Two notions of integrability

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Je me souviens qu'Andrei
admirait Kowalevskaya

Abstract. We investigate the relation between two notions of integrability, to have enough first integrals on the one hand, and to have meromorphic solutions on the other, that are present in Kowalevskaya's famous mémoire on the rigid body problem. We concentrate on the examples of the rigid body and of the system of Hénon–Heiles.

Contents

I remember Andrei sitting in my office in Strasbourg and telling me his admiration for Sofia Kowalevskaya. I am not sure whether this was because I am a woman, or because he knew I had written a long paper on the "Kowalevski top" [7] or simply because he was very enthusiastic over her brilliant and romantic personality. I would have been very happy to discuss the topic of the present paper with him.

It is with deep sadness, thinking both of Sofia and Andrei, that I reproduce (in Figure 1), the very first page of the beautiful paper [13] on the rigid body. This paper is rather famous. This is the work for which her author won the Bordin prize of the French Academy of Sciences in 1888. It seems to me that this is still very interesting and modern, in particular because of the two notions of integrability that appear here. I hope that the present paper will participate in showing how seminal and important is Kowalevskaya's contribution to the theory of integrable systems (see e.g. [8] for other aspects of her work).

SUR LE PROBLÈME DE LA ROTATION

D'UN CORPS SOLIDE AUTOUR D'UN POINT FIXE [1]

PAR

SOPHIE KOWALEVSKI

À STOCKHOLM.

§ 1.

Le problème de la rotation d'un corps solide pesant autour d'un point fixe peut se ramener, comme on sait, à l'intégration du système d'équations différentielles suivant:

$$
(1) \qquad \begin{aligned}
A\frac{dp}{dt} &= (B - C)qr + Mg(y_0\gamma'' - z_0\gamma'), & \frac{d\gamma}{dt} &= r\gamma' - q\gamma'',\\
B\frac{dq}{dt} &= (C - A)rp + Mg(z_0\gamma - x_0\gamma''), & \frac{d\gamma'}{dt} &= p\gamma'' - r\gamma,\\
C\frac{dr}{dt} &= (A - B)pq + Mg(x_0\gamma' - y_0\gamma), & \frac{d\gamma''}{dt} &= q\gamma - p\gamma'.
\end{aligned}
$$

Les constantes $A, B, C, Mg, x_0, y_0, z_0$ qui figurent dans ces équations ont la signification suivante.

A, B, C sont les axes principaux de l'ellipsoïde d'inertie du corps considéré, relativement au point fixe.

M est la masse du corps;

g l'intensité de la force de gravité;

[1] Ce mémoire est le résumé d'un travail auquel l'Académie des Sciences de Paris, dans sa séance solennelle du 24 décembre 1888, a décerné le prix Bordin élevé de 3000 à 5000 francs.

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Figure 1. The first page of [13].

I will concentrate on a very small part of the mathematics contained in [13] and limit myself to a discussion of these two notions of integrability and to try to understand their relations. Let me first explain what these two notions are. Here I will only deal with complex analytic Hamiltonian systems. The two properties are the following:

(K) The singularities of the solutions are poles (here (K) is for Kowalevskaya[1]).

(L) There are enough first integrals (here (L) is for Liouville).

They are not logically related in the sense that (as far as I know) none implies the other. However, in many significant examples, one is satisfied if and only if the other one is. Moreover, it may happen that some systems are suspected *not* to be integrable (in the Liouville sense) because they do not satisfy (K); this is the case for instance of the Hénon–Heiles system (see the discussion in [17]).

To make things more tractable, I will use "soft" versions of the two properties (K) and (L), replacing the differential (Hamiltonian) system by a *linear* differential system, the variational equation along a previously chosen particular solution:

(H) The monodromy around the singularities is trivial (here (H) is for Haine).

(MR) The Galois group is virtually Abelian (here (MR) is for Morales–Ramis).

I will discuss this more precisely (and give the appropriate definitions and explanations) on a few examples. I will of course mainly focus on the examples studied in the seminal paper [13]. The motivations for looking at these examples and at this paper are numerous. Among them:

- This is the place where the relation between the two notions of integrability I want to discuss appears for the first time.
- The problem of the rigid body is a most classical problem and I believe that we should study classical problems.

Acknowledgements. I thank all the people who made comments on previous versions of this paper, especially those who attended the talks I gave on this work in Strasbourg and in Reykjavik, and the referee for his nice but useful comments.

I am deeply grateful to Andrzej Maciejewski for his careful reading and his hunting of misprints and more serious errors.

1 The rigid body

1.1 The differential system

In the first page of her paper [13], Sofia Kowalevskaya writes the differential system describing the motion of a rigid body with a fixed point in a constant gravitation field. The equations are written in a frame which is attached to the body, the relative frame.

[1] Notice that this property is often called the "Kowalevski–Painlevé property" by the people working on integrable systems, probably in reference with the "Painlevé property" which is, for a differential equation, the fact that its *mobile* singularities are poles.

$$
\begin{aligned}
A\frac{dp}{dt} &= (B - C)qr + Mg(y_0\gamma'' - z_0\gamma'), & \frac{d\gamma}{dt} &= r\gamma' - q\gamma'',\\
B\frac{dq}{dt} &= (C - A)rp + Mg(z_0\gamma - x_0\gamma''), & \frac{d\gamma'}{dt} &= p\gamma'' - r\gamma,\\
C\frac{dr}{dt} &= (A - B)pq + Mg(x_0\gamma' - y_0\gamma), & \frac{d\gamma''}{dt} &= q\gamma - p\gamma'.
\end{aligned}
$$

Figure 2. The differential system.

The quantities M and g which appear in these equations are the mass of the body and a gravitational constant; we shall choose the units so that $Mg = 1$. The vector

$$\Gamma = \begin{pmatrix}\gamma\\ \gamma'\\ \gamma''\end{pmatrix}, \text{ that I prefer to write } \begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix}$$

is the gravity field. Let me call M the total angular momentum and Ω the angular velocity, two vectors that are related by $M = \mathcal{J}\Omega$, for a symmetric definite positive matrix $\mathcal{J}$, the inertia matrix (which reflects the shape of the body). Here[2],

$$\Omega = \begin{pmatrix}p\\ q\\ r\end{pmatrix}, \text{ and } \mathcal{J} = \begin{pmatrix}A & 0 & 0\\ 0 & B & 0\\ 0 & 0 & C\end{pmatrix}, \text{ so that } M = \begin{pmatrix}Ap\\ Bq\\ Cr\end{pmatrix}.$$

Notice (see [2, p. 141]) that A, B and C must satisfy the triangle inequalities

$$A + B \geq C, \quad B + C \geq A, \quad C + A \geq B$$

(equality holding if and only if the body is planar). The center of gravity G and the fixed point O are related by the constant vector

$$\overrightarrow{OG} = L = \begin{pmatrix}x_0\\ y_0\\ z_0\end{pmatrix}.$$

Denoting with a dot the derivation with respect to time, our differential system can also be written in a more compact form:

$$\dot{M} = M \times \Omega + L \times \Gamma, \quad \dot{\Gamma} = \Gamma \times \Omega.$$

[2] The symmetric matrix $\mathcal{J}$ is diagonalizable, so that we will assume that it is diagonal. Notice that the fact that the symmetric real matrices are diagonalizable was proven by Lagrange [14]... because he needed it to deal with the inertia matrix of the spinning top.

1.2 What was known before Kowalevskaya's paper

Let us now turn the page. Sofia Kowalevskaya reminds us (Figure 3) that, at that time, *jusqu'à présent*[3], it has been possible to solve these equations only in two special cases:

> Jusqu'à présent on n'était parvenu à intégrer ces équations que dans deux cas particuliers:
> 1) Le cas de Poisson (ou d'Euler) où l'on a $x_0 = y_0 = z_0 = 0$,
> 2) Le cas de Lagrange où l'on a $A = B$, $x_0 = y_0 = 0$.
> Dans ces deux cas l'intégration s'opère à l'aide des fonctions $\vartheta(u)$ dont l'argument est une fonction entière linéaire du temps.
> Les six quantités $p, q, r, \gamma, \gamma', \gamma''$ sont dans ces deux cas des fonctions uniformes du temps, n'ayant d'autres singularités que des pôles pour toutes les valeurs finies de la variable.

Figure 3

- the Euler–Poisson case, where $O = G$ (the center of gravity is the fixed point);
- the Lagrange case, in which the body rotates about an axis of revolution (the vector $\overrightarrow{OG}$ is the third vector of an orthonormal basis in which the constants (A, B, C) have the form (A, A, C)).

She notices that, in these two cases, the solutions can be written in terms of ϑ-functions. In (slightly more) geometric terms, this means that there is an elliptic curve present and that the solutions are linear on this curve. More importantly, she writes "the six quantities $p, q, r, \gamma, \gamma', \gamma''$ are in these two cases uniform functions of time, having no other singularities than poles for all finite values of the variable". And (Figure 4)

> Les intégrales des équations différentielles considérées conservent-elles cette propriété dans le cas général?

Figure 4

"Do the solutions of the differential equations still have this property in the general case?"

Remarks. (1) Notice that the differential system under consideration is non linear. Its solutions can thus have very complicated singularities (essential singularities, branching, logarithms,...) that may depend on the initial conditions and not only on the coefficients.

[3]Her French was much better than the so called English used nowadays in the mathematical papers (in the present one, for instance). This was the language she used, for instance, in her correspondence with Mittag-Leffler (see [8]).

(2) Moreover, if one considers time as a real variable (which is what people were doing until Kowalevskaya), there are no singularities at all: the energy is a proper function, so that nothing goes to infinity, the motion takes place on compact manifolds.

(3) This is the reason why there was no progress on the problem during about one century: to do something new, it was necessary to use complex analysis. And Sofia Kowalevskaya was one of the best specialists of this (new) topic at that time.

Then she computes and concludes that "this property", namely what we call property (K), is satisfied only in the two cases mentioned above… and in a new case, which is now called the "Kowalevski top", this is the case where

$$A = B = 2C, \quad z_0 = 0,$$

there is an axis of revolution but this is *not* the axis $\overrightarrow{OG}$, as this was the case for the Lagrange top, the axis is rather orthogonal to $\overrightarrow{OG}$.

1.3 The property (L)

Now comes the best. Let me explain this in modern geometric terms. The differential system is Hamiltonian. To avoid technicalities and the introduction of Poisson structures, let me just concentrate on the submanifold

$$W_{2\ell} = \left\{(\Gamma, M) \in \mathbb{C}^6 \mid \|\Gamma\|^2 = 1 \text{ and } \Gamma \cdot M = 2\ell\right\}.$$

This is a 4-dimensional manifold. The two equations correspond to the obvious fact that the intensity $\|\Gamma\|$ of the gravitation field is fixed (a consequence of the fact that Γ is constant in the absolute frame) and that $\Gamma \cdot M$ must also be conserved in application of the "law of areas". The manifold $W_{2\ell}$ carries a natural symplectic structure (see for instance [3]), we can consider the total energy[4]

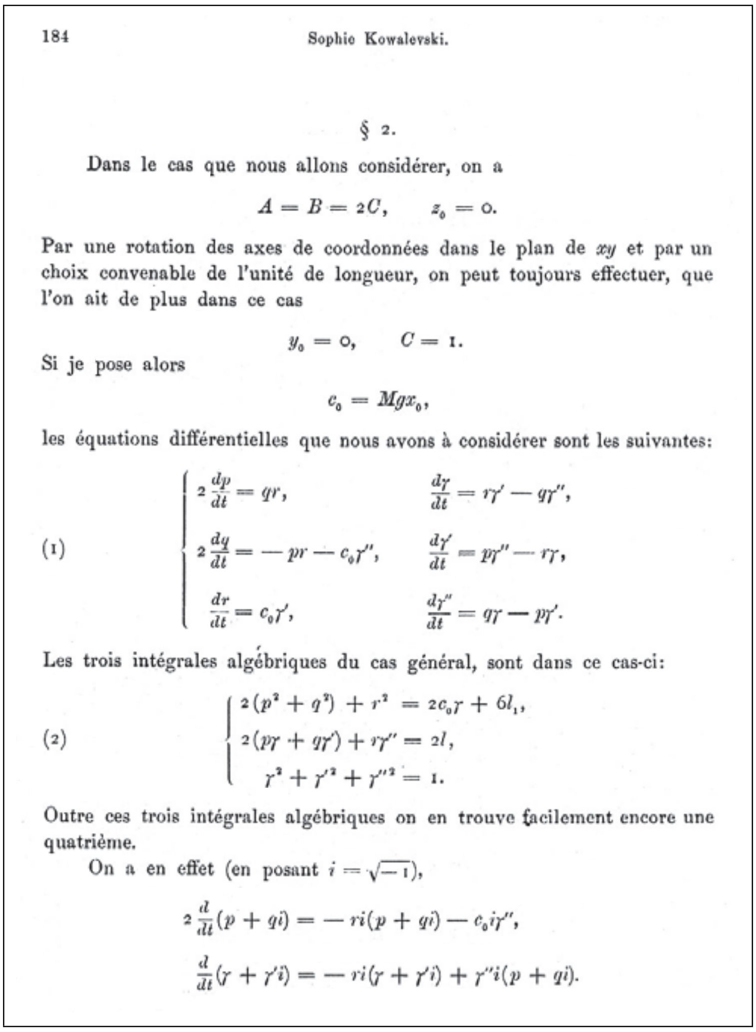
$$H(\Gamma, M) = \frac{1}{2} M \cdot \Omega - \Gamma \cdot L$$

as a function on $W_{2\ell}$ and the differential system under consideration is the corresponding Hamiltonian system. So that the total energy is also a conserved quantity (this is called $3\ell_1$ in Kowalevskaya's paper, a notation visible on Figure 5).

Notice that we have a Hamiltonian system on a 4-dimensional manifold. And remember that what Kowalevskaya wants to do is *to solve* the equations. So that she notices, incidentally, that, "in addition to these three algebraic integrals, it is easy to find a fourth one". What she was interested in was to *solve* the equations. Then she uses these four conserved quantities to actually solve the system in her case. This takes the rest of the paper. She writes the solutions, explicitly, in terms of ϑ-functions associated with a genus-2 curve.

Of course, what she notices is that the Hamiltonian system is what we call nowadays "Liouville integrable" in her case[5]. We have a Hamiltonian system with two degrees

[4] We adopt here the sign convention used in [13].



184 Sophie Kowalevski.

§ 2.

Dans le cas que nous allons considérer, on a

$$A = B = 2C, \qquad z_0 = 0.$$

Par une rotation des axes de coordonnées dans le plan de xy et par un choix convenable de l'unité de longueur, on peut toujours effectuer, que l'on ait de plus dans ce cas

$$y_0 = 0, \qquad C = 1.$$

Si je pose alors

$$c_0 = Mgx_0,$$

les équations différentielles que nous avons à considérer sont les suivantes:

$$(1) \qquad \begin{cases} 2\dfrac{dp}{dt} = qr, & \dfrac{d\gamma}{dt} = r\gamma' - q\gamma'', \\ 2\dfrac{dq}{dt} = -pr - c_0\gamma'', & \dfrac{d\gamma'}{dt} = p\gamma'' - r\gamma, \\ \dfrac{dr}{dt} = c_0\gamma', & \dfrac{d\gamma''}{dt} = q\gamma - p\gamma'. \end{cases}$$

Les trois intégrales algébriques du cas général, sont dans ce cas-ci:

$$(2) \qquad \begin{cases} 2(p^2 + q^2) + r^2 = 2c_0\gamma + 6l_1, \\ 2(p\gamma + q\gamma') + r\gamma'' = 2l, \\ \gamma^2 + \gamma'^2 + \gamma''^2 = 1. \end{cases}$$

Outre ces trois intégrales algébriques on en trouve facilement encore une quatrième.

On a en effet (en posant $i = \sqrt{-1}$),

$$2\frac{d}{dt}(p + qi) = -ri(p + qi) - c_0 i\gamma'',$$

$$\frac{d}{dt}(\gamma + \gamma' i) = -ri(\gamma + \gamma' i) + \gamma'' i(p + qi).$$

Figure 5. Liouville integrability in [13].

of freedom (that is, on a 4-dimensional symplectic manifold), so that the Kowalevski integral

$$K = \left|(p + iq)^2 + (\gamma_1 + i\gamma_2)\right|^2$$

makes it an integrable system in the Liouville sense. She does not mention this property of "Liouville integrability" as such, but of course she uses the first integrals to solve the equations.

Let me add that the two special cases already solved before [13] are of course Liouville integrable, since

- in the Euler–Poisson case, $K = \|M\|^2$ is a second first integral;
- in the Lagrange case, the momentum with respect to the axis of revolution, $K = M \cdot L$, is obviously a conserved quantity.

[5] The readers can look at [4] for instance.

1.4 What is integrability?

What Sofia Kowalevskaya showed in her paper is that, for the rigid body problem, if Property (K) is satisfied, then we are in cases where Property (L) is also satisfied (and this allows to integrate the Hamiltonian system). Hence, (K) implies (L) here.

This raises a few questions (Notice that the title of this paragraph is borrowed to [19], a book where the questions raised here are investigated):

(1) Does (K) implies (L) more generally?

(2) Does (L) implies (K) for this system?

(3) Does (L) implies (K) in general?

(4) In which cases of the rigid body is (L) satisfied?

Let us look now at the answers we know to these questions in the case of the rigid body.

Theorem 1.1 (Husson, Ziglin, Maciejewski and Przybylska). *If the rigid body is integrable in the Liouville sense, then*

- *either we are in the Euler, Lagrange or Kowalevskaya case,*
- *or* $A = B = 4C$, $z_0 = 0$ *and there is an additional first integral on* W_0.

The additional case is the Goryachev–Chaplygin case, where, only on the manifold W_0 (that is, for $2\ell = 0$),

$$K = r(p^2 + q^2) + p\gamma_3$$

is a first integral. The fact that the three and half mentioned cases are the only ones in which the system has an *algebraic* additional integral has been proved by Husson [10]. With a *meromorphic* first integral, it has been proved by Ziglin [21] using techniques which were invented by him [20] and are close to the ones I want to discuss here, namely properties of the monodromy groups of the (linearized) differential system. An alternative "Galoisian" proof has been given recently, using the Morales–Ramis criterion [17], [18], by Maciejewski and Przybylska [16].

I will come back to the Goryachev–Chaplygin case later. Notice that Kowalevskaya does not find this case with her analysis and indeed, there are non meromorphic solutions in this case. This is a Liouville integrable case that does not satisfy the Kowalevskaya condition. Note however that, for the rigid body (K) $\Rightarrow$ (L).

2 Integrability (up to order 1), two criteria

Let us now linearize the problem. We consider a complex analytic Hamiltonian system, that is, an analytic vector field X_H on an open subset of an affine space $\mathbb{C}^m$ which is the Hamiltonian vector field for some function H defined on symplectic submanifolds of $\mathbb{C}^m$.

2.1 The variational equation

We consider a special solution of the (nonlinear in general) differential system

$$\dot{x}(t) = X_H(x(t)).$$

This is a map

$$\begin{aligned} U &\longrightarrow \mathbb{C}^m, \\ t &\longmapsto \varphi^t(x_0), \end{aligned}$$

where U is a Riemann surface (a quotient of an open subset of $\mathbb{C}$), φ^t is the flow of X_H and we are describing here the special solution passing through a given point x_0.

I shall of course use the example of the rigid body, but let me add here a simple (and academic) example.

Example, a Hénon–Heiles system. In this example, the symplectic manifold is the space $\mathbb{C}^4$ itself, endowed with the symplectic form

$$\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$$

and the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}Aq_1^2 - q_1^2 q_2.$$

The Hamiltonian system is

$$\begin{cases} \dot{q}_1 = p_1, \\ \dot{q}_2 = p_2, \\ \dot{p}_1 = -Aq_1 + 2q_1q_2, \\ \dot{p}_2 = q_1^2. \end{cases}$$

Here are two special solutions of this system:

- the Riemann surface is $\mathbb{C} - \{0\}$ and

$$\begin{cases} q_1 = \dfrac{3\sqrt{2}}{t^2}, & p_1 = -\dfrac{6\sqrt{2}}{t^3}, \\ q_2 = \dfrac{3}{t^2} + \dfrac{A}{2}, & p_2 = -\dfrac{6}{t^3}; \end{cases}$$

- the Riemann surface is $\mathbb{C}$ and

$$\begin{cases} q_1 = 0, & p_1 = 0, \\ q_2 = at + b, & p_2 = a \end{cases}$$

  for some constants a and b.

Then, coming back to the general framework, we can linearize the system along the given solution. This can be defined intrinsically (see, for instance [4, § III.1]), but here we are in $\mathbb{C}^m$, so that the Hamiltonian vector field X_H itself can be considered as a map $\mathbb{C}^m \to \mathbb{C}^m$ and can be differentiated. The variational equation along the solution $x(t)$ is simply the linear differential system

$$\dot{y} = (dX_H)_{x(t)} \cdot y.$$

Example, Hénon–Heiles, continuation. Along the solutions given above, using capital letters to denote the variations of the low case letters, the linear system is

$$\begin{cases} \dot{Q}_1 = P_1, & \dot{P}_1 = \dfrac{6}{t^2} Q_1 + \dfrac{6\sqrt{2}}{t^2} Q_2, \\ \dot{Q}_2 = P_2, & \dot{P}_2 = \dfrac{6\sqrt{2}}{t^2} Q_1 \end{cases}$$

for the first special solution and

$$\begin{cases} \dot{Q}_1 = P_1, & \dot{P}_1 = (2at + b - A) Q_1, \\ \dot{Q}_2 = P_2, & \dot{P}_2 = 0 \end{cases}$$

for the second one.

2.2 Haine's criterion for (K)

In a beautiful paper [9] devoted to the investigation of geodesic flows on $\mathrm{SO}(n)$, Haine noticed the following simple and useful property:

Theorem 2.1 (Haine [9]). *If a complex analytic Hamiltonian system on $\mathbb{C}^m$ satisfies the Kowalevski property* (K)*, then the monodromy around the poles of the solutions of the variational equation along any solution is trivial.*

Let us illustrate this theorem by our simple example.

Example, Hénon–Heiles, continuation. This is really an academic example and it was already used in [1]. We consider the special solution above with a pole at 0 (as in Example 2.1) and look at the variational equation given in § 2.1. Changing of unknown functions

$$Q_1 = t^{-2} x_1, \quad Q_2 = t^{-2} x_2, \quad P_1 = t^{-3} y_1, \quad P_2 = t^{-3} y_2$$

gives the new system

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = \frac{1}{t} \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 6 & 6\sqrt{2} & 3 & 0 \\ 6\sqrt{2} & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix}.$$

In order that the solutions be univalued, it is necessary that the differences between the eigenvalues of the matrix be integral. But its characteristic polynomial is

$$\lambda^4 - 10\lambda^3 + 31\lambda^2 - 30\lambda - 72 = (\lambda + 1)(\lambda - 6)(\lambda^2 - 5\lambda + 12),$$

so that the property is obviously not satisfied and our system does not satisfy (K).

Remark. From the point of view of Sofia Kowalevskaya's paper, this system is not fully satisfactory: it is very easy to check that the special solutions I have written (in Example 2.1) are the only ones that have no other singularities that poles. Notice that the two families are very different… and, by the way, that there are no constants of integration in the solution with a pole at 0. In her paper, Kowalevskaya says (Figure 6) that the series giving the solutions in the case of the rigid body should

> séries, pour pouvoir représenter le système général d'intégrales des équations différentielles considérées, devraient contenir *cinq* constantes arbitraires.

Figure 6

contain *five* arbitrary constants. This is what is called a "principal balance" (see the papers in [19]). There are not enough constants of integration in our example: the Hénon–Heiles system has no principal balance. There are not enough meromorphic solutions.

2.3 Morales and Ramis's criterion for (L)

This is a criterion of the same nature, since it also deals with the variational equation. See the original papers [17], [18] and also [4], [5].

Theorem 2.2 (Morales and Ramis [17], [18]). *If a complex analytic Hamiltonian system on $\mathbb{C}^m$ satisfies the Liouville integrability property* (L)*, then the Galois group of the variational equation along any solution is virtually Abelian.*

Recall that to say that an algebraic group is *virtually* Abelian is to say that its neutral component is an Abelian group.

Example, Hénon–Heiles, still more academic. For instance, the variational equation along the second solution exhibited in the Hénon–Heiles examples in § 2.1 reduces to

$$\ddot{Q}_1 = (2at + 2b - A)Q_1,$$

an Airy equation. If we accept to compactify the Riemann surface $\mathbb{C}$ which is our special solution as a complex projective line, our ground field is $\mathbb{C}(t)$ and the Galois group of the Airy equation is SL(2; ℂ) (see [12]), a non Abelian connected group. Hence the Hénon–Heiles system is not Liouville integrable (at least with a rational first integral).

I like this example very much, because this is really a beautiful academic example. Firstly, I have given two completely different arguments for the seemingly different properties (K) and (L). Secondly, it shows that the Galois group is something very rich. Of course, it contains the monodromy group of the variational equation, and hence also its Zariski closure. But, in this simple example, the Riemann surface is simply connected so that there is no monodromy at all. Which does not prevent the Galois group of being huge.

Recall however that, if the variational equation has only regular singularities (which is not the case at infinity in our example), the Galois group contains nothing more than the (closure of the) monodromy group.

2.4 What is integrability? A linear version

This allows us to translate our questions (in § 1.4) into questions relative to the variational equations, namely what is the relation between property (H) and property (MR)?

$$\begin{array}{ccc} \text{Kowaleski property} & \overset{?}{\longleftrightarrow} & \text{Liouville integrability} \\ \text{Haine}\ \Big\Downarrow & & \Big\Downarrow\ \text{Morales–Ramis} \\ \text{trivial monodromy around poles} & \underset{?}{\longleftrightarrow} & \text{virtual commutativity of Galois group} \end{array}$$

Having worked on quite a few examples of applications of the (MR) criterion, I was surprised that, very often, the particular solution used to test this criterion is an elliptic curve minus a certain number of points.

There is a practical reason: when the variational equation has only regular singular points, both criteria are dealing with monodromy groups. The fact that the monodromy about the poles is trivial implies that the monodromy group is Abelian (Figure 7). Hence, if (H) is satisfied, (MR) is.

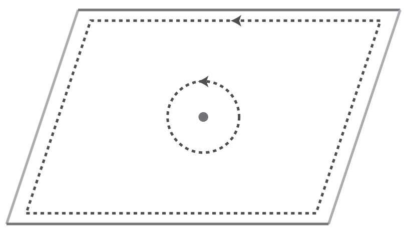


Figure 7

But there is probably a more serious reason. In general, we consider a family of Hamiltonians, H_a, say, depending on a parameter a. For some values of a, we know that the system is integrable. Integrable systems are often algebraically integrable, which means that the common levels of the first integrals are (open subsets of) Abelian varieties, in general coverings of hyperelliptic Jacobians. This is the complex algebraic counterpart of the Liouville tori in (real) symplectic topology. And, of course, the solutions of the Hamiltonian system stay on these Abelian varieties. One expects these varieties to degenerate to cases which contain elliptic curves[6].

3 Comparison of Haine's and Morales–Ramis's criteria

The case of geodesic flows of invariant metrics on the Lie group $SO(N)$ was investigated in [9]. This is related with the rigid body (the case $N = 3$ includes the Euler case). Haine shows that, the cases where (H) is satisfied are exactly those for which we know that the system is Liouville integrable.

The case of the rigid body

The case of the rigid body is slightly more involved, contrarily to what I was believing when I wrote a first version of this paper. I will only consider a special case of rigid body: the body has a rotational symmetry (an axis of revolution), which means that the inertia matrix has a double eigenvalue, and the "axis" $\overrightarrow{OG}$ lies in the "equatorial" plane, the eigenplane corresponding to the double eigenvalue.

Trying to compare (H) and (MR) in this case, I will prove:

Proposition 3.1. *Assume that in the Euler–Poisson equations, $A = B$ and $z_0 = 0$. If the solutions are meromorphic functions of time, then $A/C = 1$ or 2.*

And I will relate the proof to that of Ziglin's theorem (here Theorem 1.1) in this special case, namely:

Theorem 3.2. *Assume that in the Euler–Poisson equations, $A = B$ and $z_0 = 0$. If there exists an additional real meromorphic first integral, then $A/C = 1$, 2 or 4.*

Notice that

- the case $A/C = 1$ is a special case of a Lagrange top,
- the case $A/C = 2$ is the Kowalevski case,
- the case $A/C = 4$ is the Goryachev–Chaplygin case,

hence the three cases are known to be integrable (at least on W_0 for the last one).

[6] I am indebted to the referee for the clarification of this remark.

3.1 Choice of special solutions

We can assume that $x_0 = 1$ and $y_0 = z_0 = 0$ (by a change of coordinates) and that $C = 1$ (by a change of units).

Following Ziglin, we shall consider two families of solutions,

(1) with $p = r = \gamma_2 = 0$;

(2) or with $p = q = \gamma_3 = 0$.

Notice that such solutions will stay on the symplectic manifold W_0 (that is, $M \cdot \Gamma = 0$). The Hamiltonian system reduces to

(1) in the first case,

$$\begin{cases} A\dot{q} = -\gamma_3, \\ \dot{\gamma}_1 = -q\gamma_3, \\ \dot{\gamma}_3 = q\gamma_1; \end{cases}$$

(2) and in the second case

$$\begin{cases} \dot{r} = \gamma_2, \\ \dot{\gamma}_1 = r\gamma_2, \\ \dot{\gamma}_2 = -r\gamma_1. \end{cases}$$

The solutions are supported by

(1) the curve $\mathcal{E}_h$

$$\begin{cases} \gamma_1^2 + \gamma_3^2 = 1, \\ \frac{1}{2}Aq^2 - \gamma_1 = h; \end{cases}$$

(2) the curve $\mathcal{F}_h$

$$\begin{cases} \gamma_1^2 + \gamma_2^2 = 1, \\ \frac{1}{2}r^2 - \gamma_1 = h \end{cases}$$

(h is a parameter, the value of the total energy H on this solution). Our solution curves are thus intersections of two quadrics, hence in general elliptic curves. Figure 8 shows the shape of (the real part of) the curves $\mathcal{F}_h$ as h varies.

The picture for $\mathcal{E}_h$ is completely analogous. Two points at infinity, in the directions $(\gamma_1, \gamma_2, r) = (1, \pm i, 0)$ (or $(\gamma_1, \gamma_3, q) = (1, \pm i, 0)$) are missing on theses curves.

Remark. Notice that both $\mathcal{E}_h$ and $\mathcal{F}_h$ are smooth if and only if $h \neq \pm 1$. For $h = \pm 1$, they have an ordinary double point.

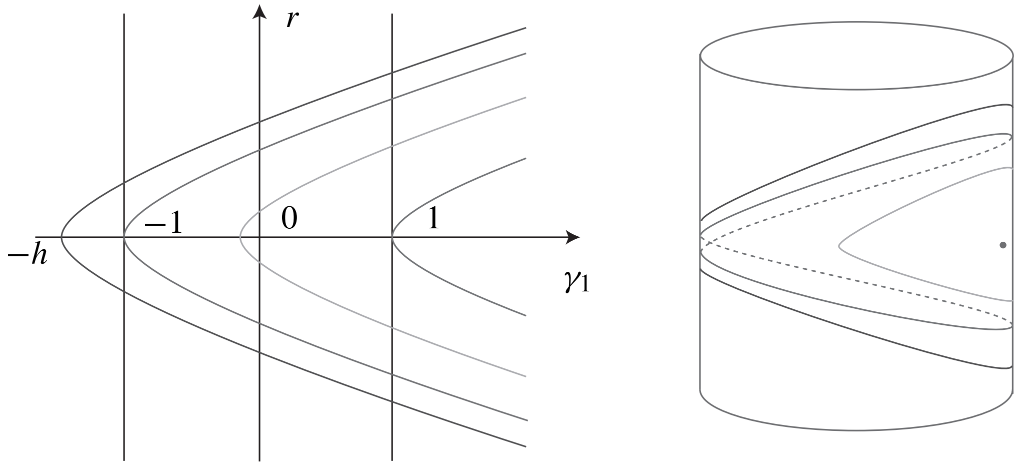


Figure 8. The curves $\mathcal{F}_h$.

3.2 The variational equation

Using as in the example above, a capital letter to denote the variations of the variable denoted by low case letters, the variational equation along a curve $\mathcal{E}_h$ is the linear differential system

$$\begin{cases} A\dot{P} = (A-1)qR, \\ A\dot{Q} = \Gamma_1, \\ \dot{R} = \Gamma_2, \end{cases} \qquad \begin{cases} \dot{\Gamma}_1 = -q\Gamma_3 - \gamma_3 Q, \\ \dot{\Gamma}_2 = \gamma_3 P - \gamma_1 R, \\ \dot{\Gamma}_3 = q\Gamma_1 + \gamma_1 Q. \end{cases}$$

Using the fact that the vector $(P, Q, R, \Gamma_1, \Gamma_2, \Gamma_3)$ is tangent to W_0, namely, the relation

$$A(\gamma_1 P + q\Gamma_2) + \gamma_3 R = 0,$$

the system reduces to the linear equation

$$\ddot{R} + \frac{q\gamma_3}{\gamma_1}\dot{R} + \left(\frac{\gamma_3^2}{A\gamma_1} + \gamma_1\right)R = 0.$$

In a similar way, along a solution of the family $\mathcal{F}_h$, we find

$$\begin{cases} A\dot{P} = (A-1)rQ, \\ A\dot{Q} = (1-A)rP - \Gamma_3, \\ \dot{R} = \Gamma_2, \end{cases} \qquad \begin{cases} \dot{\Gamma}_1 = r\Gamma_2 + \gamma_2 R, \\ \dot{\Gamma}_2 = -r\Gamma_1 - \gamma_1 R, \\ \dot{\Gamma}_3 = \gamma_1 Q - \gamma_2 P. \end{cases}$$

Differentiating the equation for $\dot\Gamma_3$ with respect to t and using the relations between our variables to simplify, we get

$$\begin{aligned}\ddot\Gamma_3 &= r(\gamma_1 P + \gamma_2 Q) + \gamma_1\Big(\frac{1-A}{A} rP - \frac{\Gamma_3}{A}\Big) - \gamma_2\Big(\frac{A-1}{A} rQ\Big)\\ &= \frac{1}{A}\left(r(\gamma_1 P + \gamma_2 Q) - \gamma_1\Gamma_3\right)\\ &= -\frac{1}{A}\Big(\frac{r^2}{A} + \gamma_1\Big)\Gamma_3.\end{aligned}$$

So that we are reduced to the investigation of the linear differential equation of order 2

$$\ddot\Gamma_3 = -\frac{1}{A}\left(\frac{r^2}{A} + \gamma_1\right)\Gamma_3$$

where r and γ_1 are solutions of the non linear system above.

3.3 About the poles

Let us now look at the monodromy around the poles. About a point at infinity of the elliptic curve $\mathcal{E}_h$ (resp. $\mathcal{F}_h$), the solutions of the non linear system have the form

$$\begin{cases}\gamma_1 = -2At^{-2}(1+tg_1(t)),\\ \gamma_3 = 2Ait^{-2}(1+tg_3(t)),\\ r = 2it^{-1}(1+tg_2(t))\end{cases} \quad \text{resp.} \quad \begin{cases}\gamma_1 = -2t^{-2}(1+tf_1(t)),\\ \gamma_2 = -2it^{-2}(1+tf_2(t)),\\ q = 2it^{-1}(1+tf_3(t))\end{cases}$$

for some functions f_i's, g_i's, that are holomorphic at 0.

Remark. Notice that these functions are of course elliptic functions of the time t (they parametrize an elliptic curve), so that we could have written them "explicitly" in terms of the Jacobi functions sn, cn and even dn. I always find it very hard for a geometer to read such formulas. This might be less impressive to write the solutions the way I wrote them here, but since we will not need more, I will content myself with these formulas, that follow directly from the differential system.

Let us look now at the indicial equations for our two linear differential equations. For the first family of solutions (along the curve $\mathcal{E}_h$), this is

$$s^2 + s + 2(1-A) = 0,$$

the difference of the two roots of which is $\sqrt{8A-7}$.

For the second differential equation (along the curve $\mathcal{F}_h$), this is

$$s^2 - s - \frac{2}{A}\Big(\frac{2}{A}+1\Big) = \Big(s + \frac{2}{A}\Big)\Big(s - \frac{2}{A} - 1\Big) = 0.$$

In this case, the two roots are $-2/A$ and $2/A + 1$.

For the monodromy around the two points at infinity of our elliptic curve $\mathcal{F}_h$ to be trivial, it is necessary that the roots of the indicial equations be integers, namely here that $2/A \in \mathbf{Z}$. Recall the triangle inequalities (§ 1) for the eigenvalues of the inertia matrix, which give here the fact that $2A \geq 1$, hence

$$A \in \left\{\frac{1}{2}, \frac{2}{3}, 1, 2\right\}.$$

The difference of the two roots of the first equation, namely $\sqrt{8A-7}$, can be an integer for A in this list only if

$$A \in \{1, 2\}.$$

In view of Haine's criterion, this concludes the proof of Proposition 3.1. □

3.4 More on the monodromy group

According to our program, let us compare this result with what the Morales–Ramis theorem tells us. To make things simpler, let us first notice that we do not need *two* poles. Obviously, the involution

$$\tau : (p, q, r, \gamma_1, \gamma_2, \gamma_3) \longmapsto (p, -q, -r, \gamma_1, -\gamma_2, -\gamma_3)$$

preserves W_0 and the total energy H. Its restriction to $\mathcal{E}_h$, still denoted τ,

$$(\gamma_1, q, \gamma_3) \longmapsto (\gamma_1, -q, -\gamma_3)$$

leaves the $\mathcal{E}_h$ and the system invariant. Moreover, it has no fixed point on the curve if $h \neq \pm 1$ (that is, when the curve is smooth). It exchanges the two points at infinity. Hence the quotient of our elliptic curve minus two points $\mathcal{E}_h$ by this involution is an elliptic curve minus one point $\mathcal{E}'_h$. The same is true of the restriction to $\mathcal{F}_h$, the quotient of which we will denote $\mathcal{F}'_h$ (this reduction was already used by Ziglin in [21]).

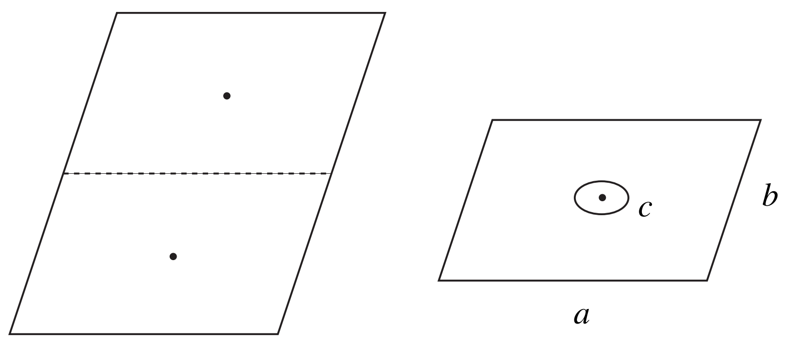


Figure 9

The fundamental group of $\mathcal{E}'_h$ is the free group on two generators a and b (Figure 9). When the monodromy about c is trivial (this is what we have discussed in Proposition 3.1), the monodromy group is Abelian. Notice that our variational equation has only regular singular points.

Corollary 3.3. *If the Galois groups of the variational equation along the curves $\mathcal{E}_h$ and $\mathcal{F}_h$ are both Abelian, then $A/C = 1$ or 2.* □

We have proved here that, in the case under consideration, the Kowalevski property implies that the Galois group is Abelian.

Unfortunately, this is not exactly what we want. Schematically, writing Gal for the Galois group and Gal° for its neutral component, what we have so far is:

$$\begin{array}{ccccc} (\mathrm{K}) & \Longrightarrow & (\mathrm{H}) & \Longleftrightarrow & \mathrm{Gal}\ \text{Abelian} \\ & & & & \Downarrow \\ (\mathrm{L}) & \Longrightarrow & (\mathrm{MR}) & = & \mathrm{Gal}^\circ\ \text{Abelian} \end{array}$$

and we would like to understand whether the vertical arrow can be reverted.

3.5 More on the monodromy group – the case of the curve $\mathcal{E}_h$

To investigate Liouville integrability, we will need an additional argument:

Lemma 3.4. *There is a non empty open interval in $]0, +\infty[$ such that, for h in this interval, there is a cycle on the curve $\mathcal{E}'_h$ the monodromy along which has two real positive distinct eigenvalues.*

Accepting this for the moment, call a the monodromy along such a cycle. Let us call λ, λ^{-1} the two positive eigenvalues, chosen so that $0 < \lambda^{-1} < 1 < \lambda$. The algebraic subgroup of $\mathrm{SL}(2;\mathbb{C})$ generated by a is conjugated with

$$H = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \mid z \in \mathbb{C}^\star \right\}.$$

An algebraic subgroup of $\mathrm{SL}(2;\mathbb{C})$ containing such a subgroup H and which is virtually Abelian is either Abelian or conjugated to a subgroup

$$G = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \mid z \in \mathbb{C}^\star \right\} \cup \left\{ \begin{pmatrix} 0 & \mu^{-1} \\ -\mu & 0 \end{pmatrix} \mid \mu \in \mathbb{C}^\star \right\}$$

(the list of all algebraic subgroups of $\mathrm{SL}(2;\mathbb{C})$ is rather short and can be found, for instance, in [17]). Hence, either Gal is connected (and we are done) or it contains an element which writes, in a basis where a is diagonal, $\begin{pmatrix} 0 & \mu^{-1} \\ -\mu & 0 \end{pmatrix}$. The subgroup H is Abelian. The commutator of the two elements a and $\begin{pmatrix} 0 & \mu^{-1} \\ -\mu & 0 \end{pmatrix}$ is then $\begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-2} \end{pmatrix}$. Our monodromy about the pole must be, at worst, such a commutator. It should thus have a real eigenvalue. But the indicial equation about the pole has real eigenvalues, so that the monodromy must have complex eigenvalues of modulus 1. A contradiction with the fact that $\lambda > 1$. Hence Gal is Abelian. The monodromy around the pole is trivial, so that $\sqrt{8A-7}$ must be an odd integer, namely

$$A = 1 + \frac{m(m+1)}{2} = 1, 2, 4, \dots .$$

Proof of Lemma 3.4 – calling to Liapounov for help. To apply this, we need to prove Lemma 3.4. To do so, I will use (at last!) the real structure of our elliptic curves $\mathcal{E}_h$ and a theorem of Liapounov (in [15], another Russian paper published in French, another paper published in Toulouse):

Theorem 3.5 (Liapounov [15]). *Let f be a periodic real function which is everywhere ≥ 0. Then the eigenvalues of the monodromy of the differential equation $\ddot{y} = f(t)y$ are real, positive and distinct.*

The proof of this theorem is rather simple and can be found in [11] (see also [6], a paper in which I have used it in a rather similar context, that of Lamé equations, elliptic curves again).

Proof of Lemma 3.4. Recall that the linear differential equation along $\mathcal{E}_h$ is

$$\ddot{R} + \frac{q\gamma_3}{\gamma_1}\dot{R} + \left(\frac{\gamma_3^2}{A\gamma_1} + \gamma_1\right)R = 0.$$

The first thing to do is to put this equation in the form $\ddot{y} = f(t)y$. But this is fairly classical and easy, just write

$$R = e^{\varphi}y, \text{ with } 2\dot{\varphi} + \frac{q\gamma_3}{\gamma_1} = 0,$$

to get the differential equation satisfied by y, namely, after a few lines of computation using the differential system satisfied by γ_1, γ_3 and q end the relations between these variables,

$$\ddot{y} = f(t)y \text{ with } f(t) = \frac{1}{2A\gamma_1^2}\left(2(1-A)\gamma_1^3 - h\gamma_1^2 + 3h\right).$$

Now it is easy to check that, on the real part of the curve $\mathcal{F}_h$ (namely for γ_1, γ_3 and q real), we have:

- if $A < 1$, f is positive when
$$0 \leq h \leq \inf\left\{9(1-A), \sqrt{\frac{3}{3-2A}}\right\};$$
- if $A > 1$, f is positive for
$$A - 1 \leq h \leq 9(A-1).$$

Hence we can apply Liapounov's theorem. □

Note that, using this argument, we have shown that, for this linear differential equation, Haine's and Morales–Ramis's criteria give the same result.

3.6 The case of the curve $\mathcal{F}_h$

This is slightly different in the case of the other curve. Recall however that, because of the Goryachev–Chaplygin case, we cannot expect Liouville integrability coincide with Kowalevski property. This is exactly what will appear in the investigation of the relation between Haine's and Morales–Ramis's criteria for the differential equation linearized along the curve $\mathcal{F}_h$.

In the Goryachev–Chaplygin case (namely, here, for $A = 4$), the indicial equation for our second differential equation (along $\mathcal{F}_h$) has roots $-1/2$ and $-3/2$, which gives eigenvalues -1 for the monodromy around the pole(s), a non trivial monodromy. Notice that the particular solution $\mathcal{F}_h$ we are considering lies in W_0, so that we are, indeed, in the Goryachev–Chaplygin case. There are non meromorphic solutions (as this can be deduced, for the original non linear system, from the fact that Kowalevskaya did not find this case, and as it was noticed by Ziglin [21, footnote p.13]). This is a case where the Galois group is not Abelian but should be virtually Abelian.

Our linear differential equation along the elliptic curve $\mathcal{F}_h$ is a Lamé equation. The monodromy and Galois groups of Lamé equations have been extensively studied (a summary of the results can be found in [17]). The results depend on the coefficients of the equation and on the elliptic curve itself. I was not able to find a direct argument (as the one provided by Liapounov's theorem above) to show that, with the information we have, Lemma 3.6, which ends the proof of Theorem 3.2, holds.

Lemma 3.6 (Maciejewski–Przybylska [16]). *Assume $A/B = B/C$ and $z_0 = 0$. If the Galois group of the variational equations along the curves $\mathcal{E}_h$ and $\mathcal{F}_h$ are virtually Abelian, then $A/C = 1$, 2 or 4.*

Remark. The non existence of a *real* meromorphic additional integral can be derived, along the lines suggested by Ziglin [22] (see also [6]), as shown in [16].

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