expect to construct global action coordinates on W° (see [12] for a general discussion).



(1) case of geodesics on quadrics (2) Euler–Arnold case

Fig. 4. Image of the momentum mapping

We thus need two things:

- a primitive of the symplectic form,
- a basis of the local system.

Geodesics on quadrics. We will use the Liouville form λ on the phase space as a (preferred!) primitive of ω. We also have a preferred basis of the local system: we use the following proposition:

Proposition 6. *Let $\mathcal{T}_C$ be a component of a regular level of the integrable system in $\mathcal{D}_n$ and let $C_1, \ldots, C_n$ be the components of the real part of the corresponding curve C defined by $z(C_i) \subset]\alpha_i, \alpha_{i+1}[$. Then the eigenvector mapping*

$$\varphi_C \colon \mathcal{T}_C \longrightarrow C_1 \times \cdots \times C_n$$

is a covering map.

Thus $(\varphi_C)_\star : H_1(\mathcal{T}_C; \mathbb{Q}) \to H_1(C_1 \times \cdots \times C_n; \mathbb{Q})$ is an isomorphism. Let $\gamma_i \in H_1(\mathcal{T}_C; \mathbb{Z})$ be such that $(\varphi_C)_\star(\gamma_i) = m_i[C_i]$. Using Theorem 1, we get

$$\int_{\gamma_i} \lambda = \frac{(-1)^n}{2} \int_{\gamma_i} \varphi_C^\star \sigma_n = \frac{(-1)^n}{2} \int_{(\varphi_C)_\star(\gamma_i)} \sigma_n = \frac{(-1)^n}{2} m_i \int_{C_i} \sigma\,.$$

Corollary 3. *There exist integral numbers $m_i \in \mathbb{Z}$ such that, if $s_i = \frac{m_i}{2} \int_{C_i} \sigma$,*

$$(s_1, \ldots, s_n) \colon W^\circ \longrightarrow \mathbb{R}^n$$

are action coordinates for the system of geodesics on quadrics. ■

Sketch proof of Proposition 6. As $\mathcal{T}_C$ and $C_1 \times \cdots \times C_n$ are both n-dimensional tori, it suffices to show that the differential $T_{(p,u)}\varphi_C$ is surjective for all $(p,u) \in \mathcal{T}_C$. This is the computation, done in [3], which shows that $\mathcal{T}_C$ is regular when C is smooth[4]. ■

REMARK 8. Corollary 3 is stated in [7] — with somewhat different motivation.

Euler–Arnold case. We have already noticed that there is no global primitive of ω on the (compact) phase space W_θ. This of course does not prevent ω to be exact on the connected components of W_θ°. However, we want to give "close" formulae. This is why we will lift everything to V, the complement of the zero section in TS^{n+1} (we use the notation of Section 2.1).

On V itself, we have an integrable system, (H, F) where H is the periodic Hamiltonian $H(x,y) = \|y\|$ (the F_i's obviously commute with H). What we will look for are action coordinates for this integrable system.

Fix a smooth curve C and call $\mathcal{T}_{C,\theta}$ the corresponding (regular) level. Let $V_{C,\theta}$ be its inverse image in $V_\theta^\circ \subset V$, so that we have a commutative diagram

$$\begin{array}{ccc} V_{C,\theta} & \xrightarrow{\widetilde{\varphi}_C} & C_0 \times \cdots \times C_n \\ \pi \big\downarrow & & \pi \big\downarrow \\ \mathcal{T}_{C,\theta} & \xrightarrow{\varphi_C} & \left(\mathrm{Pic}^{n+1} C\right)_{\mathbb{R}} \end{array}$$

in which the two vertical maps π are S^1-fibrations (onto its image for the right-hand one).

REMARK 9. It is well-known (using the Riemann–Roch theorem as in Lemma 1) that the vector space of functions on C having at most simple poles at $R_0, \ldots, R_n$ has dimension 2. For instance, here the function

$$g(z) = \frac{h + f_z(x,y)}{f_z(x,x)}$$

is a (real) nonconstant function in this vector space (notice that the analogous formula in the case of geodesics on quadrics gives a function with an extra pole at ∞, in agreement with Lemma 1!). The complex fiber is a $\mathbb{P}^1(\mathbb{C})$ ... and the real one a $\mathbb{P}^1(\mathbb{R})$.

We have, similarly to Proposition 6:

[4] The argument is sketched in the proof of Proposition 7 below.

Proposition 7. *Assume the curve C is smooth. Then φ_C is a covering map and $\widetilde{\varphi}_C$ induces an isomorphism*

$$(\widetilde{\varphi}_C)_\star : H_1(V_{C,\theta};\mathbb{Q}) \longrightarrow H_1(C_0 \times \cdots \times C_n;\mathbb{Q}) .$$

Proof.

The assertion on φ_C is proved in exactly the same way as Proposition 6 (see [3]). Let us sketch the argument for completeness. One looks at the Hamiltonian vector fields corresponding to the first integrals and at their images under the differential of φ_C. According to classical works on Lax equations (see [1], [20]), these images are expressed as elements of $H^1(C;\mathscr{O}_C)$, the tangent space to the Jacobian. It is then proved that they span this vector space by a simple residue computation.

To prove the assertion on $\widetilde{\varphi}_C$, look again at the commutative diagram

$$\begin{array}{ccc} V_{C,\theta} & \xrightarrow{\widetilde{\varphi}_C} & C_0 \times \cdots \times C_n \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow \\ \mathscr{T}_{C,\theta} & \xrightarrow{\varphi_C} & \left(\mathrm{Pic}^{n+1} C\right)_{\mathbb{R}} \end{array} .$$

Recall that all the spaces are (unions of) tori and the vertical maps are S^1-bundles. We only need to prove that, if F is a fiber of π in $V_{C,\theta}$, its image $(\widetilde{\varphi}_C)_\star [F]$ is nonzero in $H_1(C_0 \times \cdots \times C_n)$. Using Theorem 2, we have

$$\langle \sigma_{n+1}, (\widetilde{\varphi}_C)_\star [F]\rangle = \langle (\widetilde{\varphi}_C)^\star \sigma_{n+1}, [F]\rangle = 2(-1)^{n+1}\langle \lambda, [F]\rangle$$

and the latter is nonzero, as $\lambda = \sum y_j dx_j$, the Liouville form, induces a non exact 1-form on the circle F, fiber of $V_\theta \to W_\theta$. ■

As the mapping $(\widetilde{\varphi}_C)_\star : H_1(V_{C,\theta};\mathbb{Q}) \to H_1(C_0 \times \cdots \times C_n;\mathbb{Q})$ is an isomorphism, we get as above:

Proposition 8. *There exist integral numbers $m_i \in \mathbb{Z}$ such that, if $s_i = \frac{m_i}{2}\int_{C_i}\sigma$,*

$$(s_0, \ldots, s_n) \colon V^\circ \longrightarrow \mathbb{R}^{n+1}$$

are action coordinates for the integrable system (H, F) on TS^{n+1}. ■

2.6. Image of the momentum mapping

To conclude (and complete) this section, let us give the readers a precise description of the image of the momentum mapping. Firstly, as we have already mentioned it, regular values correspond to smooth curves. Critical values are thus given by

- either the polynomial Q having a multiple root,
- or one of the α_i's being a root of Q, that is, the corresponding F_i being vanishing.

In the vector space of F_i's, let us consider the affine n-dimensional subspace E given by the equations

- (in the geodesics of quadrics case) $F_1 + \cdots + F_{n+1} = 0$, because $\|p\|^2 = 1$,
- (in the Euler–Arnold case) $F_0 + \cdots + F_{n+1} = 0$ and $\sum \alpha_i F_i = \theta^2$, because

$$\sum_{i=0}^{n+1} \alpha_i F_i = \sum_{i<j} (x_i y_j - x_j y_i)^2 = \|x \wedge y\|^2 = \theta^2 .$$

The set of critical values of the momentum mapping F consists of the intersection of E with

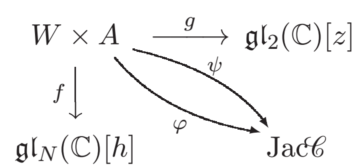
- the hyperplanes $F_i = 0$,
- and the discriminant of the polynomial Q, that is, here the hypersurface enveloped by the family of hyperplanes H_z of equation $\sum \frac{F_i}{\alpha_i - z} = 0$.

The set of regular values that are actual values of the (real) momentum mapping F is given by the conditions in Section 1.5 on the positions of the roots of Q. It consists of 2^n regions. Notice also that, between α_i and α_{i+1}, there are at most two roots of Q so that the points in the discriminant that are in the real image correspond to polynomial having (exactly) double roots — and particular, they are smooth points of this discriminant. Figure 4 above (copied from [5]) shows the images in the $n = 2$-case.

3. Further remarks and questions

3.1. Mumford's 2×2 matrices

Recall the relations between Mumford's (or Jacobi's) and Moser's matrices (see [18] and [17]) from the point of view of eigenvectors. We have maps



The map φ is our eigenvector mapping and f is the map associating with the geometric data the Moser matrix A_h of Section 2.1. Similarly, g associates with the same data the Mumford 2×2 matrix

$$R(z)\begin{pmatrix} -f_z(p,u) & f_z(p,p) \\ f_z(u,u) - 1 & f_z(p,u) \end{pmatrix} \quad \text{or} \quad R(z)\begin{pmatrix} -f_z(x,y) & f_z(x,x) \\ f_z(y,y) & f_z(x,y) \end{pmatrix}$$

the characteristic polynomial of which is $y^2 - Q(z)R(z)$. The associated spectral curve is, indeed, our curve C. Notice however that y plays the role of the eigenvalue here, in agreement with the fact that z became the spectral parameter.

Degrees. The degree of the eigenvector bundle for the Mumford matrices is easily computed by the same argument as in Section 2.3 using the degree-2 map $z\colon C \to \mathbb{P}^1$. One finds $d = n + 1$. The eigenvectors of the Mumford matrix are the multiples of $\begin{pmatrix} 1 \\ Y \end{pmatrix}$ with

$$Y = \frac{R(z)f_z(a,b) + y}{R(z)f_z(a,a)}.$$

The pole divisor[5] is given by $f_z(a,a) \neq 0$ and $f_z(a,b) + h \neq 0$ (recall that $h = \frac{y}{R(z)}$) ... the eigenvector mapping ψ in the diagram above sends the matrix of Mumford to the divisor

$$R_1 + \cdots + R_n + \infty \qquad \text{resp.} \qquad R_0 + R_1 + \cdots + R_n$$

in our two cases and thus coincides with φ (up to translation by a fixed divisor).

[5]Notice that Y is the function g of Remark 9.

3.2. The Poisson structures of Beauville

In a well known paper [9], Beauville gives a beautiful construction of a family of Poisson structures on relative Jacobians. In the hyperelliptic case considered here and with our notation, he defines a Poisson structure on $\mathscr{J}_n$ for any set $\alpha_0, \ldots, \alpha_{n+1}$ whereas we have defined a single Poisson structure on $\mathscr{J}_n$, the symplectic leaves of which are depicted by the sets $\alpha_0, \ldots, \alpha_{n+1}$. Briefly said, given $\alpha_0, \ldots, \alpha_{n+1}$,

- Beauville has a Poisson structure, the symplectic leaves of which are given by some complex numbers $c_0, \ldots, c_{n+1}$,
- we have a symplectic leaf of our Poisson structure.

Notice that the Beauville leaf depicted by $c_i = 0$ (all i) is our symplectic leaf $\alpha_0, \ldots, \alpha_{n+1}$.

Degrees. Beauville considers the degree-$(n-1)$ Jacobian. This amounts to consider the line bundle $L \otimes h^\star \mathscr{O}(-1)$ in place of the eigenline bundle L.

3.3. Other systems

The systems considered in the present paper are *very* simple. I am very tempted to assert that the construction of action coordinates given here extends to more complicated integrable systems (given by Lax equations with spectral parameters). There are, however, a few problems:

Small genus. In the examples considered here, the number of degrees of freedom coincides with the genus of the spectral curve and, thus, with the dimension of the Abelian varieties in the family. There are respectable and simple examples for which the genus of the spectral curve is too small. For instance, in the cases of the Lagrange top and of the spherical pendulum, there are two degrees of freedom and the spectral curve[6] has genus 1. This certainly prevents us to prove anything analogous with Theorems 1 or 2.

Monodromy (a side remark). In the two examples just mentioned, it is also impossible to get global action coordinates over connected components of the regular values: these components are not simply connected and there is, indeed, a monodromy obstruction (see [12]). I wonder whether the two phenomena are related. Specifically, are there examples of integrable systems given by a Lax pair with a spectral parameter and for which there is a monodromy problem although the genus of the spectral curve is (larger than or) equal to the number of degrees of freedom?

Method of proof. A good test example for this method would be that of the Kowalevski top. In this case, we know from [8] that the eigenvector mapping[7] behaves as well as possible. However, there is a serious limitation: we have used here a pedestrian and computational way to prove our results and in particular the very simple explicit determination of the eigenvector divisor in Propositions 4 and 5. The Kovalevskaya case is much more complicated and one should look

- for a more clever proof,
- but also for a natural substitute of the algebraic Liouville form α of Section 1.3 on the relative Prym.

Notice that action coordinates are known in this case. See [13], the authors of which use the genus-2 Kovalevskaya curve. The point here would be to use an eigenvector mapping.

[6]See a Lax equation for the spherical pendulum in [21].
[7]Using, as in [8], the remarkable Lax equation of [10].

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