Lectures on gauge theory and integrable systems

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Abstract

In these notes, I will describe the moduli space of flat connections on a principal bundle over a surface and its Poisson structure. I will then give examples of integrable systems on these spaces, following ideas of Goldman, Jeffrey and Weitsman, Fock and Roslyi, and Alekseev.

I will present here some examples of integrable systems, all of them defined on the moduli space of flat connections on a trivial bundle over a surface. These examples have been constructed by Goldman, Jeffrey and Weitsman, Fock, Alekseev, so that there will be nothing new in these notes. However, it seems to me that a general presentation is lacking in the literature, and that this is a pity, as so many beautiful ideas are involved.

There are various reasons why one is willing to consider such things, and the motives really depend on the author. My two starting points will be:

1. It is interesting to understand the geometry of the moduli space. There are many ways to motivate a study of this geometry (some of them coming from physics).

2. In order to understand the geometry of a Poisson manifold, it is very helpful to have an integrable system on it.

To relate points 1 and 2, notice that the moduli space is a "Poisson manifold" (or, at least, is Poisson) and that this is a very important part of its geometric structure: for instance, the moduli spaces associated with a sphere with three holes and a torus with one hole will have the same underlying topological space but of course, one feels that these spaces should be different... and actually, they are distinguished by their Poisson structures.

To comment on point 2, let me just recall the ideal situation for an integrable system on a symplectic manifold, one which is associated with a Hamiltonian torus action: then, Delzant's theorem asserts that once you know the image of the set of functions, you know the symplectic manifold (and even more!). Of course, the general situation is more complicated.

Now the theory of integrable systems is something very interesting and rich, and very extensive as well. What seems to be attractive in the case of the moduli spaces is that these spaces are complicated enough to carry some very different systems, for instance:

- Systems related to torus actions (as in Delzant's theorem), the Jeffrey-Weitsman examples.

J. Hurtubise and F. Lalonde (eds.), Gauge Theory and Symplectic Geometry, 1–48.

- Systems looking like the famous Calogero system (arising, moreover, in the very simple case of a torus with one hole), the Fock examples.

These examples belong to the same family, the family of Goldman functions. These are functions on the moduli space associated with curves drawn on the surface. Goldman has given us a lot of commuting functions, and we manage to have enough of them. There is, however, another family of examples:

- Systems (or at least families of functions) looking like algebraically integrable systems obtained by r-matrix constructions, which can be expected to relate the geometry of the moduli space to that of a family of Jacobians of algebraic curves, as in classical examples of integrable systems coming from mechanics, the Alekseev examples.

Contents of the lectures. — For the convenience of the reader, and to set the scene, I will recall a few basic facts (definitions, constructions and examples) from the theory of integrable systems. I will also explain some classical examples, in connection with the Arnold-Liouville theorem. I will then describe the moduli space and its Poisson structure from two points of view (infinite dimensional, *à la* Atiyah-Bott, finite dimensional, *à la* Fock-Roslyi). The last part is devoted to the description of Goldman's functions and to the examples.

I will not insist on giving complete proofs, but will try to give enough hints and references.

Notice that there are many other relations between integrable systems and moduli spaces, many of them very interesting and beautiful (e.g. Hitchin's systems [25]) and which will not be discussed in these notes.

Contents

References. — Although I will recall most of the basic definitions, these notes will neither try to be self-contained nor exhaustive. However, there will hopefully be enough references to guide the reader (moreover, I have added a bibliographical guide at the end of the paper).

Language. — I must apologize if this text is in English: recall that the Séminaire de Mathématiques Supérieures is supposed to be bilingual. The only serious (?) manifestation of this bilingualism could be the French joke in Figure 8.

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1 Generalities on integrable systems

1.1 Symplectic and Poisson manifolds, integrable systems

I will recall quickly some basic and classical definitions (see the bibliographical guide).

Symplectic manifolds. — A *symplectic manifold* is a manifold W endowed with a non-degenerate closed 2-form ω. Any non-degenerate 2-form allows to associate, with any function f, a vector field X_f dual to the differential i.e. such that

$$\omega(X_f,\cdot) = df(\cdot).$$

This in turn allows us to define, by $\{f,g\} = X_f \cdot g$, a skew symmetric bracket $\{\ ,\ \}$ on $\mathcal{C}^\infty(W)$ which is a derivation in each entry (i.e. satisfies the Leibniz rule).

When the 2-form is closed, this bracket satisfies the Jacobi identity, so that it endows $\mathcal{C}^\infty(W)$ with the structure of a Lie algebra. It is then called a *Poisson bracket*.

Poisson manifolds. — A *Poisson manifold* W is a manifold endowed with a Poisson bracket $\{\ ,\ \}$, namely the structure of a Lie algebra on $\mathcal{C}^\infty(W)$ obeying the Leibniz rule

$$\{f,gh\} = \{f,g\}\,h + g\,\{f,h\}\,.$$

Notice that $\{f,\cdot\}$ is then a derivation and thus a vector field, called the *Hamiltonian vector field* of f, and denoted X_f.

Recall that the Poisson bracket can also be defined as a bivector P, that is, a section of $\Lambda^2 TW$:

$$\{f,g\}(x) = \langle P_x, df(x) \wedge dg(x)\rangle.$$

Symplectic manifolds are, of course, Poisson manifolds but there are many more Poisson manifolds than symplectic manifolds.

The basic example of a Poisson manifold is the dual $\mathfrak{g}^\star$ of a Lie algebra $\mathfrak{g}$. A Poisson structure is actually what the dual vector space $\mathfrak{g}^\star$ gets from the Lie bracket of $\mathfrak{g}$, by Kirillov's magic formula

$$\langle \xi, [X,Y]\rangle$$

and more precisely, for f, $g \in C^\infty(\mathfrak{g}^\star)$ (and $\xi \in \mathfrak{g}^\star$) by

$$\{f,g\}(\xi) = \langle \xi, [df(\xi), dg(\xi)]\rangle$$

where $\langle\ ,\ \rangle$ is the canonical pairing between $\mathfrak{g}^\star$ and $\mathfrak{g}$, $df(\xi)$ and $dg(\xi)$ are linear forms on $\mathfrak{g}^\star$, thus elements of the bidual, which is identified with $\mathfrak{g}$.

There is no reason why $\mathfrak{g}^\star$ should be a symplectic manifold (half of the time, its dimension does not even have the right parity).

The symplectic foliation. — In general the Poisson bracket of a Poisson manifold W defines nothing on a submanifold $V \subset W$. If it defines a Poisson bracket, that is, if $\{f,g\}|_V$ depends only on $f|_V$ and $g|_V$, then V is said to be a Poisson submanifold. If, moreover, the restricted Poisson bracket on V can be defined by a symplectic form, then V is a symplectic submanifold.

Now, any Poisson manifold has a *symplectic foliation*, a (singular) foliation whose leaves are the maximal symplectic submanifolds. I will not discuss this in great generality, but refer the reader to [31] or [39] for instance. However, in the case of the dual $\mathfrak{g}^\star$ of the Lie algebra of a group G, this is quite simple: the symplectic leaves are just the coadjoint orbits. This is an easy exercise, one simply checks that the magic formula $\langle \xi, [X,Y]\rangle$ can be used to define a non-degenerate 2-form on the orbit through ξ.

Poisson Lie groups. — An important class of Poisson manifolds is that of *Poisson Lie groups*, i.e. Lie groups with a Poisson structure such that the multiplication is a Poisson map. Although I will use Poisson Lie groups in what follows, I will introduce the (rather cumbersome) formalism only when I will need it (see § 1.4 and § 2.3).

Momentum mappings. — Let G be a Lie group acting on the Poisson manifold W. The action is *Hamiltonian* if there exists a *momentum mapping*, that is, a Poisson mapping

$$\mu : W \longrightarrow \mathfrak{g}^\star$$

such that, for all $X \in \mathfrak{g}$, the Hamiltonian vector field of the function $\mu_X : W \to \mathbf{R}$ defined by

$$\mu_X(w) = \langle \mu(w), X\rangle$$

is $\widetilde{X}$, the fundamental vector field associated with X. Recall that to require that μ preserve the Poisson structures is equivalent to asking that the map

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathcal{C}^\infty(W) \\ Y & \longmapsto & \mu_Y \end{array}$$

be a morphism of Lie algebras (see e.g. [8]).

Integrable systems. — On a $2n$ dimensional symplectic manifold W, an *integrable system* is a set of n functionally independent functions $f_1, \dots, f_n$ which pairwise commute, i.e. such that

$$\forall i, \quad \forall j \qquad \{f_i, f_j\} = 0.$$

That the functions are *functionally independent* means that the Hamiltonian vector fields $X_{f_1}, \dots, X_{f_n}$ are independent at the points of an open dense subset of W. At any such point x, notice that they generate an isotropic subspace of $T_x W$. Thus n is the maximum possible number of commuting independent functions on W.

Why is this called a "system"? One could be interested in the differential (Hamiltonian) system defined by one of the Hamiltonian vector fields (say X_{f_1}) and view the other functions as playing an auxiliary role: as $\{f_i, f_j\} = 0$, they will be constant along the trajectories of X_{f_1}—they are said to be first integrals of X_{f_1}.

Although it gives a symmetric role to all the f_i's, the definition I have given is certainly not the best possible: very often, one is only interested in the subalgebra of $\mathcal{C}^\infty(W)$ generated by the f_i's and not in a specific set of generators. This definition will become even worse when adapted to the case of a Poisson manifold: it is not quite clear how to require that there be as many commuting functions as possible. This can be formalised, but I will not do it here. In all the examples I will consider, we will deal with symplectic leaves and integrable systems (in the above sense) on them. For a formally better definition, see [41].

Let us consider once again the case of $\mathfrak{g}^\star$. It is quite easy to check that the Casimir functions (i.e. the functions f such that $X_f \equiv 0$ or $\{f, \cdot\} \equiv 0$) are the $\mathrm{Ad}^\star$-invariant functions. In the good cases (for instance if the Lie group is compact or the Lie algebra semi-simple, which are the cases we shall consider in these notes) there are enough invariant functions to describe the symplectic leaves, which thus appear as the connected components of the common level sets of the Casimir functions. Here the maximum number of commuting functions is $n + r$ where $r = \operatorname{rk} \mathfrak{g}$ is the number of independent Casimir functions and $2n + r = \dim \mathfrak{g}$, so that $2n$ is the dimension of the generic leaves.

Examples. — A lot of examples come from classical mechanics, such as the equations of motion of a spinning top or of other rigid bodies (see e.g. [10]), or the differential equation for the geodesics on the ellipsoid and their generalisations. Some come from the mechanics of particles with various potentials, such as the Toda lattice or the Calogero systems described below (§ 1.4 and § 3.4). A family of examples (Hamiltonian torus actions) will be described in § 1.2, some others (coming from mechanics) in § 1.3, a general construction (Lax equations given by r-matrices) will be given in § 1.4. The whole of § 3 will be devoted to examples.

Action-angle coordinates. — The Arnold-Liouville theorem (see [4]) describes the situation semi-locally, in the neighborhood of a regular common level set of the commuting functions f_i's.

Theorem 1.1.1 *Let $a = (a_1, \dots, a_n) \in \mathbf{R}^n$ be a regular value of the mapping $f = (f_1, \dots, f_n) : W \to \mathbf{R}^n$. Let $\mathcal{T}_a$ be the corresponding regular level, so that $\mathcal{T}_a$ is a Lagrangian submanifold.*

(a) *Let x be a point in $\mathcal{T}_a$. If the flows of the vector fields $X_{f_1}, \dots, X_{f_n}$ starting at x are complete, the connected component of x in $\mathcal{T}_a$ is a homogeneous space of $\mathbf{R}^n$. In particular, it has coordinates $(\varphi_1, \dots, \varphi_n)$ in which the vector field X_{f_i} can be written*

$$\sum_{j=1}^{n} A_j^i(a) \frac{\partial}{\partial \varphi_j}.$$

(b) *Assume the connected component of x in $\mathcal{T}_a$ is compact. Then it is a torus and there are coordinates $(\varphi_1, \dots, \varphi_n, I_1, \dots, I_n)$ in a neighbourhood of this component such that the symplectic form can be written as*

$$\omega = \sum dI_i \wedge d\varphi_i$$

and the vector field X_{f_i} as

$$\sum_{j=1}^{n} A_j^i(I) \frac{\partial}{\partial \varphi_j}.$$

Comments. — In other words, there are "linear" *angle* coordinates φ_i, for the flows of the f_i's and dual *action* coordinates I_i depending only on the f_j's.

If one insists on compactness, there is a more beautiful way to state this theorem, due to Duistermaat [17].

It is often thought that the Arnold-Liouville theorem states that the regular levels of an integrable system are tori. What the "tori" actually are is a very interesting question[1]. Some are indeed honest tori, for instance because the system is related to a torus action (see § 1.2), some are unions of tori, for instance because they are real parts of complex Abelian varieties (see § 1.3 and § 1.4), some are discs or cylinders, some are non-compact but have natural compactifications either as Abelian varieties or as toric varieties (see § 1.4 again) or as symmetric products of curves (see [40]), some even have nothing to do with any algebraic variety...

1.2 The momentum mapping of a torus action

One good reason to discuss this family of examples is that this is the paradigm of the Arnold-Liouville theorem, as everything (regular levels, action and angle coordinates...) is both easy and explicit from this point of view.

[1] For all these topological aspects, see [10].

Hamiltonian torus actions. — Let T be a torus acting on a symplectic $2n$-manifold (W,ω). Assume that the action is Hamiltonian with momentum mapping

$$\mu : W \xrightarrow{\mu} \mathfrak{t}^\star.$$

Recall that the Atiyah [5] and Guillemin-Sternberg [24] convexity theorem asserts that if W is compact and connected the image of μ is a convex polyhedron, the convex hull of the images of the fixed points of the torus action.

Recall moreover that the main argument in the proof uses the linearization of the torus action near the fixed points to show, as noticed by Frankel [21], that any component of the momentum mapping is a Morse (-Bott) function whose indices are all even, and that this allows us to prove that all non-empty levels of μ are connected.

Half the dimension. — Notice that all the vector fields $\widetilde{Y}$ (for $Y \in \mathfrak{t}$) commute: $\mathfrak{t}$ is abelian, so that the μ_Y generate an abelian subalgebra of $\mathcal{C}^\infty(W)$. Notice also that the subspace generated by the $\widetilde{Y}_w$ in T_wW is isotropic, so that its dimension is less than or equal to n.

Let us assume for simplicity that the torus action is effective. Then $\dim T \leq \frac{1}{2}\dim W = n$. Assume now that the torus T, acting effectively, has the maximal possible dimension, that is n. We have now an integrable system on W. If one insists on giving n functions: pick a basis $(Y_1,\ldots,Y_n)$ in $\mathfrak{t}$, so that $\mu_{Y_1},\ldots,\mu_{Y_n}$ are n commuting Hamiltonians. The fact that they are independent follows from the basic

Proposition 1.2.1 *The rank of the tangent map $T_w\mu$ is the dimension of the T-orbit through w.*

Proof The tangent map $T_w\mu : T_wW \to \mathfrak{t}$ is the transpose of the map $\varphi_w : \mathfrak{t} \to T_w^\star W$

$$Y \xmapsto{\varphi_w} \left(i_{\widetilde{Y}}\omega\right)_w$$

so that the image of $T_w\mu$ is the annihilator of $\operatorname{Ker}\varphi_w$. But, ω being non-degenerate,

$$\operatorname{Ker}\varphi_w = \left\{Y \in \mathfrak{t} \mid \widetilde{Y}_w = 0\right\}$$

and this is the Lie algebra of the stabilizer T_w of w. Thus

$$\operatorname{rk} T_w\mu = \dim \operatorname{Im} T_w\mu = \operatorname{codim} \operatorname{Ker}\varphi_w = \operatorname{codim} \mathfrak{t}_w = \dim(T\cdot w). \quad \square$$

As T is abelian and the action is effective, there is an open dense subset in W consisting of orbits with trivial stabilizers, so that in particular the interior of P is non-empty[2].

[2]Discrete stabilisers would be enough.

The Arnold-Liouville theorem. — Thus regular values of the momentum mapping correspond to principal torus orbits and we know quite precisely, using the connectivity theorem of Frankel and Proposition 1.2.1, what happens at critical values: all the levels are *tori*, whose dimensions are given by the rank of μ.

Moreover, the linearisation statement in this case ($\dim T = \frac{1}{2}\dim W$) shows that the dimension of the torus $\mu^{-1}(x)$ is precisely the dimension of the open face of the polytope in which x lies. In particular the points in the interior of the polytope are exactly the regular values of μ, so that the restriction of μ

$$\{\text{points in principal orbits}\} = \mu^{-1}\left(\overset{\circ}{P}\right) \xrightarrow{\mu} \overset{\circ}{P}$$

is a submersion and thus a trivial fibration, the fibers of which are Lagrangian tori. There are natural angle coordinates given by the torus action. The corresponding action coordinates are the corresponding components of the momentum mapping (see e.g. [15] for details).

Remarks.

1. The fact that all the points in the interior of the polytope are regular values is of course true only for a half dimensional torus.

2. The symplectic volume of the manifold W is just the volume of the image polytope (computed in the lattice defined by the torus).

Example. — Consider the complex projective n-space $\mathbf{CP}^n$ with its natural symplectic (Kähler) structure and look at the n functions given in homogeneous coordinates by

$$f_i([x_0, \dots, x_n]) = \frac{|x_i|^2}{\sum_{j=0}^{n} |x_j|^2} \quad (1 \le i \le n).$$

They are the component functions of the momentum mapping μ of the T^n-action

$$(t_1, \dots, t_n) \cdot [x_0, \dots, x_n] = [x_0, t_1 x_1, \dots, t_n x_n]$$

so that they form an integrable system. The image of μ is the unit simplex in $\mathbf{R}^n$ (Figure 1). The interior points are the regular values, the corresponding regular levels are n-tori, endowed with natural angle coordinates given by the torus action, and the functions $f_1, \dots, f_n$ are action coordinates.

Delzant's theorem. — This is the perfect example of why knowledge of some integrable system on a Poisson manifold may help in understanding the geometry of this manifold. In the case of an integrable system associated with a torus action on a symplectic manifold, the theorem asserts that the image of the momentum mapping (a polytope, in this case) contains complete information on the geometry, as it determines the manifold, its symplectic form and its T^n-action (see the precise statement in [15]).

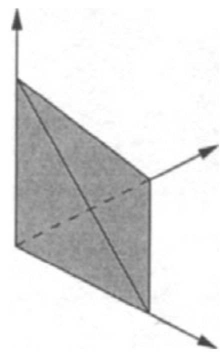

Figure 1: the projective space $\mathbf{CP}^3$

1.3 Examples from mechanics

The spinning top. — The next example is an integrable system on a Poisson manifold. Consider first the semi-direct product Lie algebra $\mathfrak{g} = \mathfrak{so}(3) \times \mathfrak{so}(3)$ consisting of pairs (X, Y) of skewsymmetric matrices with bracket

$$[(X,Y),(X',Y')] = ([X,X'],[X,Y'] - [X',Y]),$$

our Poisson manifold is the dual of $\mathfrak{g}$, identified with $\mathfrak{g}$. As a manifold, this is just $\mathbf{R}^3 \times \mathbf{R}^3$, the generic orbits are the

$$\mathcal{O}_{a,b} = \left\{(X,Y) \mid \|X\|^2 = a \text{ and } X \cdot Y = b\right\}$$

with $a > 0$ and $b \in \mathbf{R}$. Notice that $\mathcal{O}_{a,b}$ is diffeomorphic to the tangent bundle TS^2.

To define the commuting functions, we fix a vector $L \in \mathbf{R}^3$ and consider

$$\begin{cases} & & \|X\|^2 \\ & & X \cdot Y \\ H & = & Y^2 + X \cdot L \\ K & = & L \cdot X. \end{cases}$$

Notice that the first two functions are constant on the orbits so that they (Poisson) commute with everybody. The next claim is that H and K also commute, so that we will have an integrable system as soon as we know that H and K are independent. This is indeed the case. The image of the momentum mapping (in the case $a = 1$, $|b| < 2$) is shown in Figure 2.

The situation with respect to the Arnold-Liouville theorem is the following:

- The critical values are the points of the curve. The set of (actual) regular values is the component of the complement that contains the • point, with the exception of this point.

- As H is proper, the levels are compact and it turns out that they are connected, so that all regular levels are tori. The levels corresponding to smooth points of the boundary are circles. That corresponding to the double point consists of a single point.

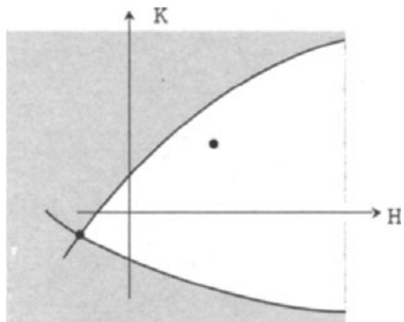

Figure 2

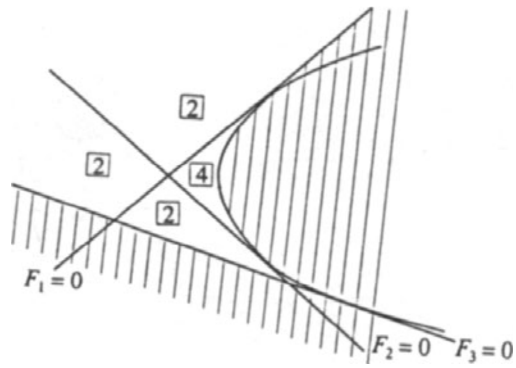

Figure 3

Notice that, up to now, this looks very much like the situation for a Hamiltonian torus action. However, there is a very big difference, as there is an isolated point in the interior of the image of the momentum mapping which is a critical value. The preimage of this point is a sphere with two points identified.

What we have got is a description of the orbit $\mathcal{O}_{a,b}$ as the total space of a singular foliation by tori.

This is still a very simple situation: for instance, all the regular levels are connected and have the same topological type. The system describes the motion of a (symmetric) spinning top[3]. Other examples in the same family (rigid bodies) would give more complicated topological pictures. The interested reader may look at [12] and [10]. These examples belong to a large family (see § 1.4). This is also the case for the next one.

Geodesics on quadrics. — It has been known since Jacobi that the motion of a free particle on a quadric in Euclidean space $\mathbf{R}^3$ is also an integrable system. The symplectic phase space in this example is again TS^2. The image of the momentum mapping is shown in Figure 3. The numbers on this picture give the numbers of Liouville tori in the preimage of a point in the given zone (see [9]).

1.4 r-matrices

Classical r-matrices. — According to Semenov-Tian-Shanski [37], a classical r-matrix is a linear self map $R : \mathfrak{g} \to \mathfrak{g}$ of a Lie algebra which satisfies a Yang-Baxter equation. Any linear map R allows us to define a new skew-symmetric bracket on $\mathfrak{g}$:

$$[X, Y]_R = [RX, Y] + [X, RY],$$

and it is said to be an r-matrix if the bracket $[\ ,\ ]_R$ satisfies the Jacobi identity (this is the Yang-Baxter equation). Then $(\mathfrak{g}, [\ ,\ ]_R)$ is a Lie algebra. Consequently, $\mathfrak{g}^\star$ is endowed with a second Poisson structure, denoted by $\{\ ,\ \}_R$.

[3]The vector L is the axis of revolution.

Integrable systems. — The first use of an r-matrix is to define integrable systems (see [37]).

Proposition 1.4.1 (a) *If φ is an invariant function on $\mathfrak{g}^\star$,*

$$X_\varphi^R(\xi) = \operatorname{ad}^\star_{Rd\varphi(\xi)} \cdot \xi.$$

(b) *If both φ and ψ are invariant, then*

$$\{\varphi, \psi\} = \{\varphi, \psi\}_R = 0.$$

Of course, X_φ^R denotes the Hamiltonian vector field associated with φ by the Poisson bracket $\{\ ,\ \}_R$, so that Part (a) of the proposition describes a Hamiltonian system, while Part (b) gives a list of commuting first integrals for this system.

If there exists on $\mathfrak{g}$ an invariant non-degenerate symmetric bilinear form, it is possible to identify $\mathfrak{g}$ with $\mathfrak{g}^\star$ and ad with $\operatorname{ad}^\star$, so that the Hamiltonian system associated with the invariant function φ becomes

$$\frac{d}{dt}A = [A, R\nabla_A \varphi].$$

Such an equation is called a Lax equation.

Proposition 1.4.1 gives a lot of commuting functions. To have an integrable system, we also need to be on a symplectic manifold of the right dimension—small enough, in fact, as the number of independent invariant functions is limited by the rank of the Lie algebra used here (we shall meet a very analogous situation in § 3). This is illustrated by the next example.

The (non-periodic) Toda lattice. — Recall now the most classical application of Proposition 1.4.1. The Lie algebra $\mathfrak{g}$ is simply $\mathfrak{sl}(n; \mathbf{C})$. The r-matrix is associated with the decomposition

$$\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$$

of $\mathfrak{g}$ as the sum of the Lie subalgebras of lower triangular and skew symmetric matrices (respectively) in the following way: if $X \in \mathfrak{sl}(n; \mathbf{C})$, write $X = X_+ + X_-$ with $X_+ \in \mathfrak{a}$, $X_- \in \mathfrak{b}$, and put[4]

$$R(X) = \frac{1}{2}(X_+ - X_-).$$

Use the Killing form $\langle X, Y \rangle = \operatorname{tr}(XY)$ to identify $\mathfrak{g}$ and $\mathfrak{g}^\star$—and also $\mathfrak{a}^\star$ and $\mathfrak{b}^\perp$. Notice that $\mathfrak{b}^\perp$ is the space of all symmetric matrices.

Consider now the invariant function on $\mathfrak{g}$ given by $f(L) = \frac{1}{2}\operatorname{tr} L^2$ and the Hamiltonian system it defines on $\mathfrak{a}^\star = \mathfrak{b}^\perp$. Of course, $df_L(X) = \operatorname{tr}(LX)$ so that $\nabla_L f = L$. As a consequence of Proposition 1.4.1 (exercise), the system

$$\dot{L} = [L, L_-]$$

(recall that $_-$ denotes the projection onto the subspace of skew symmetric matrices) is a Hamiltonian system on the space $\mathfrak{b}^\perp$ of symmetric matrices, which admits all the functions $L \mapsto \operatorname{tr}(L^k)$ $(2 \le k \le n)$ as pairwise commuting first integrals.

[4]Any decomposition of a Lie algebra as the (vector space) sum of two Lie subalgebras would give, in the same way, an r-matrix satisfying the Yang-Baxter equation.

What we have got, up to now, is a set of $n-1$ commuting Hamiltonians. We need now to look for a symplectic $2(n-1)$ submanifold Γ in $\mathfrak{b}^{\perp}$. Consider the vector space Γ of all tridiagonal traceless symmetric matrices

$$L = \begin{pmatrix} b_1 & a_1 & & & & \\ a_1 & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & a_{n-1} \\ & & & & a_{n-1} & b_n \end{pmatrix}$$

and recall that the Poisson structure we are using on the space of symmetric matrices is given by the isomorphism of $\mathfrak{b}^{\perp}$ with $\mathfrak{a}^{\star}$, so that the coadjoint action of the Lie group A of invertible lower triangular matrices is given, on $\mathfrak{b}^{\perp}$, by

$$a \cdot L = \text{ projection onto } \mathfrak{b}^{\perp} \text{ of } aLa^{-1}.$$

It is easily checked that Γ is a union of orbits, so that our system can be considered as a system on Γ. Moreover, the open subset Γ_0 of tridiagonal matrices whose non-diagonal entries never vanish, is an orbit. As it is obviously $2(n-1)$ dimensional, we are done. To summarise, the differential system

$$\frac{d}{dt}\begin{pmatrix} b_1 & a_1 & & \\ a_1 & \ddots & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ & & a_{n-1} & b_n \end{pmatrix} =$$

$$= \left[\begin{pmatrix} b_1 & a_1 & & \\ a_1 & \ddots & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ & & a_{n-1} & b_n \end{pmatrix}, \begin{pmatrix} 0 & a_1 & & \\ -a_1 & \ddots & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ & & -a_{n-1} & 0 \end{pmatrix}\right]$$

is completely integrable, with all the functions $\operatorname{tr}(L^k)$ as first integrals.

Notice that this is an example where the symplectic leaves and the levels are non-compact.

Infinite dimensional r-matrices and algebraic curves. — Although we have used duality and biduality, an important range of applications of the r-matrix approach is the case of the infinite dimensional Lie algebra $\mathfrak{g}[\lambda, \lambda^{-1}]$ where $\mathfrak{g}$ is a Lie algebra of matrices, λ is an independent parameter and R is the matrix associated as above with the decomposition

$$\mathfrak{g}[\lambda, \lambda^{-1}] = \mathfrak{g}[\lambda] + \lambda^{-1}\mathfrak{g}[\lambda^{-1}].$$

This gives Lax equations with a *spectral parameter*: the matrices have entries which are polynomials in λ, λ^{-1}. Invariant functions on $\mathfrak{g}$ give a lot of commuting functions, as they become polynomials in λ all of whose coefficients are first integrals. But they give more: if a matrix $A_\lambda(t)$ satisfies a differential equation of the form

$$\frac{d}{dt}A_\lambda(t) = [A_\lambda(t), B_\lambda(t)],$$

its eigenvalues do not depend on time and its characteristic polynomial defines an algebraic curve

$$\{(\lambda, \mu) \mid \det(A_\lambda - \mu \,\mathrm{Id}) = 0\}.$$

All the coefficients of this polynomial are constant (do not depend on t). The curve and its Jacobian can be used to linearise the flows of the various Hamiltonians, to write the solutions of the differential equations and to describe the common level sets of the first integrals. All the integrable systems of classical mechanics that can be integrated with the help of elliptic functions or more generally Abelian integrals (e.g. rigid bodies, geodesics of quadrics... mentioned in § 1.3) can be described in this way (see e.g. [35], [10], [9]).

r-matrices and Poisson Lie groups. — Suppose we are given an element

$$r \in \mathfrak{g} \otimes \mathfrak{g} = \mathrm{Hom}(\mathfrak{g}^\star, \mathfrak{g}).$$

It allows us to define a skew-symmetric bracket on $\mathfrak{g}^\star$ by

$$[\xi, \eta]^r = \mathrm{ad}^\star_{r\xi}\, \eta - \mathrm{ad}^\star_{r\eta}\, \xi.$$

Assume now that the symmetric part $\mathcal{B}$ of r in the decomposition

$$\begin{array}{ccccc} \mathfrak{g} \otimes \mathfrak{g} & = & S^2\mathfrak{g} & \oplus & \Lambda^2\mathfrak{g} \\ r & = & \mathcal{B} & + & a \end{array}$$

is non-degenerate and invariant, so that r also defines a linear map

$$R : \mathfrak{g} \xrightarrow{\mathcal{B}^{-1}} \mathfrak{g}^\star \xrightarrow{a} \mathfrak{g}$$

and thus, as above, a skew symmetric bracket $[\ ,\ ]_R$ on $\mathfrak{g}$. One easily checks that:

Proposition 1.4.2 *The bracket $[\ ,\ ]_R$ is a Lie algebra structure on $\mathfrak{g}$ if and only if the bracket $[\ ,\ ]^r$ is a Lie algebra structure on $\mathfrak{g}^\star$.*

Now a Lie algebra structure $[\ ,\ ]^r$ on $\mathfrak{g}^\star$ defines automatically a Poisson structure $\{\ ,\ \}^r$ on $\mathfrak{g}$ (using Kirillov and biduality) and this can be integrated, using Lie's third theorem, to give a Poisson Lie group structure on the group G itself (or, at least, on its universal covering):

Proposition 1.4.3 *There exists a unique connected and simply connected Poisson Lie group G whose Lie algebra is $\mathfrak{g}$ endowed with the Poisson structure $\{\ ,\ \}^r$.*

Example. — Consider the simple Lie algebra $\mathfrak{g} = \mathfrak{sl}(n; \mathbf{C})$ of traceless $n \times n$ complex matrices with its invariant symmetric bilinear form ($\pm$ Killing form) $\mathcal{B}(X, Y) = \operatorname{tr}(XY)$. Write it as the sum

$$\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_0 + \mathfrak{g}_-$$

of upper triangular, diagonal, lower triangular matrices, respectively. Call P_+, P_- the projections onto $\mathfrak{g}_+$ and $\mathfrak{g}_-$ respectively. The linear maps

$$R = \frac{1}{2}(P_+ - P_-) : \mathfrak{g} \longrightarrow \mathfrak{g}$$

or

$$r = \mathcal{B} + R\mathcal{B} : \mathfrak{g}^\star \longrightarrow \mathfrak{g}$$

give the group $SL(n; \mathbf{C})$ the structure of a Poisson Lie group. Notice that in the standard $\mathcal{B}$-orthonormal basis H_i, E_α,

$$r = \frac{1}{2} \sum_i H_i \otimes H_i + \sum_{\alpha > 0} E_\alpha \otimes E_{-\alpha}.$$

2 A Poisson structure on the moduli space of flat connections

2.1 A few generalities on the moduli space

The moduli space that I want to consider is that of flat connections on the trivial principal G-bundle on a surface S.

The surface. — The surface S is supposed to be compact, connected and oriented, but may (and will usually) have a boundary ∂S. The genus of S will be denoted by g and the number of boundary components by d.

The group. — The group G is a Lie group, real or complex. I will use (and thus assume the existence of) a non-degenerate symmetric bilinear form $\mathcal{B}$ on the Lie algebra $\mathfrak{g}$ which is invariant under the adjoint G-action. It certainly exists when G is a simple Lie group (as $SL(n; \mathbf{C})$), or is compact. The examples I have in mind are $SL(n; \mathbf{C})$ and $SU(n)$, so that I will usually assume in addition that G is simply connected—although I will use $U(1)$ to check the results obtained for more complicated Lie groups.

The bundle. — The principal bundle I will use is the *trivial* bundle $\mathrm{G} \times S$. Notice that this is not a very big restriction: when G is simply connected (e.g. $SU(n)$, $SL(n; \mathbf{C})$), all principal G-bundles on S are trivialisable[5]. However, to simplify notations, I will assume that a trivialisation is given (and fixed forever).

[5]They are classified by the homotopy classes of maps $S \to B\mathrm{G}$. If $\pi_1(\mathrm{G})$ is trivial, then because π_2 of a group is always trivial (this is a theorem of Cartan [14]), G is 2-connected and $B\mathrm{G}$ is 3-connected. For a *surface* S, any continuous map $S \to B\mathrm{G}$ is thus homotopic to a constant map and any principal G-bundle is trivialisable.

Connections. — Once the trivialisation has been fixed, there is a *trivial connection*, the one whose horizontal space at (g, x) is

$$\{0\} \times T_x S \subset T_g \mathrm{G} \times T_x S = T_{(g,x)} (\mathrm{G} \times S).$$

For any other connection, the horizontal space $\mathcal{H}_{(g,x)}$ is the graph of a linear map $T_x S \to T_g \mathrm{G}$. The invariance property leads us to a linear map $T_x S \to \mathfrak{g}$ (the Lie algebra of G), so that the space $\mathcal{A}$ of all connections can be identified with $\Omega^1(S, \mathfrak{g})$.

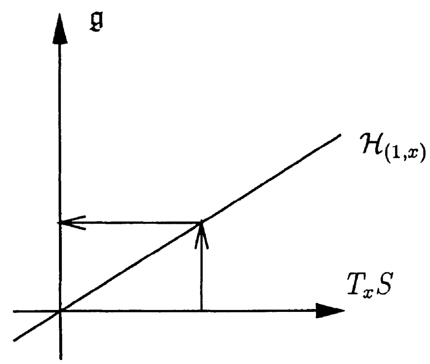

Figure 4: a connection

Remark. — Notice that $\mathcal{A}$ appears here as a vector space, simply because we have fixed an origin (the trivial connection)—otherwise, $\mathcal{A}$ is just an affine space modelled on $\Omega^1(S, \mathfrak{g})$.

Connections as derivations and the curvature. — For $A \in \mathcal{A}$, define

$$\begin{array}{rcl} d_A : \Omega^0(S, \mathfrak{g}) & \longrightarrow & \Omega^1(S, \mathfrak{g}) \\ \alpha & \longmapsto & d\alpha + [A, \alpha] \end{array}$$

where $[A, \alpha]$ is the 1-form, defined on the tangent vectors to S by

$$X \longmapsto [A(X), \alpha],$$

and similarly

$$\begin{array}{rcl} d_A : \Omega^1(S, \mathfrak{g}) & \longrightarrow & \Omega^2(S, \mathfrak{g}) \\ \varphi & \longmapsto & d\varphi + [A, \varphi] \end{array}$$

where $[A, \varphi]$ is the 2-form, defined on the pairs of tangent vectors to S by

$$[A, \varphi](X, Y) = [A(X), \varphi(Y)] - [A(Y), \varphi(X)].$$

Thus d_A is a derivation for the bracket of forms:

$$d_A[\varphi, \psi] = [d_A\varphi, \psi] \pm [\varphi, d_A\psi].$$

Now, for $\alpha \in \Omega^0(S, \mathfrak{g})$,

$$\begin{aligned} d_A \circ d_A(\alpha) &= d_A\left(d\alpha + [A, \alpha]\right) \\ &= d_A(d\alpha) + d_A[A, \alpha] \\ &= [A, d\alpha] + [d_A A, \alpha] - [A, d_A \alpha] \\ &= [d_A A, \alpha] \end{aligned}$$

so that

$$d_A \circ d_A : \Omega^0(S, \mathfrak{g}) \longrightarrow \Omega^2(S, \mathfrak{g})$$

is just the action of the 2-form $F(A) = d_A A$ with values in $\mathfrak{g}$, the *curvature form.*

The gauge group. — This is the group $\mathcal{G}$ of automorphisms of the bundle. Still using the trivialisation, we can identify it with the (functional) group of all the mappings $S \to \mathrm{G}$. Similarly, its Lie algebra $\mathbf{g}$ is that of all mappings $S \to \mathfrak{g}$.

It acts on $\mathcal{A}$, (traditionally on the right) by

$$g \cdot A = g^{-1} A g + g^{-1} dg$$

where "$g^{-1}dg$" is a symbolic notation for the 1-form on S with values in $\mathfrak{g}$ defined by

$$T_x S \ni X \longmapsto T_x g(X) \in T_{g(x)}\, \mathrm{G} \xrightarrow{\text{“}g^{-1}\text{”}} \mathfrak{g}.$$

One needs only to differentiate to get the fundamental vector field $\widetilde{\alpha}$ associated to both the element $\alpha \in \mathbf{g}$ and the $\mathcal{G}$-action. This is none other than the vector field, defined on $\mathcal{A}$ by

$$\widetilde{\alpha} = d_A \alpha \in \Omega^1(S, \mathfrak{g}) = T_A \mathcal{A}.$$

The moduli space. — The space $\mathcal{M}$ we want to consider is the quotient of the space $\mathcal{A}_{fl}$ of *flat* (i.e. with zero curvature) connections by the gauge group action:

$$\mathcal{M} = \mathcal{A}_{fl} / \mathcal{G}.$$

Example. — Assume $\mathrm{G} = U(1)$, so that $\mathfrak{g} = \mathbf{R}$, the Lie bracket is trivial and d_A is just the ordinary exterior derivative d, the curvature is dA (remember that the bundle is trivial, so that the curvature of any connection must be an exact form on S). Flat connections are closed 1-forms, the gauge group acts by translations

$$g \cdot A = A + g^{-1} dg = A + g^{\star} \sigma$$

where σ is the form dz/z on $S^1 = U(1)$. Thus $\mathcal{M}$ is the quotient of the space of closed 1-forms by the equivalence relation defined by the above translations, namely $\mathcal{M} = H^1(S; U(1))$.

The space. — From the set-theoretical or topological viewpoints, the space is quite easy to describe (even if this does not mean that it is very simple). Assign to each flat connection its holonomy[6] along a loop, and this will give you a homomorphism

$$\pi_1(S) \longrightarrow \mathrm{G}$$

which is well defined up to conjugacy by elements of G.

There is thus a one-to-one correspondence between the moduli space $\mathcal{M}$ and the quotient $\mathrm{Hom}(\pi_1 S, \mathrm{G})/\operatorname{Ad}\mathrm{G}$. This even describes $\mathcal{M}$ as a topological space, at least if the group G is compact. However, I really want to use non-compact groups (namely $SL(n; \mathbf{C})$), but this will not cause any trouble: I will use mainly functions on the moduli space, and they will appear, of course, as functions defined "upstairs", on the space $\mathrm{Hom}(\pi_1 S, \mathrm{G})$—which are invariant under conjugation. I will not really use the quotient!

Stricly speaking, to use $\pi_1(S)$ we need to have chosen a base point. However, the space $\mathrm{Hom}(\pi_1 S, \mathrm{G})/\operatorname{Ad}\mathrm{G}$ is independent of this choice, because we factored out by conjugation.

Singularities. — The space $\mathrm{Hom}(\pi_1 S, \mathrm{G})$ is itself often singular if the surface S is closed ($d = 0$): it is the hypersurface of G^{2g} defined by the equation that arises from the commutator relation in a presentation of $\pi_1(S)$. Even when $d \geq 1$ and $\mathrm{Hom}(\pi_1 S, \mathrm{G})$ is smooth, the G-action is not locally free, so that $\mathcal{M}$ is almost always singular (we will not care whether it is Hausdorff or not: as I have just said, we will deal only with *functions* on the moduli space).

Regular points correspond to representations ρ whose centraliser in G has the same dimension as the center of G. In the case of $U(1)$ or $SU(n)$, these correspond to irreducible representations. In the case of $SL(2; \mathbf{C})$, singular points come from morphisms $\rho : \pi_1 S \to SL(2; \mathbf{C})$ whose image is contained in an Abelian subgroup. This is explained in [22]. I will ignore singularities in what follows and refer the interested reader to Goldman's paper.

In the case of a simple group G, there are points at which the action is principal. The dimension of the moduli space is then $(2g + d - 1)\dim \mathrm{G} - \dim \mathrm{G} = (2g - 2 + d)\dim \mathrm{G}$.

Remarks. — First, if ∂S is not empty, $\pi_1(S)$ is a free group on $2g + d - 1$ generators (g is the genus of S, $d \geq 1$ is the number of components of the boundary) and $\mathrm{Hom}(\pi_1 S, \mathrm{G})$ is the product of $2g + d - 1$ copies of G.

This shows, among other things, that different surfaces may give the same topological space: it depends only on $2g + d - 1$. For instance the sphere with three holes ($g = 0$, $d = 3$) and the torus with one hole ($g = 1$, $d = 1$) shown in Figure 5 both give $\mathrm{G} \times \mathrm{G} / \operatorname{Ad}\mathrm{G}$ as moduli spaces. We shall see, however, that these two spaces are endowed with natural Poisson structures which are quite different.

Throughout these notes, I shall use these simple examples of spaces, but also the simple example of the group $U(1)$ already mentioned to illustrate the constructions. Let us now come back to a general genus g surface with d holes and suppose $\mathrm{G} = U(1)$. The commutativity of G leads to two important simplifications: firstly, the $\operatorname{Ad}\mathrm{G}$ action is trivial, so that the moduli space is $\mathrm{Hom}(\pi_1 S, U(1))$. Secondly, as $\pi_1 S$ has a presentation

$$\pi_1 S = \left\langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \mu_1, \ldots, \mu_d; \prod_{j=1}^{d} \mu_j \prod_{i=1}^{g} (\alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1}) \right\rangle$$

[6] See e.g. [16].

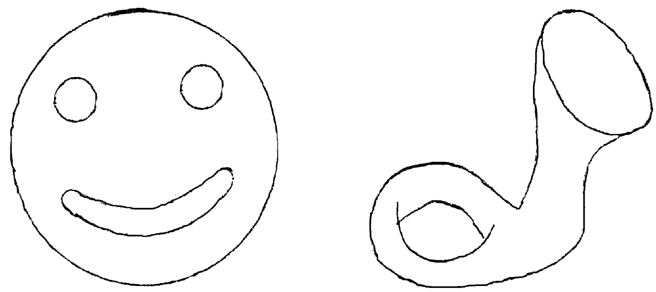

Figure 5

with a single relation, involving commutators,

$$\operatorname{Hom}(\pi_1 S, U(1)) = \begin{cases} U(1)^{2g} & \text{if } d = 0 \\ U(1)^{2g} \times U(1)^{d-1} & \text{if } d \geq 1. \end{cases}$$

This is of course the same thing as above, where we had found that the moduli space $\mathcal{M}$ was in this case $H^1(S; U(1))$, except that here we have used a basis. Notice also that this $\mathcal{M}$ can be considered as a topological version of the Jacobian.

I will describe $\mathcal{M}$ in the $SU(2)$ case in § 3.3.

A summary of notations. — The Lie group is G, its Lie algebra is $\mathfrak{g}$, the latter is endowed with an invariant non-degenerate symmetric bilinear form, that is, a linear map

$$\mathcal{B} : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbf{C}$$

(or $\mathbf{R}$ if G is a real Lie group). The gauge group is $\mathcal{G}$, its Lie algebra $\mathbf{g}$. If $\alpha \in \mathbf{g} = \Omega^1(S; \mathfrak{g})$, $\widetilde{\alpha}$ denotes the associated fundamental vector field on $\mathcal{A}$. The moduli space is $\mathcal{M}$ and also $\mathcal{A}_{fl}/\mathcal{G}$, but this $\mathcal{M}$ is only a shorthand for $\mathcal{M}_S(\mathrm{G})$, or $\mathcal{M}_{g,d}(\mathrm{G})$ (if g is the genus of S and d the number of components of the boundary ∂S), or $\mathcal{M}_{g,d}$ if the group is clear.

2.2 Construction of the Poisson structure (à la Atiyah-Bott)

The symplectic form on $\mathcal{A}$. — Following Atiyah and Bott, one defines a skew-symmetric bilinear form on the vector space $\mathcal{A}$, that can be written, as a differential form, as:

$$\omega_A(\varphi, \psi) = \int_S \mathcal{B}_\star(\varphi \wedge \psi).$$

Notice that ω is "constant" (this is just a bilinear form, ω_A does not depend on A) so that it is *closed*: this is a good reason to use the infinite dimensional description here; in the usual[7] finite dimensional descriptions, it is rather hard to prove that the forms obtained are closed (see e.g. [23] and [30].)

[7] See however the Fock and Roslyi description in § 2.3.

In the above formula, one considers the product of forms

$$\wedge : \Omega^\star(S,\mathfrak{g}) \otimes \Omega^\star(S,\mathfrak{g}) \longrightarrow \Omega^\star(S,\mathfrak{g}\otimes\mathfrak{g})$$

and $\mathcal{B}_\star$ as a mapping

$$\mathcal{B}_\star : \Omega^\star(S,\mathfrak{g}\otimes\mathfrak{g}) \longrightarrow \Omega^\star(S,\mathbf{C}).$$

It is obvious that ω is non-degenerate (being defined by two non-degenerate objects: the form $\mathcal{B}$ and the pairing of 1-forms on the surface). Now, $\mathcal{A}$ is a symplectic (vector or affine) space.

The purpose of this section is to understand what structure the form ω defines on $\mathcal{M}$. We will prove:

Theorem 2.2.1 *The symplectic form ω on $\mathcal{A}$ defines a Poisson structure on the moduli space $\mathcal{M}$, the symplectic leaves of which are obtained by fixing the conjugacy classes of the holonomies along the components of the boundary of S.*

Symplectic properties of the gauge group action. — The gauge group action preserves the symplectic form (because it is defined by the invariant form $\mathcal{B}$):

$$\begin{aligned}
(g^\star\omega)_A(\varphi,\psi) &= \omega_{g\cdot A}\left(T_A g(\varphi), T_A g(\psi)\right)\\
&= \omega_A\left(g^{-1}\varphi g, g^{-1}\psi g\right) \text{ as the } \mathcal{G}\text{-action on } \mathcal{A} \text{ is affine}\\
&= \textstyle\int_S \mathcal{B}_\star((g^{-1}\varphi g)\wedge(g^{-1}\psi g))\\
&= \omega_A(\varphi,\psi).
\end{aligned}$$

What we would like to do now is to write that the action is Hamiltonian and to describe its momentum mapping:

HOPE. — *To find, for any $\alpha \in \mathfrak{g}$, a function H_α such that the Hamiltonian vector field (the symplectic gradient of H_α) is the fundamental vector field $\tilde{\alpha}$ and such that the mapping*

$$\begin{aligned}
\mathfrak{g} &\longrightarrow C^\infty(\mathcal{A})\\
\alpha &\longmapsto H_\alpha
\end{aligned}$$

is a morphism of Lie algebras (the Lie algebra structure on $C^\infty(\mathcal{A})$ is that defined by the Poisson bracket associated with ω).

Atiyah and Bott showed in [7] that, if $\partial S = \varnothing$, a solution to this problem is given by

$$H_\alpha(A) = \int_S \mathcal{B}_\star(\alpha \wedge F(A))$$

(in other words, by the curvature, which is a momentum mapping for the gauge group action). But here, some difficulties will arise from the boundary of S:

Proposition 2.2.2 *For $\alpha \in \mathfrak{g} = \Omega^0(S, \mathfrak{g})$ and $A \in \mathcal{A}$, let*

$$H_\alpha(A) = \int_S \mathcal{B}_\star(\alpha \wedge F(A)) + \int_{\partial S} \mathcal{B}_\star(\alpha \wedge A).$$

The Hamiltonian vector field of H_α is the fundamental vector field $\widetilde{\alpha}$, but

$$\{H_\alpha, H_\beta\}(A) = H_{[\alpha,\beta]}(A) + \int_{\partial S} \mathcal{B}_\star(\alpha \wedge d\beta).$$

Proof Our candidate H_α should satisfy

$$\begin{aligned}
(T_A H_\alpha)(\varphi) &= \omega_A(\widetilde{\alpha}_A, \varphi) \\
&= \int_S \mathcal{B}_\star(\widetilde{\alpha}(A) \wedge \varphi) \\
&= \int_S \mathcal{B}_\star(d_A\alpha \wedge \varphi) \\
&= \int_S \mathcal{B}_\star d_A(\alpha \wedge \varphi) + \int_S \mathcal{B}_\star(\alpha \wedge d_A\varphi) \\
&= \int_{\partial S} \mathcal{B}_\star(\alpha \wedge \varphi) + \int_S \mathcal{B}_\star(\alpha \wedge d_A\varphi).
\end{aligned}$$

Thus we only need to check that

$$(T_A F)(\varphi) = d_A\varphi,$$

which I leave to the reader as an exercise (this is both classical and solved in [7]). Now

$$H_{[\alpha,\beta]}(A) = \int_S \mathcal{B}_\star([\alpha,\beta] \wedge F(A)) + \int_{\partial S} \mathcal{B}_\star([\alpha,\beta] \wedge A)$$

so that the result is a consequence of the invariance of $\mathcal{B}$. □

The central extension. — The formula in the previous proposition shows that the mapping $\alpha \mapsto H_\alpha$ is not exactly a morphism of Lie algebras. If α, $\beta \in \mathfrak{g}$, put

$$c(\alpha, \beta) = \int_{\partial S} \mathcal{B}_\star(\alpha \wedge d\beta).$$

This is a skew symmetric bilinear form on $\mathfrak{g}$, which has the additional property of being a *cocycle*:

$$c([\alpha,\beta],\gamma) + c([\beta,\gamma],\alpha) + c([\gamma,\alpha],\beta) = 0,$$

which is precisely what is needed in order that $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbf{C}$, endowed with the bracket

$$[(\alpha,t),(\beta,u)] = ([\alpha,\beta], c(\alpha,\beta))$$

become a Lie algebra, a central extension of $\mathfrak{g}$. Defining $H_{(\alpha,t)}(A) = H_\alpha(A) + t$, one thus gets:

Corollary 2.2.3 *The mapping*

$$\begin{array}{rcl} \widehat{\mathfrak{g}} & \longrightarrow & C^\infty(\mathcal{A}) \\ (\alpha, t) & \longmapsto & H_{(\alpha,t)} \end{array}$$

is a morphism of Lie algebras. Moreover, the Hamiltonian vector field of the function $H_{(\alpha,t)} : \mathcal{A} \to \mathbf{C}$ *is the vector field* $\widetilde{\alpha}$. □

The last assertion tells us that $\widehat{\mathfrak{g}}$ acts infinitesimally on $\mathcal{A}$ via the infinitesimal $\mathfrak{g}$-action we already know. This remark can be integrated to define a central extension $\widehat{\mathcal{G}}$ of the gauge group itself, acting on $\mathcal{A}$ via the gauge action of $\mathcal{G}$.

What I want to describe now is the momentum mapping for these actions. I must first say something about the dual of the Lie algebra $\widehat{\mathfrak{g}}$.

Proposition 2.2.4 *The pairing of* $\Omega^2(S, \mathfrak{g}) \oplus \Omega^1(\partial S, \mathfrak{g}) \oplus \mathbf{C}$ *with* $\widehat{\mathfrak{g}}$ *by*

$$(R, \varphi, z) \otimes (\alpha, t) \longmapsto \int_S \mathcal{B}_\star(\alpha \wedge R) + \int_{\partial S} \mathcal{B}_\star(\alpha \wedge \varphi) + zt$$

is non-degenerate. □

In this way, $\Omega^2(S, \mathfrak{g}) \oplus \Omega^1(\partial S, \mathfrak{g}) \oplus \mathbf{C}$ can be seen as a subspace of $\widehat{\mathfrak{g}}^\star$ and the previous study can be summarised in:

Corollary 2.2.5 *The mapping*

$$\begin{array}{rcl} \mu : \mathcal{A} & \longrightarrow & \widehat{\mathfrak{g}}^\star \\ A & \longmapsto & (F(A), A\big|_{\partial S}, 1) \end{array}$$

is a $\mathcal{G}$*-equivariant momentum mapping for the gauge group action.* □

Remark. — Notice that the central part of $\widehat{\mathcal{G}}$ does not act on $\widehat{\mathfrak{g}}^\star$, so the statement is coherent.

When the surface S is closed, the mapping μ is just the curvature, so that the space of flat connections modulo the gauge group action is a reduced level set of a momentum mapping, and thus a symplectic manifold[8], as was explained in [7].

In the general case where the surface has boundary components, the curvature is the momentum mapping for the action of a smaller group $\mathcal{G}_0$.

Proposition 2.2.6 *Let* $\mathcal{G}_0 = \{g \in \mathcal{G} \mid g|_{\partial S} = \mathrm{Id}\}$. *This is a normal subgroup of* $\mathcal{G}$. *It acts on* $\mathcal{A}$ *with momentum mapping* $A \mapsto F(A)$. *The space* $\mathcal{M}_0$ *of flat connections modulo the* $\mathcal{G}_0$*-action is symplectic.*

[8]As long as it is a manifold, of course.

Proof The Lie algebra of $\mathcal{G}_0$ is

$$\mathfrak{g}_0 = \{\alpha \in \mathfrak{g} \mid \alpha|_{\partial S} = 0\}$$

so that one can embed

$$\Omega^2(S, \mathfrak{g}) \oplus \mathbf{C} \hookrightarrow \widehat{\mathfrak{g}_0}^\star$$

and conclude that the momentum mapping for the $\mathcal{G}_0$-action is the composed map $p \circ \mu$:

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{\mu} & \widehat{\mathfrak{g}}^\star & \xrightarrow{p} & \widehat{\mathfrak{g}_0}^\star \\ A & \longmapsto & (F(A), A|_{\partial S}, 1) & \longmapsto & (F(A), 1)\,. \end{array}$$

Hence $\mathcal{M}_0 = (p \circ \mu)^{-1}(0,1)/\mathcal{G}_0$ is a symplectic reduced space. □

The Poisson structure and its symplectic foliation. — As $\mathcal{G}_0$ is normal in $\mathcal{G}$, the gauge group $\mathcal{G}$ also acts on $\mathcal{M}_0$, the action preserves the symplectic structure, and the quotient $\mathcal{M}_0/\mathcal{G}$ is, of course, our moduli space $\mathcal{M}$. Being the quotient of a symplectic space by a symplectic group action, it inherits a natural (i.e. coming from the form ω on $\mathcal{A}$) Poisson structure. The next aim is to describe its symplectic foliation. Recall the classical proposition:

Proposition 2.2.7 *Let H be a Lie group acting on a symplectic manifold W with momentum mapping*

$$\mu : W \to \mathfrak{h}^\star.$$

The Poisson bracket of W defines a Poisson bracket on W/H whose symplectic leaves are in one-to-one correspondence (via μ) with the coadjoint orbits of H in the image $\mu(W) \subset \mathfrak{h}^\star$.

Proof Let $\xi \in \mathfrak{h}^\star$ and $\mathcal{O}_\xi$ be its orbit. Consider H acting on $W \times \mathcal{O}_\xi$ by the "diagonal" map so that

$$\begin{array}{rcl} J : W \times \mathcal{O}_\xi & \longrightarrow & \mathfrak{h}^\star \\ (x, \eta) & \longmapsto & \mu(x) - \eta \end{array}$$

is the momentum mapping of the action. Perform the symplectic reduction of $W \times \mathcal{O}_\xi$ at the level 0 of J:

$$J^{-1}(0)/H = \{(x, \eta) \in W \times \mathcal{O}_\xi \mid \mu(x) = \eta\}/H \to \mu^{-1}(\mathcal{O}_\xi)/H$$

is an isomorphism, and this puts a symplectic structure on the latter space. □

Apply the proposition to $W = \mathcal{M}_0$ and $H = \mathcal{G}/\mathcal{G}_0$. Notice that there is an exact sequence

$$1 \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{G} \longrightarrow \Omega^0(\partial S, \mathrm{G}) \longrightarrow 1$$

(assuming that G is simply connected, so that any C^∞ mapping $\partial S \to \mathrm{G}$ extends to a C^∞ mapping $S \to \mathrm{G}$), so that

$$\mathcal{G}/\mathcal{G}_0 \cong \prod_{C_i \in \pi_0(\partial S)} \Omega^0(C_i, \mathrm{G})\,.$$

As usual, $\oplus_i \Omega^1(C_i, \mathfrak{g})$ can be identified with a subspace of the dual of the Lie algebra of $\mathcal{G}/\mathcal{G}_0$ by integration

$$\langle (\varphi_1, \ldots, \varphi_d), (\alpha_1, \ldots, \alpha_d) \rangle = \sum_{i=1}^{d} \int_{C_i} \mathcal{B}_\star (\varphi_i \wedge \alpha_i)$$

in such a way that the momentum mapping we are interested in is

$$\begin{array}{ccc} \mathcal{M}_0 & \longrightarrow & \bigoplus_{i=1}^{d} \Omega^1(C_i, \mathfrak{g}) \oplus \mathbf{C} \\ [A] & \longmapsto & ((A|_{C_1}, \ldots, A|_{C_d}), 1). \end{array}$$

Identify now each component C_i with the circle S^1. We now have to understand the coadjoint orbits of the loop group $L\,\mathrm{G}$ in the dual $\widehat{L\mathfrak{g}}^\star$ of the central extension $\widehat{L\mathfrak{g}}$.

I shall postpone the description of these orbits, and thus the proof of the next proposition, to the end of this section. Here is the result:

Proposition 2.2.8 *The space $\Omega^1(S^1, \mathfrak{g}) \oplus \{1\}$ can be embedded as a subspace in the dual $\widehat{L\mathfrak{g}}^\star$. The coadjoint orbits of the loop group $L\,\mathrm{G}$ in this subspace are in one-to-one correspondence with the orbits of G acting on itself by conjugation.*

The correspondence is given by the monodromy of the 1-form along S^1, so that we will eventually have proven Theorem 2.2.1. □

The case of $U(1)$. — Consider the case of the group $\mathrm{G} = U(1)$. Fixing a conjugacy class in $U(1)$ amounts to fixing an element of $U(1)$. The symplectic leaves in $\mathcal{M}_{g,d} = U(1)^{2g} \times U(1)^{d-1}$ are the $U(1)^{2g} \times \{c_1, \ldots, c_{d-1}\}$.

Different Poisson structures on the same space. — Let us turn now to our second favorite class of examples, the three holed sphere and the one holed torus, that is, the moduli spaces $\mathcal{M}_{0,3}$ and $\mathcal{M}_{1,1}$. As we have already noticed, both are copies of $\mathrm{G} \times \mathrm{G} / \mathrm{Ad}\,\mathrm{G}$.

An element of $\mathcal{M}_{0,3}$ consists of three elements M_1, M_2 and $M_3 \in \mathrm{G}$ such that $M_1 M_2 M_3 = 1$, up to simultaneous conjugation by group elements. To fix a symplectic leaf, we have to fix the conjugacy classes C_1, C_2 and C_3 of M_1, M_2 and M_3. It turns out (exercise) that the map

$$\begin{array}{ccc} C_1 \times C_2 \times C_3 & \longrightarrow & \mathrm{G} \\ (M_1, M_2, M_3) & \longmapsto & M_1 M_2 M_3 \end{array}$$

has maximal rank at a solution of the equation $M_1 M_2 M_3 = 1$ corresponding to an irreducible representation. The dimension of the symplectic leaf $\mathcal{M}_{0,3}^{C_1, C_2, C_3}$ is thus $\sum \dim C_i - 2 \dim \mathrm{G}$ (exercise).

As for an element of $\mathcal{M}_{1,1}$, it also consists of three elements A, B, $M \in \mathrm{G}$, but now the relation is $ABA^{-1}B^{-1}M = 1$ and, to fix a symplectic leaf, we need only fix the conjugacy class C of M. In the same way, it is shown that $\dim \mathcal{M}_{1,1}^{C} = \dim C$.

Generic conjugacy classes have the form G/T for a maximal torus T, and thus have dimension $\dim \mathrm{G} - \mathrm{rk}\,\mathrm{G}$, so that, for G a simple Lie group, the generic symplectic leaves have dimension $\dim G - 3\,\mathrm{rk}\, G$ in $\mathcal{M}_{0,3}$ and $\dim G - \mathrm{rk}\, G$ in $\mathcal{M}_{1,1}$: the Poisson spaces obtained are indeed very different.

Notice that this computation works only for a *simple* Lie group. Consider for instance $\mathcal{M}_{1,1}$ in the case of the commutative group $U(1)$. The only conjugacy class giving rise to a non-empty leaf is that of 1, so that $\mathcal{M}_{1,1}(U(1))$ is actually symplectic.

Description of the central extension $\widehat{L\mathfrak{g}}$. — Identify S^1 with $\mathbf{R}/2\pi\mathbf{Z}$ and define, for α, $\beta : S^1 \to \mathfrak{g}$,

$$c(\alpha, \beta) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{B}(\alpha(\theta), \beta'(\theta)) d\theta$$

(β' denotes the derivative of β with respect to θ). The Lie algebra $\widehat{L\mathfrak{g}}$ is defined by the cocycle c as above. The adjoint action of $\widehat{L\mathfrak{g}}$ on itself is actually an $L\mathfrak{g}$-action, by

$$\mathrm{ad}_\gamma(\alpha, t) = ([\gamma, \alpha], c(\gamma, \alpha))$$

and this is the infinitesimal version of the adjoint action of the group $L\,G$. To write the latter, we need a few notations: for $g : S^1 \to G$, we will consider $g^{-1}g'$ as a mapping $S^1 \to \mathfrak{g}$ and hence as an element of $L\mathfrak{g}$; a bilinear form on $L\mathfrak{g}$, which will still be denoted by $\mathcal{B}$, is defined by

$$(\alpha, \beta) \longmapsto \frac{1}{2\pi} \int_0^{2\pi} \mathcal{B}(\alpha(\theta), \beta(\theta)) d\theta$$

Now, the adjoint action is

$$\mathrm{Ad}_g(\alpha, t) = \left(\mathrm{Ad}_g \alpha, t + \mathcal{B}(g^{-1}g', \alpha)\right).$$

Let us describe now the coadjoint action, on the subspace $\Omega^1(S^1, \mathfrak{g}) \oplus \mathbf{C}$ of $\widehat{L\mathfrak{g}}^\star$:

$$\begin{aligned}
\langle \mathrm{Ad}_g^\star(\varphi, z), (\alpha, t)\rangle &= \langle(\varphi, z), \mathrm{Ad}_{g^{-1}}(\alpha, t)\rangle \\
&= \langle(\varphi, z), \left(\mathrm{Ad}_{g^{-1}} \alpha, t - \mathcal{B}\left(g'g^{-1}, \alpha\right)\right)\rangle \\
&= \int_{S^1} \mathcal{B}\left(\varphi, \mathrm{Ad}_{g^{-1}} \alpha\right) + zt - z \int_{S^1} \mathcal{B}\left(g'g^{-1}, \alpha\right)
\end{aligned}$$

so that

$$\mathrm{Ad}_g^\star(\varphi, z) = \left(\mathrm{Ad}_g \varphi - zg'g^{-1}, z\right).$$

Remark. — This is an $L\,G$-action. However, notice that the action depends, for each "level" z, on the actual value of z. The momentum mapping and the orbits we are interested in correspond to the value $z = 1$.

The last argument is copied from Pressley and Segal's book [34] (§ 4.3). Consider an element $\varphi \in \Omega^1(S^1, \mathfrak{g})$ and write it as $\varphi = \alpha d\theta$ for some $\alpha \in L\mathfrak{g}$. Solve the differential equation

$$\begin{cases} f' \cdot f^{-1} &= \alpha \\ f(0) &= 1 \end{cases}$$

at least for a function $f : \mathbf{R} \to \mathrm{G}$. As α is a function on the circle S^1, $f(\theta + 2\pi) = f(\theta)f(2\pi)$. Call $m(\varphi)$ (m for *monodromy*) the element $f(2\pi) \in \mathrm{G}$.

Let us make an element $g \in L\mathrm{G}$ act on $(\varphi, 1)$, obtaining $\widetilde{\varphi} = \mathrm{Ad}_g \varphi - (dg)g^{-1}$, that is, $\widetilde{\alpha} = \mathrm{Ad}_g \alpha - g'g^{-1}$. It is easy to check that

$$m(\mathrm{Ad}_g \varphi) = g(0)m(\varphi)g(0)^{-1}$$

and this proves Proposition 2.2.8. □

Remark. — The description we have obtained of the symplectic leaves was sketched by Atiyah in [6]. The symplectic form of the leaves is described in many papers (see e.g. [3]). I have chosen a global description of the Poisson structure (which I learned from [19] and from V. Fock) because it shows *why* the symplectic leaves are obtained by fixing the monodromy along the boundary components, so that the symplectic foliation appears in a very natural way.

It is also possible to describe the Poisson structure in the infinite dimensional setting without using the loop group, as explained by L. Jeffrey in [26]: she considers connections A that are identically zero on some neighbourhood of the boundary, which makes the approach slightly simpler but less canonical than the one presented here, as she is forced to use a parametrisation of the boundary.

2.3 The Fock-Roslyi discretisation

I will present now a finite dimensional description of the Poisson structure due to Fock and Roslyi [19], [20]. Notice that in the usual finite dimensional descriptions of the symplectic structure for the case of a closed surface (as in [22], [23] and [30]), the difficult point is to prove that the 2-form is closed. The corresponding property of a Poisson structure, namely the Jacobi identity, will hold by construction here. This is the main advantage of considering a Poisson structure rather than a symplectic form. Notice however that the following construction works only for a surface with non-empty boundary.

Discretisation of the moduli space. — A surface with a non-empty boundary can be considered (in many different ways) as a fat graph. In other words, the surface is a tubular neighborhood of a graph drawn on it. Notice that the graph itself is not sufficient to define the surface. For instance the two fat graphs in Figure 6 would give the same graph. What is needed to reconstruct the surface is a cyclic ordering of the edges at each vertex and this is what the fat on the graph gives. The idea of Fock and Roslyi in [19] and [20] is to use the fat graph as a discrete version of the surface in order to describe the moduli space $\mathcal{M}$ and to define the Poisson structure above in purely finite dimensional terms.

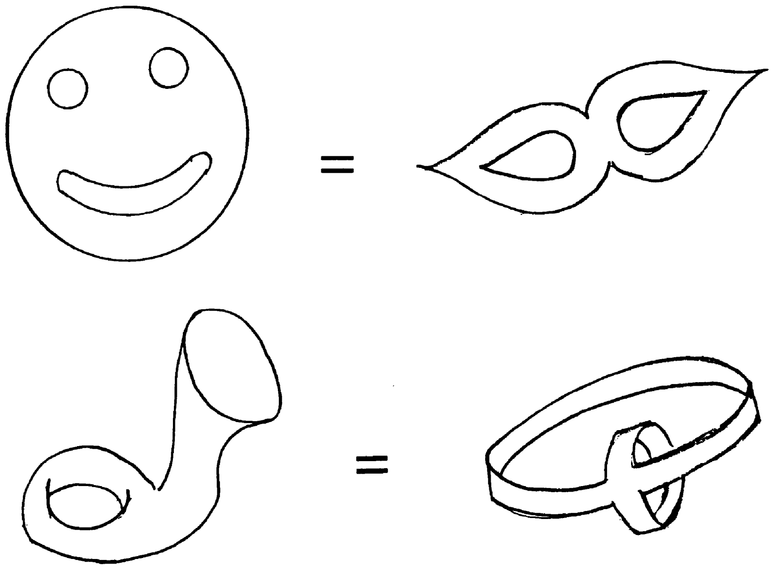

Figure 6: Fat graphs

The set $\mathcal{A}^\delta$ of *graph connections* we want to define is just the set of all maps

$$\{\text{edges}\} \longrightarrow \mathrm{G}$$

but it will be more convenient to describe it in a slightly more complicated way. Let $\mathcal{E}$ be the set of *ends* of edges. As an edge has two ends, it is endowed with an involution $\alpha \mapsto \alpha^\vee$. Let

$$\mathcal{A}^\delta = \left\{h : \mathcal{E} \to \mathrm{G} \mid h(\alpha^\vee) = h(\alpha)^{-1}\right\} \subset \mathrm{G}^{\mathcal{E}}.$$

Let $\mathcal{V}$ be the set of vertices and $\mathcal{G}^\delta$ (the *discrete gauge group*) be $\mathrm{G}^{\mathcal{V}}$. Notice that any edge ends at a vertex, so that we have a map $\mathcal{E} \to \mathcal{V}$. I will denote the vertex associated with α by $[\alpha]$ or simply by α. With these notations, the gauge group $\mathcal{G}^\delta$ acts on the space of connections $\mathcal{A}^\delta$ by the map defined by

$$\begin{array}{ccc} \mathcal{G}^\delta \times \mathcal{A}^\delta & \longrightarrow & \mathcal{A}^\delta \\ (g(v), h(\alpha)) & \longmapsto & (\alpha \mapsto g(\alpha^\vee)^{-1} h(\alpha) g(\alpha)) \end{array}$$

(see Figure 7). For instance, if $\alpha^\vee$ and α are at the same vertex v, the corresponding copy of G acts by conjugation.

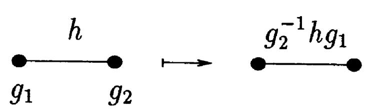

Figure 7

Let δ be the "discretisation map": its version

$$\delta : \mathcal{A}_{fl} \longrightarrow \mathcal{A}^\delta$$

sends the connection A to the collection of its holonomies along the edges (this time we are considering the end α as the origin of an oriented edge), while its version

$$\delta : \mathcal{G} \longrightarrow \mathcal{G}^{\delta}$$

sends the map $g : S \to \mathrm{G}$ to the collection of its values on the vertices.

Proposition 2.3.1 *The discretisation map δ intertwines the two actions, in other words the diagram*

$$\begin{array}{ccc} \mathcal{G} \times \mathcal{A}_{fl} & \longrightarrow & \mathcal{A}_{fl} \\ \downarrow \delta & & \downarrow \delta \\ \mathcal{G}^{\delta} \times \mathcal{A}^{\delta} & \longrightarrow & \mathcal{A}^{\delta} \end{array}$$

commutes. Moreover, it defines a one-to-one correspondence

$$\delta : \mathcal{A}_{fl}/\mathcal{G} \longrightarrow \mathcal{A}^{\delta}/\mathcal{G}^{\delta}.$$

In the case where the graph is reduced to only one vertex, the right hand side is just $\mathrm{Hom}(\pi_1 S, \mathrm{G})/\operatorname{Ad}\mathrm{G}$ and we recover the statement of § 2.1. The proof of this slightly more general statement is not much harder. □

The Poisson structure on $\mathcal{A}^{\delta}$ and $\mathcal{G}^{\delta}$. — The next step is to define the Poisson structure. The idea is to construct Poisson structures both on $\mathcal{A}^{\delta}$ and $\mathcal{G}^{\delta}$, compatible in the sense that

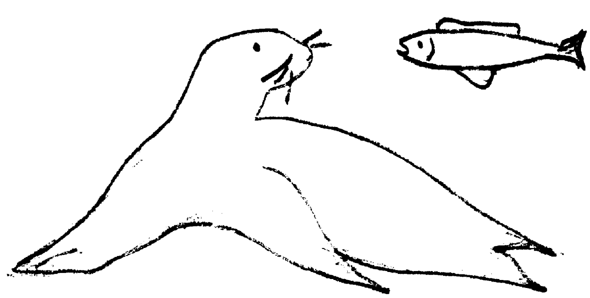

Figure 8: the Poisson structure considered by Fock

the action of the gauge group $\mathcal{G}^{\delta}$ is a Poisson action, that is:

- $\mathcal{G}^{\delta}$ is a Poisson Lie group
- $\mathcal{A}^{\delta}$ is a Poisson manifold
- $\mathcal{G}^{\delta} \times \mathcal{A}^{\delta} \to \mathcal{A}^{\delta}$ is a Poisson mapping.

As everything is built from copies of G, it is easy to see that one should use Poisson Lie group structures on G. Choose, for each vertex v, an r-matrix $r_v \in \mathfrak{g} \otimes \mathfrak{g}$. The only compatibility assumption is that the symmetric parts of all these r-matrices must coincide with the symmetric form $\mathcal{B}$.

As in § 1.4, use $\mathcal{B}$ to consider r_v as a map $r_v : \mathfrak{g} \to \mathfrak{g}$. To simplify notations, fix also a $\mathcal{B}$-orthonormal basis $(X_1, \ldots, X_N)$ in $\mathfrak{g}$ so that r_v is actually a matrix $(r_v^{i,j})_{1 \le i,j \le N}$.

Poisson structure on $\mathcal{A}_\delta$ (via cilia). — Consider first a single vertex v and all the ends of edges at v. We need now to order them. This is described by Fock and Roslyi in two steps:

- Fix a cyclic ordering at v. This is what is needed to reconstruct the surface from the graph (compare Figure 6 and the discussion there): this is the fat graph.
- Fix an actual ordering by fixing an "origin". This is achieved by a "ciliation of the graph": each vertex is endowed with a *cilium* as in Figure 9. This ciliated fat graph is what is needed to define the Poisson structure.

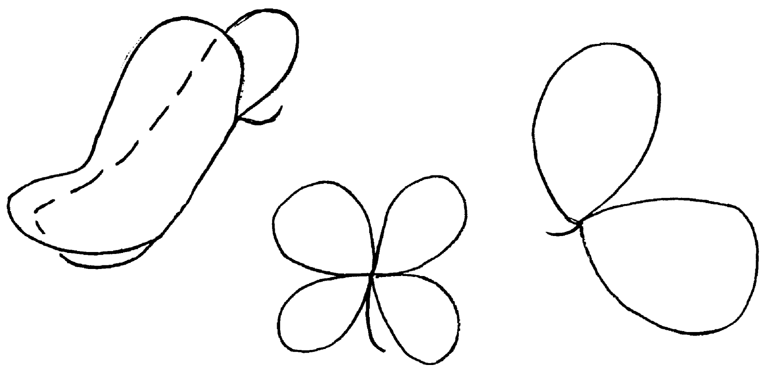

Figure 9: ciliated fat graphs

Having fixed an order, define a bivector on the product of copies of G corresponding to the given vertex v by the following formula:

$$P_v = \sum_{\alpha<\beta} r_v^{i,j} X_i^\alpha \wedge X_j^\beta + \frac{1}{2}\sum_{\alpha} r_v^{i,j} X_i^\alpha \wedge X_j^\alpha$$

where the sums range over the ends of edges at v, sums over repeated indices being understood, and where the vector fields X_i^α are defined below.

With any $X \in \mathfrak{g}$, associate two vector fields L_X and R_X on G (respectively the left- and right-invariant vector fields defined by X). Consider then the embedding

$$\begin{array}{ccc} \mathrm{G} & \longrightarrow & \mathrm{G}\times\mathrm{G} \\ g & \longmapsto & (g, g^{-1}) \end{array}$$

so that the vector $L_X(g) - R_X(g^{-1})$ is tangent to the image of G: we will consider it as a vector in $T_g\,\mathrm{G}$ and will denote this vector field on G by X. This way, any $X \in \mathfrak{g}$ and any end of edge α define a vector field X^α.

Recall that $(X_1, \ldots, X_N)$ was a $\mathcal{B}$-orthonormal basis of $\mathfrak{g}$, so that the X_i^α appearing in the definition of P_v are just the vector fields associated with X_i and α.

The Poisson structure on $\mathcal{A}^\delta$ is defined by the bivector

$$P = \sum_{v\in\mathcal{V}} P_v.$$

Notice that $X^{(v)} = \sum_{\alpha\in v} X^\alpha$ is the fundamental vector field for the gauge action of the factor $G \subset \mathcal{G}^\delta$ corresponding to the given vertex v and associated with the vector $X \in \mathfrak{g}$. Notice also that ($X_1, \ldots, X_N$ being a $\mathcal{B}$-orthonormal basis) the terms in P_v can be reordered in such a way that

$$P_v = \tilde{r}_v^{i,j} X_i^{(v)} \otimes X_j^{(v)} + \sum_{\alpha,\beta\in v} (\alpha,\beta)_v X_i^\alpha \otimes X_i^\beta$$

where $\tilde{r}$ is the skewsymmetric part of r, $(\alpha,\beta)_v = 1$ if $\alpha > \beta$, 0 if $\alpha = \beta$ and -1 if $\alpha < \beta$.

The first term involves the whole matrix r_v, but only fundamental vector fields of the gauge action, while the second involves only the symmetric part $\mathcal{B}$ of the r-matrix. This leads naturally to:

Proposition 2.3.2 *The gauge action $\mathcal{G}^\delta \times \mathcal{A}^\delta \to \mathcal{A}^\delta$ is a Poisson action. The resulting Poisson structure on the quotient $\mathcal{A}^\delta/\mathcal{G}^\delta$ does not depend on the r-matrix chosen to define it.*

The structure obtained on $\mathcal{A}^\delta/\mathcal{G}^\delta$ is also independent of the graph we have chosen, as we have:

Theorem 2.3.3 *The natural bijection $\delta : \mathcal{A}_{\mathfrak{fl}}/\mathcal{G} \to \mathcal{A}^\delta/\mathcal{G}^\delta$ is a Poisson map.*

Let us sketch now a proof of this theorem. We look first at the *functions* on $\mathcal{A}^\delta$. As $\mathcal{A}^\delta$ is a product of copies of G, this is quite easy. Choose an end for each edge of the graph, thus defining a subset $\mathcal{E}_1 \subset \mathcal{E}$. With each $\alpha \in \mathcal{E}_1$, associate a linear representation

$$\pi_\alpha : \mathrm{G} \longrightarrow \mathrm{End}(V_\alpha)$$

and to the other end $\alpha^\vee \notin \mathcal{E}_1$ of the same edge, associate $\pi_{\alpha^\vee} = \pi_\alpha^\star$. This way

$$V = \bigotimes_{\alpha\in\mathcal{E}} V_\alpha = \bigotimes_{\alpha\in\mathcal{E}_1} \mathrm{End}(V_\alpha).$$

Consider now the set $\mathcal{V}$ of vertices of the graph. To each $v \in \mathcal{V}$, associate a linear form

$$C_v \in \bigotimes_{\alpha\in v} V_\alpha^\star$$

so that

$$\bigotimes_{v\in\mathcal{V}} C_v \in \bigotimes_{v\in\mathcal{V}} \left(\bigotimes_{\alpha\in v} V_\alpha^\star\right) = \bigotimes_{\alpha\in\mathcal{E}} V_\alpha^\star = V^\star.$$

Call the data (C, π) a *colouring* of the graph. To any colouring is associated a function $\varphi = \varphi(C, \pi)$ defined by the natural pairing between V and $V^\star$:

$$\begin{array}{llcl} \varphi : & \mathcal{A}^\delta & \longrightarrow & \mathbf{C} \\ & (h : \mathcal{E} \to \mathrm{G}) & \longmapsto & \left\langle \bigotimes_{v\in\mathcal{V}} C_v, \bigotimes_{\alpha\in\mathcal{E}_1} \pi_\alpha(h(\alpha)) \right\rangle. \end{array}$$

This way we have all the functions on $\mathcal{A}^\delta$. To get gauge invariant functions (i.e. in order that φ be invariant under the $\mathcal{G}^\delta$ action), we need only assume that

$$C_v : \bigotimes_{\alpha\in v} V_\alpha \longrightarrow \mathbf{C}$$

is $\otimes\pi_\alpha$-invariant.

Now any colouring (C,π) of a graph γ drawn on S also defines a function

$$\psi = \psi(C,\pi) = \psi(\gamma, C, \pi)$$

on the space $\mathcal{A}$ of *all* connections, in such a way that, for the discretisation map δ associated with the graph γ, $\varphi \circ \delta = \psi$. To prove the theorem, what we need to show is that

$$\{\varphi, \varphi'\}(\delta(A)) = \{\psi, \psi'\}(A).$$

This is achieved by Fock and Roslyi by two direct computations, allowing them to express both sides of this equation by the same formula.

For the left hand side, we need only to use the formula for the bivector P given above and to notice that the terms involving the skewsymmetric part $\widetilde{r}$ of the r-matrix do not contribute for gauge invariant functions, so that we can forget them.

For the right hand side, the computation is a bit more subtle, since it first uses functions associated with transversal (and in particular different) graphs.

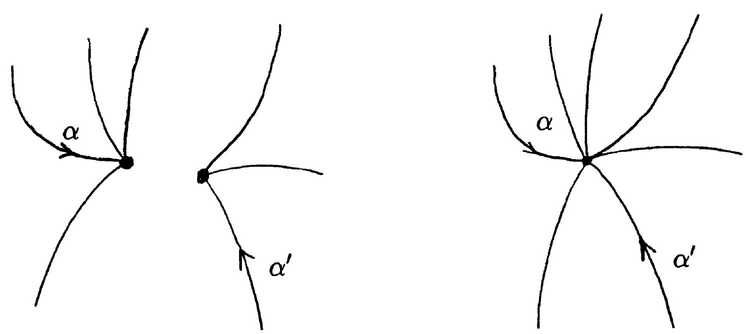

Figure 10

Suppose (γ, C, π) and (γ', C', π') are two coloured graphs. For each of them, choose an end of edge, say $\alpha \in \mathcal{E}$, $\alpha' \in \mathcal{E}'$. Then construct a new coloured graph

$$(\gamma, C, \pi) \cup_{\alpha,\alpha'} (\gamma', C', \pi'),$$

which is obtained from γ and γ' by identifying the vertices $[\alpha]$ and $[\alpha']$ (Figure 10). The two colourings are unchanged, except for the new vertex v, which we colour by

$$C_v = \left(C_{[\alpha]} \otimes C'_{[\alpha']}\right) \mathcal{B}^{\alpha\alpha'}$$

where $\mathcal{B}^{\alpha\alpha'} : V_\alpha \otimes V'_{\alpha'} \to V_\alpha \otimes V'_{\alpha'}$ is the image of $\mathcal{B} \in \mathfrak{g} \otimes \mathfrak{g}$ under the infinitesimal representation

$$\pi_\alpha \otimes \pi'_{\alpha'} : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \operatorname{End}\left(V_\alpha \otimes V'_{\alpha'}\right)$$

considered here in the natural way as an endomorphism of

$$\left(\bigotimes_{\beta \in [\alpha]} V_\beta\right) \otimes \left(\bigotimes_{\beta' \in [\alpha']} V'_{\beta'}\right).$$

With these notations, one proves:

Proposition 2.3.4 *Let (γ, C, π), (γ', C', π') be two transversal coloured graphs on the oriented surface S. Then*

$$\{\psi(\gamma, C, \pi), \psi(\gamma', C', \pi')\} = \sum_{x \in \gamma \cap \gamma'} \varepsilon(x) \psi\left((\gamma, C, \pi) \cup_{\alpha(x), \alpha'(x)} (\gamma', C', \pi')\right)$$

where $\alpha(x) \in \mathcal{E}$, $\alpha'(x) \in \mathcal{E}'$ are ends of the edges of γ, γ' that intersect at x, and $\varepsilon(x)$ is the intersection number of the oriented edges $[\alpha(x), \alpha(x)^\vee]$ and $[\alpha'(x), \alpha'(x)^\vee]$.

Remark. — I will not give a proof here. Notice that this is very close to Goldman's formula [23].

The last step applies this proposition to two functions defined by colourings of the *same* graph. Fortunately, we are dealing with flat connections, so that it is possible to deform a graph to make it transversal to itself. Using the cyclic order and the cilium at each vertex, there is even a more or less standard way to do it, shown in Figure 11, which gives

- an intersection point x_α in the middle of each edge, with $\varepsilon(x_\alpha) = 1$,
- an intersection point $x_{\alpha,\beta}$ for each pair (α, β) of edges starting at the same vertex and such that $\alpha \geq \beta$, with $\varepsilon(x_{\alpha,\beta}) = -1$.

Hence Proposition 2.3.4 allows us to compute the Poisson bracket of any two functions on $\mathcal{A}_{fl}$ defined by two colourings of the same graph, giving eventually:

$$\begin{aligned} \{\psi(\gamma, C, \pi), \psi(\gamma, C', \pi')\} &= \sum_{\alpha \in \mathcal{E}_1} \psi\left((\gamma, C, \pi) \cup_{x_\alpha, x_\alpha} (\gamma, C', \pi')\right) \\ &\quad - \sum_{v \in \mathcal{V}} \sum_{\alpha, \beta \in v, \alpha \geq \beta} \psi\left((g, C, \pi) \cup_{\alpha, \beta} (\gamma, C', \pi')\right) \end{aligned}$$

which turns out to give the result.

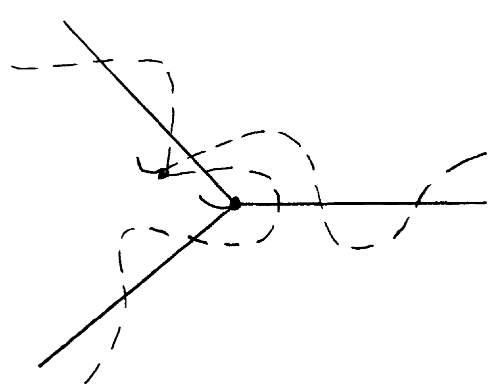

Figure 11

The example of $U(1)$. — I want to emphasise the fact that the Poisson structure on $\mathcal{A}^\delta$ is *not* a product structure. This is illustrated by the example of $\mathcal{M}_{1,1}(U(1))$. The torus with one hole is represented by the graph with one vertex v, ciliated as on the left of Figure 9: the cyclic order is clockwise and the origin is at the cilium. Of course, there is no non-trivial Poisson structure on $U(1)$, the only possible r-matrix is the symmetric

$$\mathrm{Id} : \mathfrak{u}(1) \longrightarrow \mathfrak{u}(1).$$

The bivector $P = P_v$ is simply

$$P_v = \sum_{\alpha<\beta} X^\alpha \wedge X^\beta$$

(the gauge action is indeed trivial). Notice also that, in this case, $X^{\alpha^\vee} = -X^\alpha$.

Now the space is just $U(1) \times U(1)$, but to understand this, we have to choose an end for each edge, that is, to orient the edges. Then the bivector P_v can be written:

$$\begin{aligned} P &= X_1 \wedge X_2 + X_1 \wedge X_4 \\ &= X_1 \wedge (X_2 + X_4) + (X_2 - X_4) \wedge X_3 \\ &= 2X_2 \wedge X_3, \end{aligned}$$

as $X_4 = -X_2$, so that: although defined by a (trivial) Poisson structure on $U(1)$, the Poisson structure is non-trivial. Notice that it is even a symplectic structure, which fits very well with the remarks following Theorem 2.2.1.

Another useful exercise is to perform the same computation for the graph on the right of Figure 9 (corresponding to $\mathcal{M}_{0,3}$). The bivector P_v is now:

$$\begin{aligned} P_v &= X_1 \wedge X_3 + X_1 \wedge X_4 + X_2 \wedge X_3 + X_2 \wedge X_4 \\ &= (X_1 + X_2) \wedge (X_3 + X_4) \\ &= 0, \end{aligned}$$

as $X_2 = -X_1$ and $X_4 = -X_3$, which shows the role played by the ordering... and which also fits very well with our previous remarks.

The daisy with m petals. — Consider now the moduli space $\mathcal{M}_{0,m+1}(\mathrm{G})$. The surface it comes from is described by the fat graph shown in the center of Figure 9, the ciliation of which will be used to describe the Poisson structure.

Fixing an orientation on each edge, we get an identification of $\mathcal{A}^\delta$ with G^m:

$$\begin{array}{ccc} \mathrm{G}^m & \longrightarrow & \mathcal{A}^\delta \subset \mathrm{G}^{2m} \\ (h_1, \dots, h_m) & \longmapsto & (h_1, h_1^{-1}, \dots, h_m, h_m^{-1}) \end{array}$$

so that the bivector P_v gives the following formula for the Poisson bracket of two functions on G^m:

$$\{F, G\} = \sum_{a<b} \langle r, \nabla_a F \wedge \nabla_b G \rangle + \frac{1}{2} \sum_a \langle r, \nabla_a F \wedge \nabla_a G \rangle$$

where $\nabla_a F$ is the mapping $\mathrm{G}^m \to \mathfrak{g}^\star$ defined by

$$\langle (\nabla_a F)(h), X \rangle = \begin{cases} \dfrac{d}{dt} F(\exp(tX)\cdot h)\Big|_{t=0} & \text{if } a \text{ is odd} \\[2ex] \dfrac{d}{dt} F(h\cdot \exp(tX))\Big|_{t=0} & \text{if } a \text{ is even} \end{cases}$$

for $a \in \{1, \ldots, 2m\}$, $h \in \mathrm{G}^m$, $X \in \mathfrak{g}$.

Remark. — Using similar geometric ideas, Fock also has a description of structures on the moduli space of projective structures on a surface (see [18]).

3 Examples of integrable systems via the moduli space

Let us now look for integrable systems. We will first show two constructions of functions on the moduli space $\mathcal{M}$ (the first and more famous one due to Goldman, the second to Alekseev), then prove that some of them actually commute. We will then derive some examples of integrable systems.

3.1 Constructions of commuting functions on $\mathcal{M}$

Goldman's functions. — They are usually described in the following way. Consider both an invariant function $f : \mathrm{G} \to \mathbf{R}$ or $\mathbf{C}$ and a simple closed curve C on S, and define a map

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{f_C} & \mathbf{R} \text{ or } \mathbf{C} \\ [\rho] & \longmapsto & f(\rho(C)) \end{array}$$

where ρ is a representative $\pi_1 S \to \mathrm{G}$ of the class $[\rho]$ and C is considered as an element of $\pi_1 S$ in any obvious way, the invariance of f implying that the result does not depend on the choices made.

Remark. — For C a simple closed curve, let $\mathcal{G}_C$ be the subgroup of the gauge group $\mathcal{G}$ consisting of mappings $g : S \to \mathrm{G}$ that are identically equal to 1 on C. The subgroup $\mathcal{G}_C$ is, of course, normal. Call $\mathcal{M}_C = \mathcal{A}_{fl}/\mathcal{G}_C$. The map

$$\begin{array}{rcl} \mu_C : \mathcal{M}_C & \longrightarrow & \widehat{L\mathfrak{g}}^\star \\ A & \longmapsto & (A\,|_C, 1) \end{array}$$

(with the notations of § 2.2), defined using a diffeomorphism $C \to S^1$, is the momentum mapping for the $\mathcal{G}_C$-action on $\mathcal{M}_C$ and thus defines a mapping

$$\Phi_C : \mathcal{M} \longrightarrow \widehat{L\mathfrak{g}}^\star / L\,\mathrm{G},$$

which associates, to the gauge equivalence class of the flat connection A, its holonomy along C. The Goldman functions are just the quantitative (numerical) versions the mapping of Φ_C: invariant functions on G more or less describe the conjugacy classes in G. The flow of a function f_C (as defined above) will preserve the levels of the map Φ_C. Such a level is a set of flat connections whose holonomy along C is in a given conjugacy class (notice that f_C depends only on the homotopy class of the free loop C). So we have the following proposition:

Proposition 3.1.1 *Let C be a simple closed curve on S. The flow of f_C acts on the class $[A] \in \mathcal{M}$ of a flat connection by some element of the stabiliser of $\mathrm{G}_{m(\Phi_C([A]))}$ of its holonomy along C.*

Remark. — The Goldman function f_C also belongs to the Fock and Roslyi framework, namely they are among the functions given by colourings of graphs used in the proof of 2.3.3. Consider a graph on the surface S for which the closed curve C appears as an edge $[\alpha, \alpha^\vee]$ with a single vertex v. Put $V_\beta = 0$ for all $\beta \neq \alpha, \alpha^\vee$. Now the invariant function $f : \mathrm{G} \to \mathbf{C}$ can be written as a composition

$$\mathrm{G} \xrightarrow{\pi_\alpha} \mathrm{End}(V_\alpha) = V_\alpha^\star \otimes V_\alpha = V_{\alpha^\vee} \otimes V_\alpha^\star \xrightarrow{C_v} \mathbf{C}$$

for some representation $\mathrm{G} \to \mathrm{End}(V_\alpha)$ and some linear map C_v. Thus f_C is our previous $\psi(\gamma, C, \pi)$.

After having defined these functions, Goldman proved two beautiful theorems:

- He computed the Poisson brackets of any two functions f_{C_1}, g_{C_2}, roughly speaking in terms of the intersections of the two curves C_1 and C_2 (this is very similar to Proposition 2.3.4).
- He gave an explicit formula for the flow of f_C, viewed as a Hamiltonian on $\mathcal{M}$.

I will use mainly a special case of the first theorem mentioned:

Theorem 3.1.2 *Let C_1 and C_2 be two disjoint simple closed curves on the surface S. Then, for any two invariant functions f and g on the group G,*

$$\{f_{C_1}, g_{C_2}\} = 0.$$

Remark. — Goldman considers a *closed* surface S, but this does not make any difference here.

Corollary 3.1.3 *Let C be a simple closed curve on the surface S. Then for any two invariant functions f and g on the group G, $\{f_C, g_C\} = 0$.*

Proof Once we know that the Poisson structure is that given by the Fock-Roslyi construction, and in particular does not depend on the fat graph chosen to represent the surface, this is quite obvious: choose a graph decomposition of the surface in which the two given curves appear as unions of edges *without common vertex*. As the Poisson structure is the sum of its components for the different vertices, the Poisson bracket of the two functions vanishes. □

Remark. — Along the same lines, but with more care, the Fock-Roslyi description can be used to prove Goldman's general result, i.e. to compute $\{f_{C_1}, g_{C_2}\}$.

As for the second theorem, I will use only a special case, which is a quantitative version of Proposition 3.1.1, in § 3.3.

Alekseev's functions. — We will now construct, following [1], a lot of functions at the same time, which will depend on a presentation of the fundamental group $\pi_1 S$, with generators α_i, β_j and μ_k. The relation is

$$\mu_1 \cdots \mu_d \alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \cdots \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1} = 1.$$

Figure 12 shows free loops representing the generators (one should add a base point and connect everything to it).

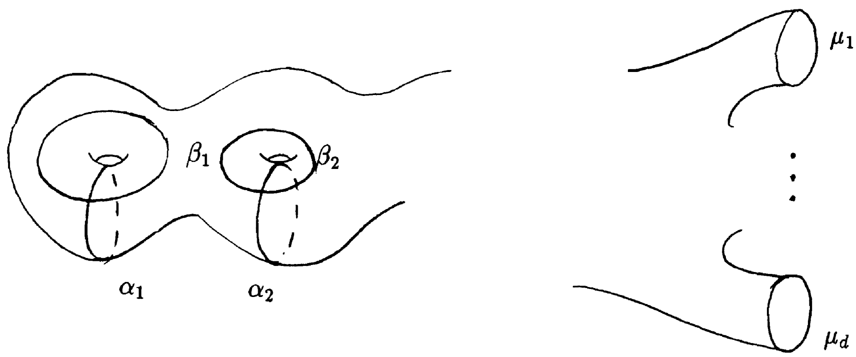

Figure 12

The functions considered by Alekseev belong to two different families.

- First, Goldman's functions f_{α_i} associated with simple curves representing α_i.
- Second, a new sort of objects, that I will describe now. Denote a representation ρ by the images of the generators, that is, write $A_i = \rho(\alpha_i)$, $B_j = \rho(\beta_j)$ and $M_k = \rho(\mu_k)$ and group terms in the relation as follows:

$$M_1 \cdots M_d A_1 \left(B_1 A_1^{-1} B_1^{-1}\right) \cdots A_g \left(B_g A_g^{-1} B_g^{-1}\right) = \mathrm{Id}.$$

In other words, write

$$M_{d+2i-1} = A_i, \quad M_{d+2i} = B_i A_i^{-1} B_i^{-1}$$

so that our relation becomes

$$M_1 \cdots M_{d+2g} = \mathrm{Id}.$$

Add a "spectral parameter" λ and consider the polynomial matrix

$$M(\lambda) = (M_1 + \lambda\,\mathrm{Id}) \cdots (M_{d+2g} + \lambda\,\mathrm{Id}).$$

If f is any invariant function on $SL(n; \mathbf{C})$ (typically, $\mathrm{tr}(M^k)$), it can be evaluated on the polynomial $M(\lambda)$ and expanded as a polynomial

$$f(M(\lambda)) = \sum \varphi_i \lambda^i.$$

The second family of functions I want to consider is the set of all such φ_i (all i, all f).

We already know that the functions of the first family commute (this is 3.1.2 and 3.1.3). It is also a consequence of what we have done before (Proposition 3.1.1) that functions of the first family commute with functions of the second.

Fix conjugacy classes $\Gamma_1, \ldots, \Gamma_d$ in G and consider the symplectic leaf $\mathcal{M}^{\Gamma_1,\ldots,\Gamma_d} \subset \mathcal{M}_{g,d}$. Consider a simple closed curve C disjoint from the boundary. The map Φ_C of § 3.1 can be considered as a map

$$\Phi_C : \mathcal{M}^{\Gamma_1,\ldots,\Gamma_d} \longrightarrow \widehat{L\mathfrak{g}}^\star / L\,\mathrm{G}\,.$$

Now, cut the surface along the curve C and look at what happens. The new surface has two more boundary components. By fixing a conjugacy class Γ in G, one can consider the symplectic leaf

$$\mathcal{M}^{\Gamma_1,\ldots,\Gamma_d,\Gamma,\Gamma^{-1}} \subset \mathcal{M}_{g-1,d+2}.$$

The following proposition is now more or less obvious:

Proposition 3.1.4 *The natural map*

$$\Phi_C^{-1}(\Gamma) \longrightarrow \mathcal{M}^{\Gamma_1,\ldots,\Gamma_d,\Gamma,\Gamma^{-1}}$$

is the symplectic reduction of the level Γ *in* $\mathcal{M}^{\Gamma_1,\ldots,\Gamma_d}$.

Taking for C any "α curve", we see that the first family of functions will commute with the members of the second. Moreover, having cut S along the α curves, we are led to the study of the case of a sphere with $2g + d$ holes. Recall that our polynomial is simply

$$M(\lambda) = \prod_{i=1}^{d+2g} (M_i + \lambda\,\mathrm{Id})\,.$$

The next result is then:

Proposition 3.1.5 *The algebra of functions on* $\mathcal{M}_{0,d}(SL(n;\mathbf{C}))$ *generated by the coefficients of the polynomials* $f(M(\lambda))$, *for* f *any invariant function on* $SL(n;\mathbf{C})$, *is abelian.*

Corollary 3.1.6 *The algebra of functions on* $\mathcal{M}_{g,d}(SL(n;\mathbf{C}))$ *generated by the coefficients of the polynomials* $f(M(\lambda))$ *and the* $g(A_i)$, *for* f, g *any invariant functions on* $SL(n;\mathbf{C})$, *is abelian.*

That the functions commute is a direct computation based on the formula for the Poisson bracket in the case of a daisy with m petals given in § 2.3.

3.2 Integrable systems?

Counting commuting Goldman functions. — Recall that g is the genus and d the number of boundary components of S. Assume $d \geq 1$ if $g = 1$ and $d \geq 3$ if $g = 0$. Then the maximum number of disjoint (non-trivial) curves on S is $3g-3+d$ (this yields a *trinion*[9] decomposition,

[9]The beautiful French terminology turns out to be very inconvenient when translated into English, as French people simply wear "pantalons" while the British need "*pairs* of pants". In any case, a trinion is just a sphere with three holes.

see Figure 13), so that Goldman gives us $(3g-3+d)\,\mathrm{rk}\,\mathrm{G}$ (hopefully independent) commuting functions on $\mathcal{M}$. This will be an integrable system... provided it is defined on a symplectic manifold of dimension $2(3g-3+d)\,\mathrm{rk}\,\mathrm{G}$. Notice that the Goldman functions defined by the boundary components are Casimir functions: this is more or less equivalent to saying that the symplectic leaves are obtained by fixing the conjugacy classes of the holonomy along the boundary. This is the reason why I have not included them in the enumeration here. I will nevertheless use the boundary curves in §3.3.

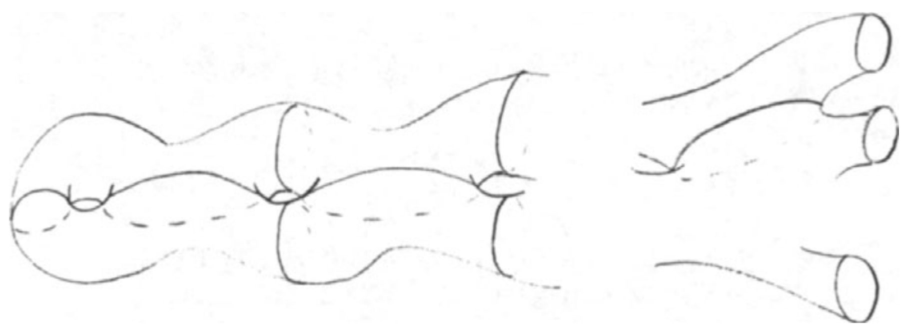

Figure 13: a trinion decomposition

Notice that we have two degrees of freedom here:

- we can of course choose the Lie group G,
- but we also have the choice of the symplectic manifold.

In the case of a closed surface ($d = 0$) the moduli space itself is a symplectic manifold. If moreover the Lie group G is simple, we have seen that the dimension of $\mathcal{M}_{g,0}$ is $(2g-2)\dim \mathrm{G}$ and we have $(3g-3)\,\mathrm{rk}\,\mathrm{G}$ functions. Thus Goldman gives us an integrable system if and only if

$$\frac{\dim \mathrm{G}}{\mathrm{rk}\,\mathrm{G}} = 3.$$

This happens with $\mathrm{G} = SU(2)$ or $SL(2)$.

Another way to have enough functions is to consider a surface with non-empty boundary and choose a small enough symplectic leaf. Consider for instance the moduli space $\mathcal{M}_{1,1}(SL(n;\mathbf{C}))$. Goldman allows us to use one curve and $n-1$ invariant functions... and this might give us (provided the functions are independent) an integrable system on any symplectic leaf of dimension $2(n-1)$.

In both cases, I will show that the Goldman functions are actually independent.

Counting Alekseev functions. — It seems that there are enough independent Alekseev functions to give an integrable system, at least in the $\mathrm{G} = SL(n;\mathbf{C})$ case ([2]), but I am not yet able to provide a proof of this. According to the reduction process explained above, it would be enough to check the $g = 0$ case.

What I plan to do now is to concentrate on Goldman functions and to explain what happens in the $SU(2)$ case, following Jeffrey-Weitsman [27], [28]: we will see a beautiful toric geometry on the moduli space (as in § 1.2). I will then follow the second idea and explain Fock's description of the Calogero (actually the Ruijsenaars) system.

3.3 The $SU(2)$ case, the polytopes of Jeffrey and Weitsman

Generalities on the $SU(2)$ case. — In this section, $G = SU(2)$. Let me begin with a few specific features of $\mathcal{M}$ in this case:

- Recall that all elements are diagonalisable. More precisely, for any $A \in SU(2)$, there is a basis of $\mathbf{C}^2$ in which A takes the form

$$A \sim \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

  for some $\theta \in [0, \pi]$, and, provided $A \neq \pm \mathrm{Id}$ (which is equivalent to $\operatorname{tr} A \neq \pm 2$), the stabiliser of A in $SU(2)$ is simply the $U(1)$ involved in the choice of the first eigenvector.

- As $SU(2)$ is compact, regular points of $\mathcal{M}$ correspond to irreducible representations of $\pi_1 S$.

- Moreover, there are rather few singular points: if all the matrices have a common eigenvector, they have two, since the second eigenline is just the orthogonal complement of the first. In other words, they are diagonalisable in the same basis—and in particular they commute.

Let me make the last remark more precise. Assume $\rho : \pi_1 S \to SU(2)$ is a reducible representation. Let $M_1, \ldots, M_d$ be the images in $SU(2)$ of the loops around the holes of S. As $\rho(\pi_1 S)$ is commutative, the relation in $\pi_1 S$ gives only $M_1 \cdots M_d = \mathrm{Id}$. Write now all the M_j's in a common basis of eigenvectors

$$M_j = \begin{pmatrix} e^{i\varepsilon_j\theta_j} & 0 \\ 0 & e^{-i\varepsilon_j\theta_j} \end{pmatrix}$$

for $\varepsilon_j = \pm 1$ (recall that I assume $\theta_j \in [0, \pi]$), so that the relation becomes simply $\sum \varepsilon_j \theta_j \equiv 0 \bmod 2\pi$.

Now the conjugacy class of M_j is well defined by θ_j, so that we have:

Proposition 3.3.1 *Let $\theta_1, \ldots, \theta_d \in [0, \pi]$ be such that, for all $\varepsilon_j \in \{\pm 1\}$, $\sum \varepsilon_j \theta_j \not\equiv 0 \bmod 2\pi$. Then the symplectic leaf defined by the conjugacy classes of the diagonal matrices $(e^{i\theta_1}, e^{-i\theta_1})$, $\ldots, (e^{i\theta_d}, e^{-i\theta_d})$ consists of regular points of $\mathcal{M}_{g,d}(SU(2))$ and is a smooth compact symplectic manifold of dimension $6g + 2d - 6$.*

Example. — Consider the case of the sphere with three holes ($g = 0$, $d = 3$). The moduli space is

$$\mathcal{M}_{0,3} = \{(M_1, M_2, M_3) \in SU(2) \times SU(2) \times SU(2) \mid M_1 M_2 M_3 = \mathrm{Id}\} / SU(2).$$

By conjugation, it may be assumed that

$$M_1 = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & e^{-i\theta_1} \end{pmatrix} \qquad 0 \le \theta_1 \le \pi.$$

We can then conjugate M_2 by a diagonal element of $SU(2)$. So we may assume that

$$M_2 = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \qquad \text{with } b \in \mathbf{R}^+$$

in other words, there exists $\beta \in [0, \pi]$ such that

$$\begin{cases} a &= \cos\theta_2 - i\sin\theta_2\cos\beta \\ b &= \sin\beta. \end{cases}$$

The condition that

$$M_1 M_2 \text{ be conjugate to } \begin{pmatrix} e^{i\theta_3} & 0 \\ 0 & e^{-i\theta_3} \end{pmatrix}$$

is then

$$\mathrm{Re}\left(e^{i\theta_1}a\right) = \cos\theta_3$$

or

$$\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\beta = \cos\theta_3.$$

This is reminiscent of spherical trigonometry: one may solve this for $\cos\beta$ if and only if θ_1, θ_2 and θ_3 are the lengths of the edges of a spherical triangle (see Figure 14). The representation is irreducible if and only if the triangle is an actual triangle (i.e. is not flat), so that we can state:

Proposition 3.3.2 *The moduli space $\mathcal{M}_{0,3}(SU(2))$ can be identified with the set of isometry classes of spherical triangles. The map*

$$\begin{array}{ccc} \mathcal{M}_{0,3} & \longrightarrow & \mathbf{R}^3 \\ (M_1, M_2, M_3) & \longmapsto & \dfrac{1}{\pi}(\theta_1, \theta_2, \theta_3) \end{array}$$

defines a diffeomorphism onto the tetrahedron

$$0 \le t_i \le 1, \quad |t_1 - t_2| \le t_3 \le t_1 + t_2, \quad \sum t_i \le 2.$$

Two more remarks about this seemingly trivial example. Firstly, the singularities of $\mathcal{M}_{0,3}$ are those of the tetrahedron (i.e. the boundary points). Secondly, as the conjugacy classes of the monodromy along the boundary components are precisely given by the θ_i's, the symplectic leaves are just points here: the Poisson structure is trivial.

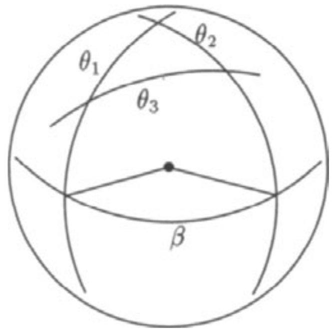

Figure 14

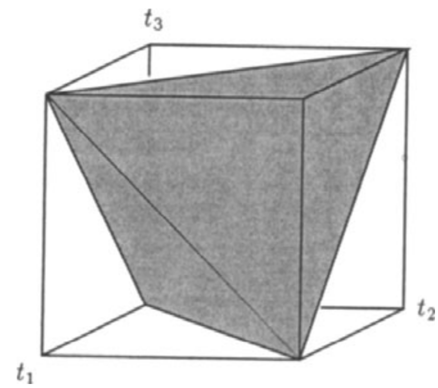

Figure 15: $\mathcal{M}_{0,3}(SU(2))$

Goldman's functions. — Let us turn to Goldman's functions in this case. The only invariant function is the trace, which defines a function f_C and a flow on $\mathcal{M}$ for any curve C. The first remark is a very special case of Proposition 3.1.1:

Proposition 3.3.3 *All the orbits of the flow of f_C are periodic.*

Proof The stabiliser of a generic element of $SU(2)$ is a circle. □

Remark. — The Hamiltonian system defined by the function f_C on $\mathcal{M}$ is *superintegrable*, in the sense that there are many ("count" the curves not intersecting C!) functions commuting with f_C (of course they do not pairwise commute). The trajectories of the flow lie on the common level sets of all these functions. As there are enough functions, these common level sets are 1-dimensional. A good reason for the flow to be periodic.

Now we normalise so that this periodicity corresponds to an S^1-action. Define $h_C : \mathcal{M} \to [0,1]$ by

$$h_C([\rho]) = \frac{1}{\pi} \arccos\left(\frac{1}{2} \operatorname{tr} \rho(C)\right)$$

(up to the factor $1/\pi$, this is just the θ in a matrix corresponding to C as above, namely, $e^{i\theta}$ is an eigenvalue). Let $\mathcal{U}_C \subset \mathcal{M}$ be the open subset consisting of all the classes of representations $\rho : \pi_1 S \to SU(2)$ such that $\rho(C) \neq \pm \operatorname{Id}$, so that h_C is smooth on $\mathcal{U}_C$ and $h_C(\mathcal{U}_C) \subset (0,1)$.

Proposition 3.3.4 *The Hamiltonian $h_C : \mathcal{U}_C \to \mathbf{R}$ is periodic.*

Proof This is a quantitative version of 3.1.1 and 3.3.3 and a special case of Goldman's result described in § 3.1. The function h_C is the Hamiltonian of the S^1-action

$$(z \cdot \rho)(\gamma) = \begin{cases} \rho(\gamma) & \text{if } \gamma \cap C = \varnothing \\ \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \rho(\gamma) & \text{if } \gamma \cdot C = 1 \end{cases}$$

in the case where the curve C does not disconnect the surface. When $S - C$ has two components S_1 and S_2 say, if $\gamma \cap C = \varnothing$, $(z \cdot \rho)(\gamma) = \rho(\gamma)$ if γ is homotopic to a curve in S_1 say, and $(z \cdot \rho)(\gamma)$ is the conjugate $\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \rho(\gamma) \begin{pmatrix} \bar{z} & 0 \\ 0 & z \end{pmatrix}$ if γ is homotopic to a curve in S_2. □

Trinion decompositions and torus actions. — Let us now fix $3g+d-3$ curves on the surface S as in Figure 13. Together with the d boundary components of S, this gives us a set $\mathcal{C}$ of $3g + 2d - 3$ curves which divide the surface into $2g + d - 2$ trinions (see Figure 13). To each curve $C \in \mathcal{C}$, associate the corresponding θ_C, h_C. Let $h : \mathcal{M} \to \mathbf{R}^{\mathcal{C}} = \mathbf{R}^{3g+2d-3}$ be the map whose components are the h_C ($C \in \mathcal{C}$). To each trinion in the decomposition is associated a tetrahedron in the subspace $\mathbf{R}^3 \subset \mathbf{R}^{\mathcal{C}}$ corresponding to its three boundary curves. Let

$$\mathcal{U}_{\mathcal{C}} = \bigcap_{C \in \mathcal{C}} \mathcal{U}_C.$$

Proposition 3.3.5 *The restriction to $\mathcal{U}_{\mathcal{C}}$ of the mapping*

$$h : \mathcal{M} \longrightarrow \mathbf{R}^{\mathcal{C}}$$

is the momentum mapping for a $T^{3g+2d-3}$-action on the Poisson manifold $\mathcal{U}_{\mathcal{C}} \subset \mathcal{M}_{g,d}$. The closure of the image of h is the convex compact polyhedron $P_{\mathcal{C}}$ obtained by taking the product of the tetrahedra corresponding to the trinions defined by $\mathcal{C}$ and intersecting it with the hyperplanes corresponding to the gluing of two trinions.

Proof That the image is included in the polyhedron $P_{\mathcal{C}}$ is clear. We have only to check that all the interior points are actually obtained. Now any point of $P_{\mathcal{C}}$ gives a point in all the tetrahedra corresponding to the trinions defined by $\mathcal{C}$, that is, according to 3.3.2, a family of representations of the fundamental groups of the various trinions, or a gauge class of flat connections on each trinion (one even constructs local sections of the momentum mapping this way). One then checks that there exist either morphisms $\pi_1 S \to SU(2)$ or flat connections on S that extend the given data. □

Remarks. — This also proves that the Goldman functions under consideration are independent, as $P_{\mathcal{C}}$ is a polytope with non-empty interior.

Notice also that Proposition 3.3.5 is not an application of the convexity theorem (§ 1.2) even when restricted to the intersection of a symplectic leaf with $\mathcal{U}_{\mathcal{C}}$, as this is not a compact smooth symplectic manifold.

Let me make this remark more precise. Choose $t = (t_1, \ldots, t_d)$, $t_i \in [0, 1]$ such that $\sum \varepsilon_i t_i \not\equiv 0 \bmod 2$ and cut the polyhedron by the affine subspace defined by t (corresponding to the boundary curves). This gives a compact convex polyhedron $P_{\mathcal{C}}^t$ which is the image of the compact smooth symplectic leaf $\mathcal{M}^t$ under the mapping h. However, nobody claims that $P_{\mathcal{C}}^t$ is the image of the momentum mapping of a torus action on $\mathcal{M}^t$: this is the image of h and also the closure of the image of the momentum mapping for a torus action on $\mathcal{M}^t \cap \mathcal{U}_{\mathcal{C}}$ (which is, indeed, the restriction of h). Obviously, if the genus of S is large enough, there can be some matrices equal to $\pm\,\mathrm{Id}$ in an irreducible representation of $\pi_1 S$.

In the sequel of this section, I will discuss some examples, in order to explain the meaning of this result.

Examples. — Notice that Proposition 3.3.2 is a special case of 3.3.5, the case of $\mathcal{M}_{0,3}$. As the Poisson structure is trivial, a diffeomorphism can be a momentum mapping.

Consider now (once again) our second favorite example, the torus with one hole, that is, the space $\mathcal{M}_{1,1}$. The moduli space is 3-dimensional, as is $\mathcal{M}_{0,3}$. But now, the momentum mapping is for a T^2 -action and takes its values in $\mathbf{R}^2$. With the notations defined by Figure 16, the image of the momentum mapping is a triangle (intersection of the basic tetrahedron with the plane $t_1 = t_2$). The symplectic leaves are defined by t_3. Notice that all of them are smooth surfaces, provided $t_3 \neq 0, 1$. As this implies that $t_1 = t_2 \in (0,1)$, $\mathcal{U}_{\mathcal{C}} \cap \mathcal{M}^t = \mathcal{M}^t$. Thus the symplectic leaves are indeed endowed with a Hamiltonian S^1-action, and so are 2-spheres, the image of the corresponding momentum mapping is the intersection of the triangle with the line $t_3 =$ const (Figure 16).

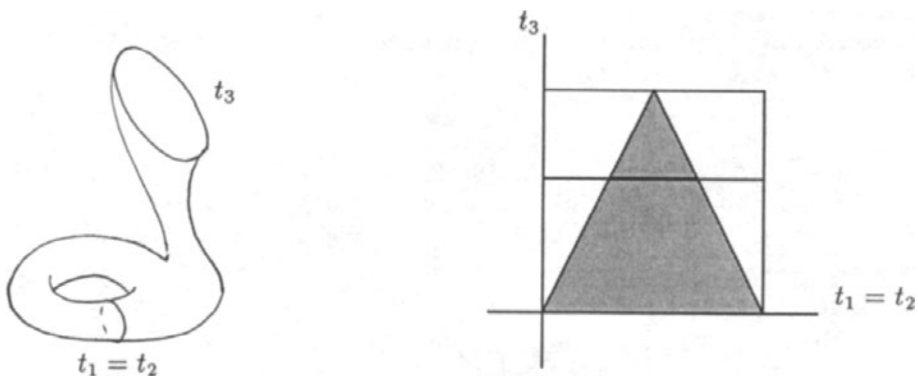

Figure 16: the moduli space $\mathcal{M}_{1,1}(SU(2))$

The next example is a torus with two holes, that is, the moduli space $\mathcal{M}_{1,2}$. Using the trinion decomposition shown in Figure 17 and Proposition 3.3.5, one gets a momentum mapping whose image is a 4 dimensional polytope. To obtain a symplectic leaf, fix the numbers t_3 and t_3' corresponding to the boundary components. The leaf is then mapped onto the rectangle shown in Figure 17 (and, again according to Delzant's theorem [15], this is an $S^2 \times S^2$).

Dependence on the decomposition. — There can exist many different trinion decompositions for a given surface. Consider the simple example of a closed genus 2 surface ($g = 2$, $d = 0$). It has two trinion decompositions as shown in Figure 18, which also shows the closures of the images of the corresponding momentum mappings $h : \mathcal{M}_{2,0} \to \mathbf{R}^3$. The two polytopes are quite different. For instance the pyramid cannot be obtained as the image of the momentum mapping of a torus action on a smooth compact symplectic manifold as it has a vertex with valency 4, which is forbidden by the linearisation theorem for torus actions.

Let me make a last remark on these two examples: the tetrahedron in Figure 18 turns out to be the same as the one in Figure 1. To compare them, we must be rather careful with the integral lattices the tori define in $\mathbf{R}^3$. In Figure 1, this is the standard integral lattice in

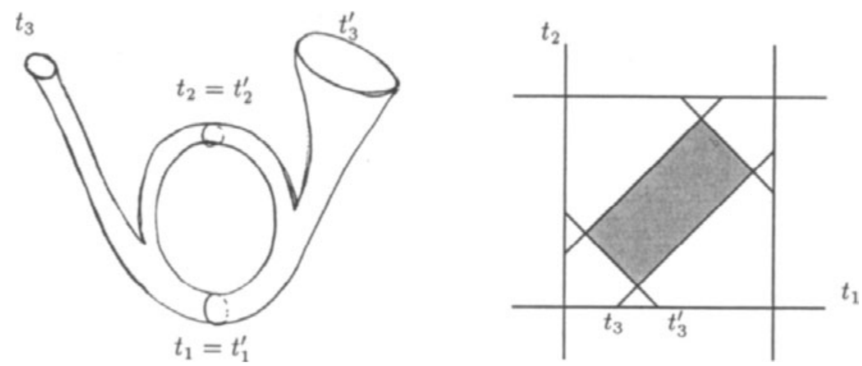

Figure 17: a symplectic leaf in the moduli space $\mathcal{M}_{1,2}(SU(2))$

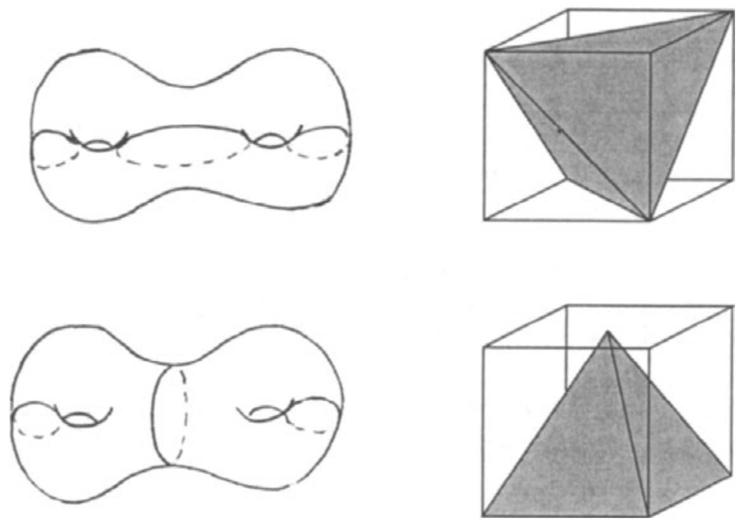

Figure 18: torus actions on $\mathcal{M}_{2,0}$

$\mathbf{R}^3$. The T^3-action on $\mathcal{M}_{2,0}$ I have written is not effective (see [28]), so that the lattice consider in Figure 18 is... precisely the one generated by the vertices of the tetrahedron.

For the sake of completeness, let me recall that, following [33], $\mathcal{M}_{2,0}(SU(2))$ is homeomorphic with $\mathbf{CP}^3$.

Applications. — The system on $\mathcal{M}_{g,d}$ and the torus action on $\mathcal{U}_{\mathfrak{C}}$ may help to understand the Poisson and/or symplectic geometry of the moduli space, as I have explained in the previous examples.

As a first application, Proposition 3.3.5 and its symplectic specialisations make it very easy to compute the volume of the symplectic moduli spaces, as this is simply the volume of the image polytope (with respect to the *ad hoc* latti ce, see § 1.2). For instance, an argument by induction on the genus g which is based on the decomposition of a surface with one hole shown in Figure 19 and on the volume of the rectangle of Figure 17 allows J effrey and

Weitsman to give a simple proof of a theorem of Donaldson's, which states that the volume of the symplectic leaf defined by t in $\mathcal{M}_{g,1}$, as a function of t, is a Bernoulli polynomial (see [29]).

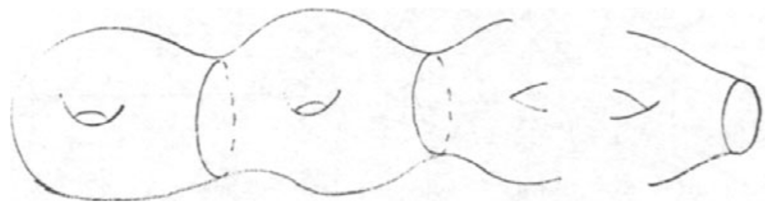

Figure 19

The torus actions on the moduli spaces are also used by Jeffrey and Weitsman to understand the cohomology of $\mathcal{M}$. Consider in particular the case of a closed surface, so that $\mathcal{M}$ is symplectic. They compute the number of integral points in $mP_{\mathcal{C}}$ (this is where you have to be very careful with the definition of the integral lattice—which I was not). It turns out that this number is given by the Verlinde formula[10]. Now this can be interpreted in the following way. The torus action defines a complex structure on $\mathcal{U}_{\mathcal{C}}$, and being integral, the symplectic form defines a complex line bundle $\mathcal{L}$ on $\mathcal{U}_{\mathcal{C}}$. Now, if we were in ordinary toric geometry (i.e. with complete complex toric manifolds), the dimension of the space of holomorphic sections of $\mathcal{L}^{\otimes m}$ would be the number of integral points in $mP_{\mathcal{C}}$ and thus would also be given by the Verlinde formula! (see [27] for the detail).

3.4 The Ruijsenaars system, according to Fock

In this section, I will concentrate on the moduli space $\mathcal{M}_{1,1}(SL(n;\mathbf{C}))$. As we are on a torus with one hole, Goldman gives us, on any symplectic leaf, $n-1$ commuting functions ($n-1 = \operatorname{rk} SL(n;\mathbf{C})$), corresponding to one curve and to the invariant functions $\operatorname{tr} A^m$ ($1 \leq m \leq n-1$). This will (hopefully) give an integrable system as soon as we find a symplectic leaf of dimension $2(n-1)$. This is the same idea we have used to introduce the example of the Toda lattice in § 1.4: count the functions and look for a small enough symplectic leaf.

The symplectic manifold. — Fix a complex number x, put $y = x^{-(n-1)}$ and consider the conjugacy class C_x of the diagonal matrix $(x,\dots,x,y)$ in $SL(n;\mathbf{C})$. It defines a symplectic leaf

$$\mathcal{M}^x = \left\{(A,B) \in SL(n;\mathbf{C}) \times SL(n;\mathbf{C}) \mid ABA^{-1}B^{-1} \in C_x\right\} / SL(n;\mathbf{C})$$

in our moduli space.

[10] So that in particular, it does not depend on the actual trinion decomposition chosen!

A simple way to compute the dimension of $\mathcal{M}^x$ is to solve the equation

$$ABA^{-1} - B^{-1} \in C_x,$$

that is,

$$\operatorname{rk}\left(ABA^{-1} - xB\right) \leq 1.$$

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Write everything in a basis relative to which A is diagonal so that, if $B = (b_{i,j})$,

$$ABA^{-1} - xB = \left(\left(\lambda_i\lambda_j^{-1}\right) b_{i,j} - xb_{i,j}\right).$$

This will have rank ≤ 1 as soon as all its 2×2 minors vanish and one can solve

$$b_{i,j} = \frac{\sqrt{b_{i,i}b_{j,j}}(1-x)}{\lambda_i\lambda_j^{-1} - x}$$

so that $(\lambda_1, \ldots, \lambda_n; b_{1,1}, \ldots, b_{n,n})$ with $\prod \lambda_i = 1$ and $\det((b_{i,j})) = 1$ are local coordinates on $\mathcal{M}^x$. In particular,

$$\dim \mathcal{M}^x = 2(n-1)$$

and $\lambda_1, \ldots, \lambda_{n-1}$ are independent functions, so that the $\operatorname{tr} A^m$ $(1 \leq m \leq n-1)$ are also independent.

Remark. — Induction on n allows us to compute the determinant:

$$\det B = x^{n(n-1)/2} \prod_i b_{i,i} \prod_{j \neq i} \frac{\lambda_i - \lambda_j}{x\lambda_i - \lambda_j}.$$

The integrable system. — We have $n-1$ commuting functions on the $2(n-1)$ dimensional symplectic leaf $\mathcal{M}^x$, the $\operatorname{tr} B^m$ for $1 \leq m \leq n-1$ (but we don't know that they are independent). Using the formulas for the Poisson bracket, Fock computes the brackets of the coordinate functions and eventually finds that

- $\{\lambda_i, \lambda_j\} = 0$, this is obvious: the eigenvalues of A must commute as the $\operatorname{tr} A^m$ commute,
- $\{b_{i,i}, b_{j,j}\} = \dfrac{b_{i,i}b_{j,j}(\lambda_i + \lambda_j)}{\left(\lambda_i\lambda_j^{-1} - x\right)\left(\lambda_j\lambda_i^{-1} - x\right)(\lambda_i - \lambda_j)}$,
- $\{\lambda_i, b_{j,j}\} = \lambda_i b_{j,j}\delta_{i,j}$.

Then, to get coordinates conjugate to the eigenvalues λ_i, one can multiply $b_{i,i}$ by an *ad hoc* factor. It turns out that

$$s_i = x^{(n-1)/2}\left(\prod_{k \neq i} \frac{(\lambda_k - \lambda_i)(\lambda_i - \lambda_k)}{(\lambda_k - x\lambda_i)(\lambda_i - x\lambda_k)}\right)^{\frac{1}{2}} b_{i,i}$$

gives

$$\begin{aligned} \{s_i, s_j\} &= 0 \\ \{\lambda_i, s_j\} &= \lambda_i s_j \delta_{i,j}. \end{aligned}$$

Using the formula for $\det B$ above, one finds that the function (the Hamiltonian) $H(A, B) = \operatorname{tr}(B + B^{-1})$ can be written, in terms of the new coordinates, as

$$H = \sum_i (s_i + s_i^{-1}) x^{(n-1)/2} \left(\prod_{k \neq i} \frac{(\lambda_k - \lambda_i)(\lambda_i - \lambda_k)}{(\lambda_k - x\lambda_i)(\lambda_i - x\lambda_k)} \right)^{\frac{1}{2}}.$$

This is known as Ruisjenaars' Hamiltonian (see [36], where s_i is written e^{θ_i}), a generalisation of the Calogero system.

A bibliographical guide

In addition to the references given in the text, I give here some (mainly basic) references, without trying to be exhaustive.

On symplectic and Poisson geometry. — The basic references I know best on symplectic geometry are [4], [8] and [11] (advertisement). On both symplectic and Poisson geometry, see [31]. On Poisson manifolds, see [39] and the references given there. Specifically on Poisson Lie groups, see [32] and [38].

On integrable systems. — There are plenty[11] of papers on the subject. I highly recommend the survey [35] and the collection of papers [13]. Examples of applications of the r-matrix approach to the topology of some mechanical systems are discussed in [10].

On bundles, connections and holonomy. — See [16].

On moduli spaces. — See [7], [22], [23], [6].

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