AN INTRODUCTION TO FROBENIUS MANIFOLDS, MODULI SPACES OF STABLE MAPS AND QUANTUM COHOMOLOGY

MICHÈLE AUDIN

Abstract. We give the definition of Dubrovin's Frobenius manifolds and present two families of examples, that of unfoldings of singularities and that of quantum cohomology. Defining the latter, we also explain the construction of Kontsevich's spaces of stable maps. We emphasize on geometric constructions and examples.

Contents

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Introduction

The aim of this paper is to give an introduction to the geometry of Frobenius manifolds and especially to that of two important families of examples:

- parameter spaces of unfoldings of singularities
- quantum cohomology.

I find it very impressive that these two appearently very different theories can be presented as two examples of a single notion (even more impressive is the fact that they are related by mirror symmetry, see below). I also enjoy very much the idea of making geometrical constructions in the cohomology of a manifold. This is what I want to explain in this paper.

Frobenius manifolds. At first sight, the definition of a Frobenius manifold looks like some French cooking recipes (see *e.g.* [Cou98, Dub95]): melt a connection, add a product, sprinkle with a vector field, stir in a metric, taste and add a potential before serving. However, the concept itself is quite natural and, in some sense, classical: singularists know this structure since K. Saito's paper [Sai83]. The beautiful idea in Dubrovin's definition and extensive study is that the same structure appears in many other areas of mathematics.

Mirror symmetry. In the list above, the two main ingredients are a ring structure on the (sheaf of) vector fields and a flat structure. In the example of unfoldings, the product is very natural and quite easy to describe whereas the flat structure is more mysterious. In the cohomology case, the flat structure is the natural flat structure of the complex vector space $H^\star(X;\mathbb{C})$ but the definition of the product involves the construction of the Gromov-Witten invariants, a rather elaborate theory.

In the terminology of physicists (or field theoretists), the former examples are called B-models while the latter are A-models. The mirror conjecture postulates a correspondence between A- and B-models (see Givental's work in [Giv95a, Giv96] and in [Voi96, Voi98, BdCPP98, Pan98]).

Spaces of stable maps. Having in view the description of the Frobenius structure on (quantum) cohomology, another aim of this paper is to be a comprehensive introduction to Kontsevich's moduli spaces of stable maps, a very clever and efficient tool to enumerate curves in algebraic manifolds.

They are of great use in the definition of the Gromov-Witten invariants and the theory of quantum cohomology, which is "simply" the structure of a Frobenius manifold on the complex cohomology of an algebraic manifold.

In turn, this structure itself gives information on the invariants. For instance, the associativity of the products allows to enumerate the rational degree-d plane curves through $3d-1$ points: starting from 1 line through 2 points, one gets 1 conic through 5 points (not a surprise) and recursively, *e.g.* $26,312,976$ sextics through 17 points, as was shown by Kontsevich (a classical but unavoidable application).

Geometry. Unfoldings of singularities of functions belong to the world of caustics and Lagrangian projections. Let M be the space of parameters of the versal unfolding. For instance, if the singular function is $f(z)=z^{n+1}$, the unfolding is

$$F_\xi(z)=z^{n+1}+\xi_{n-1}z^{n-1}+\cdots+\xi_0$$

and M is the space of coordinates $(\xi_0,\ldots,\xi_{n-1})$.

It is classical in singularity theory (see [AVGZ85]) that the critical points of F_ξ can be organised in a Lagrangian submanifold in the cotangent bundle $T^\star M$. The very same thing happens in all Frobenius manifolds. In the case of quantum cohomology, we are now discussing a Lagrangian subvariety in the cotangent bundle of the cohomology of a manifold. This conception turns out to be very useful to understand what happens, *e.g.* in the Givental and Kim computation of the (small) quantum cohomology ring of flag manifolds (in [GK95]).

There are now many papers on Frobenius manifolds (*e.g.* [Dub95, Hit97, Sab98, Man96, Man98]) and also many papers on quantum cohomology ([MS94, RT95, Aud97, FP96] to name a few). I cannot claim originality for the contents of the present paper. However, I have tried to emphasize on geometric descriptions and examples. Also, I have not tried to give the most general possible results, but, as this is a survey paper, rather to give a flavour of the possible methods and results.

I have chosen the algebro-geometric way to Gromov-Witten invariants because this is the simplest to describe (avoiding generalities on almost compex structures and pseudo-holomorphic maps). However, I have also chosen to avoid too much algebraic generalities[1] (I have not mentioned stacks ... except in the present sentence) so that the paper should be readable by an honest geometer or topologist. For the same reasons, when discussing unfoldings, I have concentrated on the simple case of z^{n+1} and contented myself with short indications of the general case.

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Part 1. Frobenius manifolds

1. Baby-example: Frobenius structures on $\mathbb{C}^2$

The structure of a Frobenius manifold consists of a few data together with several compatibility conditions. The interesting examples themselves being rather elaborate, I have chosen to describe first a baby-example, namely what a Frobenius structure on $\mathbb{C}^2$ is. This will (hopefully) motivate the precise definition given in section 2.

A product on vector fields. Part of the structure consists of a ring structure, or, better to say, of a family of ring structures on $\mathbb{C}^2$, parametrized by $\mathbb{C}^2$ itself. In general, given a ring structure on $\mathbb{C}^2$, there exists two vectors u and v and a complex number a such that

$$\begin{cases} u \text{ is the identity} \\ v^2 = au. \end{cases}$$

We want this product to depend on a point in $\mathbb{C}^2$, so that we will assume that in some coordinates (t_0, t_1) on $\mathbb{C}^2$, the two vectors u and v are

$$u = \frac{\partial}{\partial t_0}, \quad v = \frac{\partial}{\partial t_1}$$

and that a is a function $\mathbb{C}^2 \to \mathbb{C}$. The product of two elements at the point $t = (t_0, t_1)$ is given by the formula:

$$\left(x_0\frac{\partial}{\partial t_0} + x_1\frac{\partial}{\partial t_1}\right) \star_t \left(x_0'\frac{\partial}{\partial t_0} + x_1'\frac{\partial}{\partial t_1}\right) = (x_0x_0' + a(t)x_1x_1')\,\frac{\partial}{\partial t_0} + (x_1x_0' + x_0x_1')\,\frac{\partial}{\partial t_1}.$$

This product may also be described by the matrix of 1-forms

$$\Omega_t = \begin{pmatrix} dt_0 & a(t)dt_1 \\ dt_1 & dt_0 \end{pmatrix}$$

which satisfies

$$\Omega_t(\alpha) \cdot \beta = \alpha \star_t \beta$$

for any two vector fields α and β.

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[1] Reading [FP96] was especially useful.

A closedness condition. We require that the form Ω be closed, namely that the function a depend only on t_1. As we are on $\mathbb{C}^2$, closed forms are exact, so that there exists a matrix S of functions such that $\Omega = dS$:

$$S(t) = \begin{pmatrix} t_0 & b(t_1) \\ t_1 & t_0 \end{pmatrix}$$

where b is a primitive of a.

A flat metric. We consider now the symmetric bilinear form

$$g((t_0, t_1), (t_0', t_1')) = \frac{1}{2}(t_0 t_1' + t_1 t_0'),$$

in other words the "metric" $dt_0 dt_1$. Notice that Ω and S are symmetric with respect to g. Thus, S is given by the second derivative of a function: there exists a function $\Psi : \mathbb{C}^2 \to \mathbb{C}$ such that the bilinear forms $g(S(t)\cdot, \cdot)$ and $(d^2\Psi)_t$ coincide. For example,

- If $a(t_1) = t_1^m$,
$$\Psi(t_0, t_1) = \frac{1}{2} t_0^2 t_1 + t_1^{m+3}.$$
- If $a(t_1) = \exp \frac{2}{r} t_1$,
$$\Psi(t_0, t_1) = \frac{1}{2} t_0^2 t_1 + C \exp \frac{2}{r} t_1$$
(the constant C is a simple function of r).

Homogeneity. It is worthwhile noticing that, in the first example, the function Ψ is quasi-homogeneous: give degree 1 to t_1 and degree $\frac{m}{2} + 1$ to t_0. This is less obvious, but we shall see below that the second function also satisfies a kind of homogeneity property.

2. Definition of a Frobenius manifold

We replace now the complex analytic manifold $\mathbb{C}^2$ of the previous examples by a complex analytic manifold M. The letter ξ will denote a point in the manifold M.

The product on vector fields. It will be replaced by a ring structure on each tangent space $T_\xi M$. Equivalently, this is a morphism

$$\begin{array}{rcl} \Omega : TM & \longrightarrow & \operatorname{End}(TM) \\ (\xi, \alpha) & \longmapsto & (\beta \mapsto (\alpha \star_\xi \beta)) \end{array}$$

satisfying some conditions expressing the commutativity and associativity of $\star_\xi$.

The closedness condition. Without additional choice, the symbol d makes no sense for a morphism like Ω. We will choose a flat connection ∇ on M in order to state an analogue of the closedness condition in the baby-example above.

The flat metric. The next ingredient will replace $dt_0 dt_1$. This is a metric on M, that is, a non degenerate bilinear form g_ξ on each tangent space. We will require that g be flat, namely, that, in some local coordinates, it is constant (as is $dt_0 dt_1$).

Notice that g is complex bilinear and thus is not a metric in the usual Riemannian sense. We will nevertheless call it a metric, as this terminology has proved very convenient. As this is the case for a Riemannian metric, there is a Levi-Civitá connection associated with g, we will require that this is the flat connection ∇.

In the baby-example above, we have used that the matrix Ω was symmetric with respect to g. We will impose the same condition in general. Notice that, taking the commutativity of $\star_\xi$ into account, this means that:

$$g(\alpha \star \beta, \gamma) = g(\alpha, \beta \star \gamma),$$

a very natural compatibility condition.

Notice also that, in the baby-example, the unit of the ring structures, namely the vector field $\dfrac{\partial}{\partial t_0}$, is constant in the flat coordinates (t_0, t_1). We thus require the same property in general.

Homogeneity. We will also require the existence of a vector field $\mathcal{E}$ such that our data be homogeneous with respect to $\mathcal{E}$.

Here is the formal definition of a Frobenius manifold.

Definition 2.1. A *Frobenius manifold* is a complex analytic manifold M endowed with

- a commutative ring structure on the sheaf of vector fields, the product at the point ξ of M being denoted $\star_\xi$ and the unit 1_ξ,
- a flat metric g, the Levi-Civitá connection of which denoted ∇.
- a vector field $\mathcal{E}$, the Euler vector field

satisfying a couple of compatibility conditions, namely:

(1) $g(\alpha\star\beta,\gamma) = g(\alpha,\beta\star\gamma)$.
(2) $\nabla 1 = 0$.
(3) The 1-form $\Omega : TM \to \operatorname{End}(TM)$ defined by $\Omega(\alpha)\cdot\beta = \alpha\star\beta$ satisfies $d^\nabla\Omega = 0$.
(4) The vector field $\mathcal{E}$ acts by conformal transformations of g and by rescalings of $\star_\xi$ on $T_\xi M$, namely

$$\mathcal{E}\cdot g(\alpha,\beta) - g\left([\mathcal{E},\alpha],\beta\right) - g\left(\alpha,[\mathcal{E},\beta]\right) = Dg(\alpha,\beta)$$

and

$$[\mathcal{E},\alpha\star\beta] - [\mathcal{E},\alpha]\star\beta - \alpha\star[\mathcal{E},\beta] = d_1\alpha\star\beta$$

for some constants D, d_1.
(5) $\nabla(\nabla\mathcal{E}) = 0$.

Remarks 2.2. (1) Here, the term "metric" is used with the meaning explained above: this is a (complex) non degenerate bilinear form.
(2) The mapping $\alpha\mapsto g(\alpha,1)$ defines a 1-form θ on M. Conversely, the 1-form θ and the product $\star$ determine g by $g(\alpha,\beta) = \theta(\alpha\star\beta)$.
(3) The compatibility condition between the metric and the multiplication means that Ω takes values in the *symmetric* (with respect to the metric g) endomorphisms of TM.
(4) If one defines a 3-tensor c by the equation

$$c(\alpha,\beta,\gamma) = g(\alpha\star\beta,\gamma),$$

the closedness condition $d^\nabla\Omega = 0$ is equivalent to the fact that the 4-tensor $(\nabla_\delta c)(\alpha,\beta,\gamma)$ be (completely) symmetric.
(5) One usually assumes that the scaling constant d_1 is non zero (and thus that the Euler vector field is non trivial). It is then possible to assume that $d_1 = 1$ by rescaling $\mathcal{E}$.
(6) Condition $\nabla 1 = 0$ says that the unit vector field is constant in flat coordinates. We will usually choose these coordinates in such a way that $1 = \partial/\partial t_0$. This is the reason why we do not insist that the metric g be *diagonal* in flat coodinates: the unit is often a null-vector.
(7) I have not been very precise on the homogeneity property in the baby-example above. The readers will probably want to check that

$$\mathcal{E} = t_0\frac{\partial}{\partial t_0} + \left(\frac{m}{2}+1\right)t_1\frac{\partial}{\partial t_1}$$

is an Euler vector field in the polynomial case (where $a(t_1) = t_1^m$) and that this is the case of

$$\mathcal{E} = t_0\frac{\partial}{\partial t_0} + r\frac{\partial}{\partial t_1}$$

in the exponential case (where $a(t_1) = \exp\dfrac{2}{r}t_1$).

The flat pencil of connections. The connection ∇ and the 1-form Ω can be put together in a family $\nabla^t = \nabla + t\Omega$ of connections on M (or a connection on $M \times \mathbb{P}^1$, see [Sab98]), the Dubrovin connection:

$$\nabla^t_\alpha(\beta) = \nabla_\alpha \beta + t\Omega(\alpha)\cdot\beta.$$

The commutativity of the products $\star_t$ is equivalent to the fact that ∇^t is torsionless. The flatness of ∇^t for all t's is expressed by two equations, that correspond to the closedness of Ω (or the symmetry of ∇c) and the associativity of the product (respectively). Thus our ∇^t is flat for all t.

Formality. We will also need to use the notion of a *formal* Frobenius manifold: there are flat coordinates $(t_0,\dots,t_{N-1})$ and the functions on M (especially the potential) are the formal series in $(t_0,\dots,t_{N-1})$ (see [Man96]).

The two main families of Frobenius manifolds are the parameter spaces of unfoldings of singularities and quantum cohomology. The second is rather elaborate and we will devote to it the third part of this paper. To begin with, let us look at the first family.

3. Examples: unfoldings

The manifold. This is the space M of all complex polynomials of the form

$$F_\xi(z) = z^{n+1} + \xi_{n-1}z^{n-1} + \cdots + \xi_0.$$

This is a complex affine space of dimension n. The tangent space at any point F_ξ (or ξ) is the vector space of all polynomials of degree less than or equal to $n-1$.

The product. We identify the tangent space at ξ with the algebra $\mathbb{C}[z]/F'_\xi$. This gives the sheaf of vector fields a ring structure, the product $\alpha \star_\xi \beta$ being the remainder of $\alpha\beta$ in the Euclidean division by F'_ξ.

The metric. It is constructed as in Remark 2 above *via* the product and the 1-form θ defined by

$$\theta_\xi(\alpha) = -\operatorname{Res}_\infty \frac{\alpha\, dz}{F'_\xi}.$$

Notice that, if the roots $(a_1,\dots,a_n)$ of the degree-n polynomial F'_ξ (in other words the critical points of F_ξ) are distinct, this can also be expressed as

$$\theta_\xi(\alpha) = \sum_{i=1}^n \frac{\alpha(a_i)}{F''_\xi(a_i)}.$$

From this equality, one deduces that

$$g_\xi(\alpha,\beta) = \theta_\xi(\alpha \star_\xi \beta)$$

is non degenerate (at least when the critical points are distinct).

Flatness. To show that g is flat is less straightforward. The only way I know is to exhibit coordinates in which it has constant coefficients. The idea is to solve the equations

$$\begin{cases} w^{n+1} &= F_\xi(z) \\ w &= z + O(z^{-1}) \end{cases}$$

near $z=\infty$ and to expand z as a Laurent series in w:

$$z = w + \frac{t_{n-1}}{w} + \cdots + \frac{t_0}{w^n} + O\left(\frac{1}{w^{n+1}}\right),$$

defining $t_0,\dots,t_{n-1}$ (a basis of the algebra of symmetric polynomials). The claim is that $t_0,\dots,t_{n-1}$ are the flat coordinates we are looking for. That they are (global) coordinates is a simple consequence of the definition, which implies that

$$\xi_i = -t_i + A_i(t_{i+1},\dots,t_{n-1}) \quad 0 \le i \le n-1$$

or equivalently, that

$$t_i = -\xi_i + B_i(\xi_{i+1}, \dots, \xi_{n-1}) \quad 0 \le i \le n-1.$$

That they are flat coordinates follows from the computation:

$$\frac{\partial}{\partial t_i}(F_\xi) = F'_\xi(z(w,t))\frac{\partial z}{\partial t_i} = F'_\xi(z(w,t))w^{-n+i}$$

so that

$$g\left(\frac{\partial}{\partial t_i}(F_\xi), \frac{\partial}{\partial t_j}(F_\xi)\right) = -\operatorname{Res}_{z=\infty} F'_\xi(z(w,t))w^{-2n+i+j}dz.$$

But $F'_\xi(z)dz = (n+1)w^n dw$, and thus

$$g\left(\frac{\partial}{\partial t_i}(F_\xi), \frac{\partial}{\partial t_j}(F_\xi)\right) = -(n+1)\operatorname{Res}_{w=\infty} w^{-n+i+j}dw = (n+1)\delta_{i+j,n-1}.$$

This shows

- that g is non degenerate everywhere
- and that it is constant in the t_i's:

$$g = (n+1)(dt_0 dt_{n-1} + dt_1 dt_{n-2} + \cdots).$$

Thus the latter are flat coordinates. □

Euler vector field. Here $\mathcal{E}$ is the polynomial F_ξ itself. More precisely, $\mathcal{E}_\xi$ is the image of F_ξ in $\mathbb{C}[z]/F'_\xi$, that is, the remainder of F_ξ in the Euclidean division by F'_ξ. We will show later (in Remark 5.9) why $\mathcal{E}$ rescales the product. To check that it acts by conformal transformations of the metric, let us write it in coordinates.

To begin with, let us make the Euclidean division of F_ξ by F'_ξ, to express $\mathcal{E}$ in the coordinates $(\xi_0, \dots, \xi_{n-1})$:

$$F_\xi(z) - \frac{z}{n+1}F'_\xi(z) = \sum_{j=0}^{n-1} \frac{n-j+1}{n+1}\xi_j z^j,$$

in other words,

$$\mathcal{E}_\xi = \sum_{j=0}^{n-1} \frac{n-j+1}{n+1}\xi_j \frac{\partial}{\partial \xi_j}.$$

Let us now express this vector field in terms of the flat coordinates $(t_0, \dots, t_{n-1})$. Recall that

$$t_i = -\xi_i + B_i(\xi_{i+1}, \dots, \xi_{n-1}) \quad 0 \le i \le n-1.$$

Give ξ_j the degree $n-j+1$ — so that $F_\xi(z)$ is homogeneous of degree $n+1$. The function B_i is then homogeneous of degree $n-i+1$. Let us now compute:

$$\begin{aligned}
\mathcal{E}\cdot t_i &= \sum_{j=0}^{n-1}\frac{n-j+1}{n+1}\xi_j\frac{\partial t_i}{\partial \xi_j}\\
&= -\frac{n-i+1}{n+1}\xi_i+\sum_{j=i+1}^{n-1}\frac{n-j+1}{n+1}\xi_j\frac{\partial B_i}{\partial \xi_j}\\
&= \frac{n-i+1}{n+1}t_i-\left(\frac{n-i+1}{n+1}B_i-\sum_{j=i+1}^{n-1}\frac{n-j+1}{n+1}\xi_j\frac{\partial B_i}{\partial \xi_j}\right)\\
&= \frac{n-i+1}{n+1}t_i
\end{aligned}$$

according to the homogeneity of B_i. Thus $\mathcal{E}$ has the expression

$$\mathcal{E}=\sum_{j=0}^{n-1}\frac{n-j+1}{n+1}t_j\frac{\partial}{\partial t_j}$$

in flat coordinates. As we have seen it above,

$$g=(n+1)\sum dt_i\,dt_{n-1-i}.$$

One easily deduces that $\mathcal{E}$ acts by conformal mappings.

Remark 3.1. Notice that this computation implies also that $\nabla(\nabla\mathcal{E})=0$.

And thus we have proved:

Proposition 3.2. *The space M of polynomials $z^{n+1}+\xi_{n-1}z^{n-1}+\cdots+\xi_0$, endowed with the products $\star_\xi$, the metric g and the vector field $\mathcal{E}$, has the structure of a Frobenius manifold.* □

Other singularities. More generally, if $f:\mathbb{C}^k\to\mathbb{C}$ is a germ with an isolated singularity at 0, the parameter space of the versal unfolding F_ξ of f can be endowed with a Frobenius structure. Let me summarize very briefly its main features.

- The ring structure at ξ is given by the choice of an isomorphism between the tangent space $T_\xi M$ and the Jacobian algebra
$$\mathbb{C}\{z_1,\ldots,z_k\}\,/\left\langle\frac{\partial F_\xi}{\partial z_1},\ldots,\frac{\partial F_\xi}{\partial z_k}\right\rangle.$$
- The flat metric is given by an analogue of the residue form, namely
$$\begin{aligned}
\theta_\xi:\mathbb{C}\{z_1,\ldots,z_k\} &\longrightarrow \mathbb{C}\\
\alpha &\longmapsto \left(\frac{1}{2i\pi}\right)^k\int_\Gamma\frac{\alpha\,dz_1\wedge\cdots\wedge dz_k}{\dfrac{\partial F_\xi}{\partial z_1}\cdots\dfrac{\partial F_\xi}{\partial z_k}}
\end{aligned}$$
where Γ is the so-called *distinguished boundary* of a small enough neighbourhood of the critical points of F_ξ:
$$\Gamma=\left\{z\mid\left|\frac{\partial F_\xi}{\partial z_i}\right|=\varepsilon_i\right\}\quad \varepsilon_i>0.$$

The value of $\theta_\xi(\alpha)$ depends only of the class of α in the Jacobian algebra, so that it is possible to consider the family of θ_ξ, letting ξ vary, as a 1-form θ. At a point ξ where the μ critical points $a_1, \ldots, a_\mu$ of F_ξ are distinct (and thus non degenerate), it is possible to write (see § 5 of [AVGZ85]):

$$\theta_\xi(\alpha) = \sum_{i=1}^{m+1} \frac{\alpha(a_i)}{\mathcal{H}_\xi(a_i)}$$

where $\mathcal{H}_\xi(a_i)$ is the value at a_i of the Hessian of F_ξ.

At these points, it is clear that the bilinear form

$$(\alpha, \beta) \longmapsto \theta_\xi(\alpha\beta)$$

is non degenerate. It can be shown that this is true everywhere. Moreover, a theorem of K. Saito [Sai83] asserts that there is an isomorphism of vector bundles over $\mathbb{C}^\mu$, which, at any ξ, is an isomorphism between $T_\xi\mathbb{C}^\mu$ and the Jacobian algebra, and which transports the bilinear form defined by θ to a *flat* metric on $\mathbb{C}^\mu$.

4. The potential and the associativity equations

Let us go back to a general Frobenius manifold M and use *flat* coordinates $(t_0, \ldots, t_{N-1})$, so that we assume that M is $\mathbb{C}^N$ with a constant metric and that $d^\nabla = d$.

Construction of the potential. As we have noticed it in the baby-example, Ω, being closed, has a primitive

$$S : \mathbb{C}^N \longrightarrow \mathrm{End}(\mathbb{C}^N).$$

Since the 1-form Ω defines g-symmetric endomorphisms of $\mathbb{C}^N$, we may assume that the primitive S satisfies the same property:

$$S : \mathbb{C}^N \longrightarrow \mathrm{Sym}_g(\mathbb{C}^N).$$

Such a mapping can be considered as the second derivative of a function

$$\Psi : \mathbb{C}^N \longrightarrow \mathbb{C}$$

in the sense that

$$\left(d^2\Psi\right)_\xi (\beta, \gamma) = g\left(S(\xi)\cdot\beta, \gamma\right).$$

By definition, the third derivative of Ψ is essentially the form Ω: we have

$$\left(d^3\Psi\right)_\xi (\alpha, \beta, \gamma) = g\left(\left(dS\right)_\xi(\alpha)\cdot\beta, \gamma\right) = g(\alpha \star_\xi \beta, \gamma)$$

or, in (flat) coordinates:

$$\frac{\partial^3\Psi}{\partial t_i \partial t_j \partial t_k} = g\left(\frac{\partial}{\partial t_i} \star \frac{\partial}{\partial t_j}, \frac{\partial}{\partial t_k}\right).$$

Equivalently, the 3-tensor c is given by

$$c = \sum \frac{\partial^3\Psi}{\partial t_i \partial t_j \partial t_k} dt_i dt_j dt_k.$$

The "invariance" properties with respect to the Euler vector field can then be stated very simply in terms of the potential Ψ, namely

$$\mathcal{E}\cdot\Psi = d_\Psi \Psi + \text{ terms of degree } \leq 2$$

for some degree d_Ψ.

Remark 4.1. If the flat coordinates are chosen in such a way that

$$1 = \frac{\partial}{\partial t_0}$$

(see Remarks 2.2), the metric g can be written

$$g = \sum \frac{\partial^3 \Psi}{\partial t_0 \partial t_j \partial t_k} dt_j dt_k.$$

Remark 4.2. The "trivial" case where Ψ is a polynomial of degree 3 in the flat coordinates corresponds to products $\star_\xi$ which do not depend on ξ.

Associativity equations. As the products $\star_\xi$ are given by the third derivatives of Ψ, their associativity can be expressed as a system of (non linear) partial differential equations on the function Ψ, the celebrated WDVV[2] equation:

$$\sum_{k,n} \frac{\partial^3 \Psi}{\partial t_i \partial t_j \partial t_k} g^{k,n} \frac{\partial^3 \Psi}{\partial t_\ell \partial t_m \partial t_n} = \sum_{k,n} \frac{\partial^3 \Psi}{\partial t_\ell \partial t_j \partial t_k} g^{k,n} \frac{\partial^3 \Psi}{\partial t_i \partial t_m \partial t_n}$$

where the matrix $\left(g^{k,n}\right)$ is the inverse of the matrix $(g_{i,j})$ of g, defined by

$$g_{i,j} = g\left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j}\right).$$

WDVV in dimension 3. Notice that the associativity conditions (WDVV equations) are empty in dimension 2. Let us try a 3-dimensional example. As usual, $\partial/\partial t_0$ is the unit. Choose the flat metric

$$g = dt_1^2 + dt_0 dt_2$$

so that, assuming there is a potential Ψ,

$$\begin{aligned}
\frac{\partial}{\partial t_1} \star \frac{\partial}{\partial t_1} &= \frac{\partial^3 \Psi}{\partial t_1^2 \partial t_2} \frac{\partial}{\partial t_0} + \frac{\partial^3 \Psi}{\partial t_1^3} \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \\
\frac{\partial}{\partial t_1} \star \frac{\partial}{\partial t_2} &= \frac{\partial^3 \Psi}{\partial t_1 \partial t_2^2} \frac{\partial}{\partial t_0} + \frac{\partial^3 \Psi}{\partial t_1^2 \partial t_2} \frac{\partial}{\partial t_1} \\
\frac{\partial}{\partial t_2} \star \frac{\partial}{\partial t_2} &= \frac{\partial^3 \Psi}{\partial t_2^3} \frac{\partial}{\partial t_0} + \frac{\partial^3 \Psi}{\partial t_1 \partial t_2^2} \frac{\partial}{\partial t_1}
\end{aligned}$$

The associativity equation

$$\left(\frac{\partial}{\partial t_1} \star \frac{\partial}{\partial t_1}\right) \star \frac{\partial}{\partial t_2} = \frac{\partial}{\partial t_1} \star \left(\frac{\partial}{\partial t_1} \star \frac{\partial}{\partial t_2}\right)$$

gives the PDE

$$\frac{\partial^3 \Psi}{\partial t_2^3} = \left(\frac{\partial^3 \Psi}{\partial t_1^2 \partial t_2}\right)^2 - \frac{\partial^3 \Psi}{\partial t_1^3} \frac{\partial^3 \Psi}{\partial t_1 \partial t_2^2}.$$

It turns out that this equation is related to the Painlevé VI equation (see [Dub95, Hit97, Man96]). We shall meet examples of solutions of this equation later (in § 14).

[2] for Witten, Dijkgraaf, Verlinde, Verlinde.

Remark 4.3. It can be shown (and the readers should check) that the PDE above is the only associativity equation in this dimension, namely that the condition

$$\left(\frac{\partial}{\partial t_2} \star \frac{\partial}{\partial t_2}\right) \star \frac{\partial}{\partial t_1} = \frac{\partial}{\partial t_2} \star \left(\frac{\partial}{\partial t_2} \star \frac{\partial}{\partial t_1}\right)$$

does not give anything new.

The case of unfoldings. The expression of the Euler vector field given above (in § 3) for the unfolding of z^{n+1} shows that the potential is a polynomial in the flat coordinates in this case (I will not be more specific in this paper).

5. Massive Frobenius manifolds and canonical coordinates

The "flat" coordinates we have used so far were designed to put the metric g and the connection ∇ in a simple form. In this section, we introduce another set of local coordinates, the so-called "canonical" coordinates, in which this is the product $\star$ that has a very simple form.

Canonical coordinates. Local coordinates $(x_1, \dots, x_N)$ are said to be *canonical* if they satisfy the equalities

$$\frac{\partial}{\partial x_i} \star \frac{\partial}{\partial x_j} = \delta_{i,j} \frac{\partial}{\partial x_i}.$$

On an open subset $\mathcal{U}$ of M where such coordinates exist, the bundle $TM \to M$ splits as the sum

$$TM = \bigoplus_{i=1}^{N} W_i$$

of the line bundles generated by the vectors $\frac{\partial}{\partial x_i}$ and the rings $T_\xi M$ are decomposed as sums of 1-dimensional summands. Notice that the $\frac{\partial}{\partial x_i}$ are common eigenvectors of the multiplication $\star_\xi$.

Proposition 5.1. *If the ring $(T_{\xi_0} M, \star_{\xi_0})$ is semi-simple, there exists canonical coordinates in a neighborhood of ξ_0.*

Proof. The assumption implies that all the endomorphisms $\Omega_\xi(\alpha)$ are simultaneously diagonalisable for ξ in a suitable neighborhood $\mathcal{V}$ of ξ_0. Let us consider their common eigenvectors and the associated eigenvalues.

There exists vector fields $(w_1, \dots, w_N)$ on $\mathcal{V}$ such that

$$w_i \star w_j = \delta_{i,j} w_i$$

which form a basis of eigenvectors of the $\Omega_\xi(\alpha)$ at any ξ in $\mathcal{V}$:

$$\alpha \star_\xi w_i = \mu_i(\xi, \alpha) w_i.$$

The eigenvalue $\mu_i(\xi, \cdot)$ is a linear form on $T_\xi M$. In other words, μ_i is a 1-form on $\mathcal{V}$. Let us use a lemma (which will be proved below).

Lemma 5.2. *The 1-forms μ_i are closed.*

If this is the case, they are locally exact. Let $x_1, \dots, x_N$ be primitives of $\mu_1, \dots, \mu_N$ (repectively) on an open subset $\mathcal{U}$ of $\mathcal{V}$. They form a system of local coordinates on $\mathcal{U}$ which are canonical: by construction, $(\mu_1(\xi, \cdot), \dots, \mu_N(\xi, \cdot))$ is the basis of $T_\xi^\star$ dual to $(w_1(\xi), \dots, w_N(\xi))$, so that

$$w_i = \frac{\partial}{\partial x_i}.$$

□

Proof of the lemma. Let us choose a distinguished trivialisation of the tangent bundle of M restricted to $\mathcal{U}$, so that we can use the symbol d in place of d^∇ and that the symbols $d\Omega$, dw_i make sense. Differentiate the relation

$$w_i \star w_i = w_i$$

to get

$$2(dw_i) \star w_i = dw_i.$$

Write the vector dw_i in the basis $(w_1, \ldots, w_N)$:

$$dw_i = \sum_j a_i^j w_j$$

for some matrix (a_i^j) of 1-forms, and get, multiplying both sides by w_j for $j \neq i$:

$$\sum_j a_i^j w_j = dw_i = 2(dw_i) \star w_i = 2(\sum a_i^j w_j) \star w_i = 2a_i^i w_i$$

and thus $a_i^j = 0$ for all i and j, so that $dw_i = 0$. Write now

$$\Omega \cdot w_i = \mu_i w_i.$$

Once again, differentiate this relation:

$$(d\Omega) \cdot w_i + \Omega \cdot (dw_i) = (d\mu_i) w_i + \mu_i dw_i$$

and use the assumption $d\Omega = 0$ to get

$$(d\mu_i) w_i = 0.$$

□

Remark 5.3. Notice that, as everything is analytic, semi-simplicity at some point ξ_0 implies semi-simplicity at generic points (see also Proposition C.1 in Appendix C).

Definition 5.4. A Frobenius manifold M is *massive* if the ring $(T_\xi M, \star_\xi)$ is semi-simple at some point ξ.

Remarks 5.5. (1) In canonical coordinates, the unit vector field is

$$1 = \sum_{i=1}^N \frac{\partial}{\partial x_i}.$$

(2) A vector field given (in canonical coordinates) by a formula such as $\sum x_i \dfrac{\partial}{\partial x_i}$ rescales the product $\star_\xi$ in the sense that

$$[\mathcal{E}, \alpha \star \beta] - [\mathcal{E}, \alpha] \star \beta - \alpha \star [\mathcal{E}, \beta] = \alpha \star \beta.$$

Canonical coordinates for the unfolding of z^{n+1}. Let M be the Frobenius manifold described in § 3, namely the parameter space of the unfolding of z^{n+1}. In this example, the critical values of the polynomial F_ξ are canonical coordinates. This is a consequence of the next proposition.

Proposition 5.6. *Let $\xi \in \mathbb{C}^n$ be such that the polynomial F_ξ' has simple roots $(a_1, \ldots, a_n)$. The eigenvalues of the multiplication by α at ξ are the values $(\alpha(a_1), \ldots, \alpha(a_n))$ of α at the critical points.*

Proof. Let $(w_i(\xi))_{1 \leq i \leq n}$ be the polynomials defined by

$$w_i(\xi)(a_j) = \delta_{i,j},$$

namely

$$w_i(\xi)(z) = A_i \frac{F_\xi'(z)}{z - a_i} \qquad \text{with} \qquad A_i = \frac{1}{\prod_{j \neq i}(a_i - a_j)}.$$

Notice that, the a_i's being distinct, the polynomials $w_i(\xi)$ form a basis of $T_\xi M$.

Let α be a polynomial of degree less than or equal to $n-1$. By the very definition of the product,

$$\alpha(z)w_i(z) = (\alpha \star_\xi w_i)(z) + F'_\xi(z)Q(z)$$

for some polynomial Q. Computing both sides at a_j, we get

$$(\alpha \star_\xi w_i)(a_j) = \delta_{i,j}\alpha(a_j)$$

so that $\alpha \star w_i$ is a scalar multiple of $w_i(\xi)$. Thus $w_i(\xi)$ is an eigenvector of the multiplication at ξ and the corresponding eigenvalue is the form $\alpha \mapsto \alpha(a_i)$. □

Remark 5.7. This implies that the ring $(T_\xi M, \star_\xi)$ is semi-simple for generic ξ.

Corollary 5.8. *The critical values $x_i = F_\xi(a_i)$ form a set of canonical coordinates.*

Proof. We must check that $\mu_i(\xi,\alpha) = (dx_i)_\xi(\alpha)$. Look at the function

$$\begin{array}{rcl} F : \mathbb{C}\times M & \longrightarrow & \mathbb{C} \\ (z,\xi) & \longmapsto & F_\xi(z) \end{array}$$

and at its differential:

$$(dF)_{(a_i,\xi)}(Z,\alpha) = F'_\xi(a_i)Z + (d_\xi F)_{(a_i,\xi)}(\alpha).$$

By definition, $F'_\xi(a_i) = 0$. As $\xi \mapsto F_\xi(z)$ is a linear map, it coincides with its differential, thus the latter term is $\alpha(a_i)$. Eventually

$$(dF)_{(a_i,\xi)}(Z,\alpha) = \alpha(a_i),$$

that is, $(dx_i)_\xi = \mu_i(\xi,\cdot)$. □

Remark 5.9. Notice that, by the very definition of the polynomials w_i, any polynomial P satisfies

$$P = \sum_{i=1}^n P(a_i)w_i \qquad \mod F'_\xi.$$

For instance,

$$F_\xi = \sum_{i=1}^n F_\xi(a_i)w_i \qquad \mod F'_\xi.$$

Writing as above $\mathcal{E}_\xi = F_\xi \mod F'_\xi$, we see that the vector field $\mathcal{E}$ is given, in canonical coordinates, by

$$\mathcal{E}_\xi = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}.$$

This is thus indeed an Euler vector field: it rescales the multiplication.

6. Canonical coordinates and symplectic geometry

This section is not absolutely essential for the logical understanding of the paper, but it seems to me that it gives the right interpretation of some of the constructions above. It will also be useful to look at the so-called "small quantum product" (see § 16).

Remark 6.1. As the careful readers will notice it, the constructions in this section do not use the full Frobenius structure, but only the product and the flat connection. They use the assumption that the product be semi-simple at some point.

In the case where M is a massive N-dimensional Frobenius manifold, consider the graphs of the eigenvalues forms $\mu_1,\ldots,\mu_N$ as submanifolds in $T^\star M$. To say that these forms are closed is to say that this submanifold is *Lagrangian*.

Lagrangian submanifolds of a cotangent bundle. The total space of the cotangent bundle of a manifold M carries a canonical 1-form, the Liouville form λ: recall that, if

$$\pi : T^{\star}M \longrightarrow M$$

denotes the projection and (ξ, φ) denotes a point in $T^{\star}M$ (consisting of a point ξ of M and a linear form $\varphi : T_{\xi}M \to \mathbb{C}$),

$$\lambda_{(\xi,\varphi)}(X) = \varphi(T_(\xi, \varphi)\pi(X)).$$

It is a "universal" 1-form in the sense that, if α is a 1-form on M that we consider as a section of the projection π:

$$\alpha : M \longrightarrow T^{\star}M,$$

then the Liouville form satisfies the magic formula

$$\alpha^{\star}\lambda = \alpha.$$

The derivative $\omega = d\lambda$ is a symplectic form (*i.e.* a non degenerate closed 2-form on $T^{\star}M$).

A submanifold L of $T^{\star}M$ is called *Lagrangian* if it is totally isotropic (that is to say that ω vanishes on L) of maximal dimension (half the dimension of $T^{\star}M$, that is, L and M have the same dimension). More generally, an immersion

$$f : L \longrightarrow T^{\star}M$$

is *Lagrangian* if $\dim L = \dim M$ and $f^{\star}\omega = 0$.

Generating functions and families. The Liouville form λ defines a closed 1-form on Lagrangian submanifolds. For instance, the magic formula for the Liouville form shows that the graph of a 1-form α is a Lagrangian submanifold of $T^{\star}M$ if and only if α is a closed form. The simplest examples of Lagrangian submanifolds are thus the graphs of exact 1-forms, that is, the graphs of the differentials of functions

$$F : M \longrightarrow \mathbb{C}.$$

Such a function F is called a *generating function* for the Lagrangian submanifold it defines.

The graphs of differentials of functions are very special Lagrangian submanifolds, since the graph L of dF obviously has the property that the composition

$$L \subset T^{\star}M \xrightarrow{\pi} M$$

is a diffeomorphism. To construct more general examples, one slightly modifies the construction, adding parameters to the function F.

Consider a function

$$\begin{array}{rcl} F : \mathbb{C}^k \times M & \longrightarrow & \mathbb{C} \\ (z_1, \ldots, z_k, \xi) & \longmapsto & F_{\xi}(z). \end{array}$$

The graph of its differential dF is a Lagrangian subvariety of $T^{\star}\left(\mathbb{C}^k \times M\right)$.

Assume now that this Lagrangian is transversal to the conormal bundle of the projection $\mathbb{C}^k \times M \longrightarrow M$: in local coordinates $(\xi_1, \ldots, \xi_N)$ on M, this is to say that the $k \times (k+N)$-matrix

$$\left(\left(\frac{\partial^2 F}{\partial z_i \partial z_j}\right)\left(\frac{\partial^2 F}{\partial z_i \partial \xi_j}\right)\right)$$

has rank k at the points (z, ξ) such that $d_z F = 0$. The intersection of the graph of dF with the conormal bundle of the projection is the image of the submanifold

$$\Lambda = \left\{(z, \xi) \in \mathbb{C}^k \times M \mid d_z F = 0\right\} \subset \mathbb{C}^k \times M.$$

The mapping

$$\begin{array}{rcl} \Lambda & \longrightarrow & T^{\star}M \\ (z, \xi) & \longmapsto & (\xi, d_{\xi}F) \end{array}$$

is a Lagrange immersion. This is indeed more general than the case of generating functions as any germ of Lagrange immersion in $T^\star M$ can be described in this way (see *e.g.* [Wei77]).

Examples 6.2. (1) Let M be the space of all polynomials F_ξ and let F be the evaluation map

$$\begin{array}{rcl} F : \mathbb{C} \times M & \longrightarrow & \mathbb{C} \\ (z, \xi) & \longmapsto & F_\xi(z) \end{array}$$

considered in the proof of Corollary 5.8. The submanifold $\Lambda \subset \mathbb{C} \times M$ is the set of critical points:

$$\Lambda = \left\{ (z, \xi) \mid F'_\xi(z) = 0 \right\}.$$

The Lagrangian submanifold L of $T^\star M$ is

$$L = \left\{ (\xi, p) \mid p = (d_\xi F)_{(z,\xi)} \text{ and } F'_\xi(z) = 0 \right\}.$$

This is the union of the graphs of the forms μ_i, as shown in the proof of Corollary 5.8. Notice however that the Lagrangian submanifold L is well defined and smooth even over points ξ such that the critical points of F_ξ are not distinct.

(2) Let us look at the $n = 2$ case. The polynomial F_ξ is

$$F_\xi(z) = z^3 + \xi_1 z + \xi_0$$

and the Lagrangian is

$$L = \left\{ (\xi_0, \xi_1, p_0, p_1) \in \mathbb{C}^4 \mid p_0 = 1 \text{ and } p_1^2 = -\frac{\xi_1}{3} \right\}.$$

The left part of Figure 1 shows the intersection of L with a plane $\xi_0 =$ constant.

(3) More generally, unfoldings of singularities define Lagrangian submanifolds (see [AVGZ85]) in the very same fashion.

Spectral covers. Let us go back to the case of an analytic N-dimensional manifold M endowed with products $\star_\xi$ and thus with an $\mathrm{End}(TM)$-valued 1-form

$$\begin{array}{rcl} \Omega : TM & \longrightarrow & \mathrm{End}(TM) \\ (\xi, \alpha) & \longmapsto & \alpha \star_\xi \cdot \end{array}$$

and note that commutativity and associativity of $\star_\xi$ imply:

Lemma 6.3. *The* 1*-form* Ω *satisfies* $\Omega \wedge \Omega = 0$.

Proof. By definition, $\Omega_\xi(\alpha) \cdot (\Omega_\xi(\beta) \cdot \gamma) = \alpha \star_\xi (\beta \star_\xi \gamma)$, so that

$$\begin{aligned} (\Omega \wedge \Omega)_\xi (\alpha, \beta) \cdot \gamma &= \frac{1}{2} \left[\Omega_\xi(\alpha), \Omega_\xi(\beta)\right] \cdot \gamma \\ &= \frac{1}{2} [\alpha \star_\xi (\beta \star_\xi \gamma) - \beta \star_\xi (\alpha \star_\xi \gamma)]. \end{aligned}$$

□

Lemma 6.3 can be rephrased as "(TM, Ω) is a *Higgs pair*".

It turns out that a Higgs pair defines a subvariety $L \subset T^\star M$ such that the projection

$$L \subset T^\star M \xrightarrow{\pi} M$$

is a degree-N map. The construction of L, the *spectral cover*, uses the eigenvalues of Ω. Spectral covers originate from Hitchin's famous paper [Hit87], in which M was a curve.

Definition of the spectral cover. There is an abstract and general algebro-geometrical definition. See [DM95] in general and [Aud98] for the case at hand. In this paper, we will content ourselves with a down-to-earth description: the subvariety L will be defined as the union of the graphs of the eigenvalue forms considered above (in § 5).

More precisely, assume that at some point $\xi_0 \in M$, the algebra $(T_{\xi_0}M, \star_{\xi_0})$ is semi-simple, or, equivalently, that all the $\Omega_{\xi_0}(\alpha)$ for $\alpha \in T_{\xi_0}M$ are diagonalisable (recall that the $\Omega_\xi(\alpha)$ commute so that they can be diagonalised simultaneously). Such a point ξ_0 is called a *semi-simple point*. Over a neighborhood of a semi-simple point ξ_0, the spectral cover is simply the union of the graphs of the eigenvalue forms.

Over a neighbourhood $\mathcal{U}$ of ξ_0, the bundle $TM \to M$ then splits as a sum

$$TW = \oplus_{i=1}^N W_i$$

of line bundles that are the eigenline bundles of Ω. Equivalently, on $\mathcal{U}$, there is a basis

$$\xi \longmapsto (w_1(\xi), \dots, w_N(\xi))$$

of common eigenvectors of the multiplication $\star_\xi$.

To each W_i and each $\alpha \in T_\xi M$ corresponds an eigenvalue $\mu_i(\alpha, \xi)$ of $\Omega_\xi(\alpha)$. Such a μ_i can be considered as a local section of $T^\star M$ and altogether, the graphs of the μ_i's form the N sheets of the spectral cover L.

The abstract definition works without any semi-simplicity assumption. It works for non semi-simple points, as are the parameters ξ for which the polynomials F'_ξ have mutiple roots. Moreover, the spectral cover L has a very beautiful property (which is almost tautological in the abstract definition, see *e.g.* [Aud98]): the ring of functions on L is the ring sheaf defined by the products $\star_\xi$:

Proposition 6.4. *The ring sheaf of vector fields on M is the sheaf of regular functions on the spectral cover L associated with (TM, Ω).* □

Remark 6.5. The spectral cover (and the projection onto M) may have singularities over the complement of the set of semi-simple points (see Examples 6.2). However, according to Proposition 6.4, these singularities are relatively simple, since the structural sheaf of L is the sheaf of local sections of a vector bundle (namely, TM).

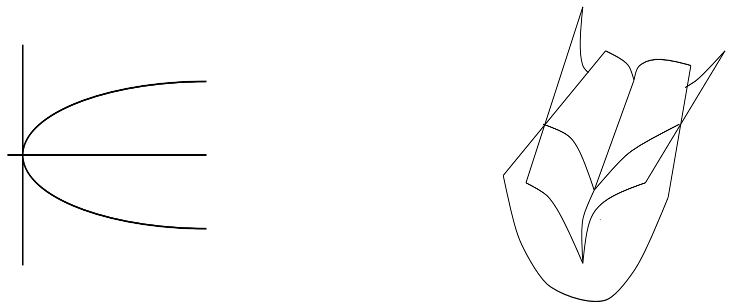
The Lagrange property. Let us assume now that M is endowed with a flat connection as above. The following property comes from [GK95, Giv96] and is, in our context, a simple rephrasing of the existence of canonical coordinates.

Proposition 6.6. *Assume $d^\nabla \Omega = 0$. If there exists a semi-simple point in M, the spectral cover is a Lagrangian subvariety of $T^\star M$.* □

Examples 6.7.
(1) Let us go back to our polynomial $F_\xi(z) = z^3 + \xi_1 z + \xi_0$ (Examples 6.2). At the points $(\xi_0, 0)$, the product is not semi-simple. This produces a singularity of the projection on the ξ-plane, the fold visible on the left part of Figure 1.
(2) For the space of unfoldings of a more general singularity, the singularities of the projection $L \to M$ are again the points where the product is not semi-simple, the so-called *discriminant* of the Frobenius manifold. The right part of Figure 1 shows a typical discriminant (that of the D_4-singularity).
(3) In general, the spectral cover itself may have singularities over points where the product is not semi-simple. In the baby-example of $\mathbb{C}^2$, the spectral cover is the Lagrangian

$$L = \left\{(t_0, t_1, p_0, p_1) \mid p_0 = 1 \text{ and } p_1^2 = a(t_1)\right\}.$$

For example, if $a(t_1) = At_1^m$ (for $m \geq 2$) the spectral cover is singular at the points $(t_0, 0)$ (the first example above corresponds to $m = 1$).



(1) a fold (2) a discriminant

Figure 1

Symplectic reduction. One can define more generally and in the same way a spectral cover in $T^\star M$ for a morphism

$$\Omega : TM \longrightarrow \mathrm{End}(E)$$

where E is any vector bundle on M and Ω satisfies $\Omega \wedge \Omega = 0$. Consider in particular the inclusion of a submanifold

$$j : B \longrightarrow M.$$

Then $j^\star T^\star M$ is a co-isotropic subvariety of $T^\star M$, and the tangent sheaf to M is endowed with a product $\star_\xi$ such that the spectral cover is a Lagrangian subvariety of $T^\star M$. At least at semi-simple points, the spectral cover is locally the graph of a (closed) 1-form and thus is transversal to $j^\star T^\star M$.

It is a simple form of the symplectic reduction process that the intersection $L \cap (j^\star T^\star M)$ projects to a *Lagrangian* subvariety $L_B \subset T^\star B$. Of course, we have:

Proposition 6.8. *The Lagrangian subvariety L_B is the spectral Lagrangian in $T^\star B$ associated with the morphism $j^\star \Omega : TB \to \mathrm{End}(j^\star TM)$ over B.*

Proof. The spectral cover for

$$j^\star \Omega : TB \longrightarrow \mathrm{End}(j^\star E)$$

is defined by the eigenvalues $\mu_i(\alpha, \xi)$ of $\Omega_\xi(\alpha)$ for $\xi \in B$, $\alpha \in T_\xi B \subset T_\xi M$. In other words, one considers the section μ_i of $T^\star M$ as a section of $T^\star B$ by restriction and this is the definition of the symplectic reduction explained above. □

7. Other examples of Frobenius manifolds

We have already said that parameter spaces of unfoldings are Frobenius manifolds. Notice that this provides a Frobenius structure on any complex space $\mathbb{C}^n$, the latter being the parameter space of the unfolding of z^{n+1} investigated above. Notice also that, as the computation of the Euler vector field in § 3 shows it, the potential of this structure is a polynomial in the flat coordinates.

Orbit spaces of Coxeter groups. The space of degree-$(n+1)$ polynomials

$$F_\xi(z) = z^{n+1} + \xi_{n-1} z^{n-1} + \cdots + \xi_0$$

is also isomorphic to the quotient

$$\left\{ (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \mid \sum z_i = 0 \right\} / \mathfrak{S}_{n+1}$$

by the action of the symmetric group $\mathfrak{S}_{n+1}$, using the map

$$(z_1, \ldots, z_{n+1}) \longmapsto \prod_{i=1}^{n+1}(z - z_i).$$

In other words, our space of polynomials, as all spaces of parameters of versal unfoldings, is the space of orbits of a Coxeter group. These spaces have natural Frobenius structures (see [Dub95]) and a conjecture of Dubrovin asserts that all polynomial solutions of the WDVV equations should be potentials of such structures.

Hurwitz spaces. The polynomial z^{n+1}, the unfolding of which we have investigated in detail, generates another family of examples, that of Hurwitz spaces. Simply notice that a degree-$(n+1)$ polynomial in one variable is also an $(n+1)$-fold covering of $\mathbb{P}^1$ with a single pole (of multiplicity $n+1$).

Recall that Hurwitz spaces are moduli spaces of coverings of $\mathbb{P}^1$ (or of meromorphic functions on curves) with prescribed degree. Dubrovin investigates, in Chapter V of [Dub95], a variant of these spaces: a constraint on the structure of the pole divisor is added. Precisely, given non negative integers $g, d_1, \ldots, d_m$, define $M_{g,d_1,\ldots,d_m}$ as the set of isomorphism classes of data $(\Sigma, z_1, \ldots, z_m, f)$ where

- Σ is a smooth genus-g curve,
- $(z_1, \ldots, z_m)$ is an ordered set of m distinct points on Σ
- and f a meromorphic function on Σ with pole divisor

$$(f)_\infty = \sum (d_i + 1) z_i.$$

For example, the parameter space of the unfolding of z^{n+1} is $M_{0,n}$. Dubrovin constructs a Frobenius structure on $M_{g,d_1,\ldots,d_m}$ generalising the one we have shown in the z^{n+1} case. As in Proposition 5.6, the critical values of the meromorphic functions are canonical coordinates.

Landau-Ginzburg *vs* field theory models. In the two families of examples above, the structure of the Frobenius manifold M comes from a function F defined on some bundle over M:

- $M = \mathbb{C}^\mu$ and $F : \mathbb{C}^\mu \times \mathbb{C}^k \to \mathbb{C}$ is the unfolding itself,
- M is the Hurwitz space, $\mathcal{C} \to M$ the "universal curve" over M and $F : \mathcal{C} \to \mathbb{C}$ is the "universal function", namely the evaluation mapping $(\Sigma, z_1, \ldots, z_m, f, z) \mapsto f(z)$.

The function F is a generating function for the spectral cover. These examples are called "Landau-Ginzburg models", the function F is the "superpotential" of the model.

There are other examples of Frobenius manifolds where you start from the potential Ψ to describe the structure — these are the "field theory models". This is the case of quantum cohomology.

Roughly speaking, in Landau-Ginzburg models, the easy part of the structure is the multiplication, while in field theory models, this is the metric.

Cohomology. The next example is that of the (even part of the) cohomology of a compact manifold X: $M = H^\star(X; \mathbb{C})$. The product is the cup-product $\smile$ and the metric is defined by Poincaré duality. This is very simple, but unfortunately not very interesting, and this for two reasons:

(1) the product is constant (does not depend on any point of M), the potential is a cubic form (see Remark 4.2),
(2) the cup-product is far from being semi-simple, being nilpotent.

Quantum cohomology. However, the cohomology of X is a very simple manifold, being a complex vector space. The metric defined by Poincaré duality is obviously flat, being constant in the natural (linear) coordinates. Fortunately, there is another product (under certain assumptions on the manifold X) on the sheaf of vector fields of M, the quantum product $\star_\xi$, which is often semi-simple. The idea is to define it through the *Gromov-Witten potential:*

$$\Psi(\xi) = \sum_{m\geq 3} \frac{1}{m!} \sum_A \Psi_A(\xi^{\otimes m}),$$

in which expression ξ is a point in $M = H^\star(X;\mathbb{C})$, the summation is over all dimension-2 homology classes $A \in H_2(X;\mathbb{Z})$ and $\Psi_A(\xi^{\otimes m})$ designates a Gromov-Witten invariant, heuristically described as the number of certain rational curves of class A in X and rigorously defined in the second and third parts of this paper. We will see, for instance, that the potential

$$\Psi(t_0, t_1) = \frac{1}{2} t_0^2 t_1 + \exp t_1$$

found above (in § 1) in dimension 2 describes the Frobenius structure on the cohomology of $\mathbb{P}^1$.

Part 2. Moduli spaces of stable maps

8. Introducing Gromov-Witten invariants

Let X be a (smooth) complex projective manifold. A "rational curve" in X is, roughly speaking, a genus-0 curve in X and more precisely a map

$$u : \mathbb{P}^1 \longrightarrow X$$

from the complex projective line to X. To be completely correct, we must identify two maps which differ by an automorphism of $\mathbb{P}^1$ (a change of parametrization).

Such a curve defines a homology class

$$A = u_\star[\mathbb{P}^1] \in H_2(X;\mathbb{Z}).$$

The idea in the construction of Gromov-Witten invariants is to *count* the rational curves in a given homology class in X that satisfies certain incidence conditions. The basic example is that of the complex projective plane ($X = \mathbb{P}^2$) and of the homology class represented by the straight lines. The first result on Gromov-Witten invariants goes back at least to Euclide:

Theorem 8.1. *Through any two points goes one and only one line.*

It asserts that the space of all rational curves of degree 1 that go through two points consists of one single curve.

Going back to the general case, the aim is, given $A \in H_2(X;\mathbb{Z})$ and some subvarieties $Z_1, \ldots, Z_m \subset X$, to give a rigorous meaning to "the number of rational curves in the class A meeting $Z_1, \ldots, Z_m$ somewhere".

The strategy is to define a suitable "moduli" space of maps $u : \mathbb{P}^1 \to X$. To impose incidence conditions, we will add to the data m points $z_1, \ldots, z_m$ of $\mathbb{P}^1$, getting a space $\overline{\mathcal{M}}_{0,m}(X, A)$ (the "0" is for "rational", see below) with an evaluation map

$$\begin{array}{rcl} \mathrm{ev}_m : \overline{\mathcal{M}}_{0,m}(X,A) & \longrightarrow & X^m \\ (z_1, \ldots, z_m, u) & \longmapsto & (u(z_1), \ldots, u(z_m)). \end{array}$$

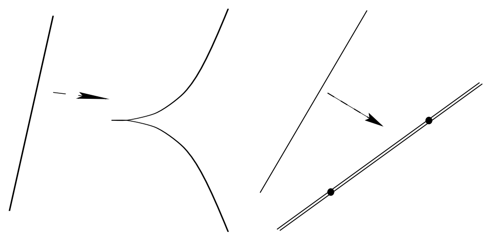
Denoting the cohomology classes dual to $Z_1, \ldots, Z_m$ by $\xi_1, \ldots, \xi_m$, our number will be expected to be the evaluation on the fundamental class:

$$\Psi_A(\xi_1 \otimes \cdots \otimes \xi_m) = \big\langle (\mathrm{ev}_m)^\star(\xi_1 \otimes \cdots \otimes \xi_m), \big[\overline{\mathcal{M}}_{0,m}(X,A)\big]\big\rangle \in \mathbb{Z}.$$

The main difficulty will be to define the moduli spaces $\overline{\mathcal{M}}_{0,m}(X,A)$ ... and to prove that they do have a fundamental class (under certain assumptions, as this is not true in general).

Singular curves. To define the space $\overline{\mathcal{M}}_{0,m}(X, A)$, we will be forced to consider also *singular* curves. By this, I do not mean that the *image* curves are singular. This can certainly happen for the maps $\mathbb{P}^1 \to X$ we have already considered. Think for example of the maps

- $t \mapsto (t^2, t^3)$, which parametrises a cuspidal cubic (the source curve is smooth)
- $t \mapsto (t^2, 0)$, which parametrizes a degenerate conic consisting of a double line, and which is nowhere injective or even immersive (see Figure 2).



(1) singular cubic (2) double line

FIGURE 2. Singular maps

What I mean is that even the *source* curve cannot always be smooth. Think for instance of the plane conics (degree-2 curves), the simplest example after that of straight lines.

Consider the conic of (affine) equation $xy = \varepsilon$. When ε tends to 0, it tends to a degenerate conic, the union of the two lines $x = 0$ and $y = 0$. Of course, the union of two distinct lines cannot be parametrized by a map $u : \mathbb{P}^1 \to \mathbb{P}^2$. We will thus be forced to add maps whose source space is a singular curve to our space of maps.

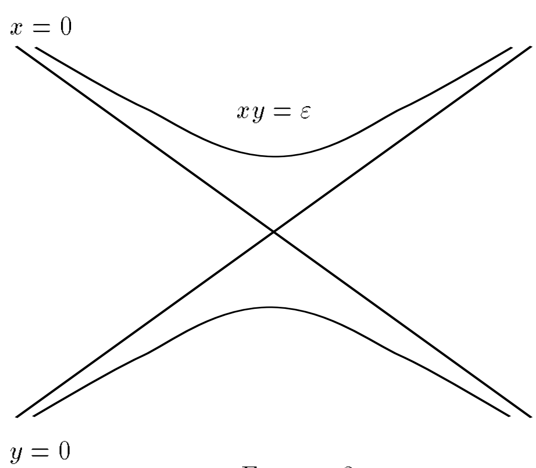


FIGURE 3

Notation and convention.

- Projective spaces. The homology class of a projective line in the complex projective space $\mathbb{P}^n$ is denoted L, it is a generator of $H_2(\mathbb{P}^n;\mathbb{Z})$. The degree-2 cohomology class dual to a hyperplane is denoted p, so that
$$H^\star(\mathbb{P}^n;\mathbb{Z}) = \mathbb{Z}[p]/p^{n+1}$$
and that $\langle p, L\rangle = 1$. Euclide's theorem above can be restated as "in $\mathbb{P}^2$, $\Psi_L(p^2\otimes p^2) = 1$".
- Homology and cohomology. I will assume that:
  - The homology and cohomology therories used satisfy the "easy" Künneth formula, *e.g.*
  $$H^\star(X\times Y) = H^\star(X)\otimes H^\star(Y)$$
  for instance, because this is cohomology with rational coefficients.
  - The second homology group $H_2(X;\mathbb{Z})$ has no torsion. Otherwise, what I will denote by $H_2(X;\mathbb{Z})$ is the quotient by the torsion subgroup.
  - Moreover, but this is only for simplicity[3], I want the cup-product to be strictly commutative (and not commutative in the graded sense). The only issue is to consider only the even part calling $H^\star(X)$ the ring $H^{\text{even}}(X)$.

9. Spaces of stable maps

Let X be a complex projective manifold. Recall that, for any embedding $X\subset\mathbb{P}^N$, the pull-back of the standard (or Fubiny-Study) Kähler form on $\mathbb{P}^N$ is a Kähler form on X. We will use the fact that X is Kähler when necessary.

A *stable map* consists firstly of a package $(\Sigma,\vec{z},u)$, where

- Σ is a genus-g complex curve with, at worse, ordinary double points (nodes),
- $\vec{z} = (z_1,\dots,z_m)$ is an (ordered) m-tuple of distinct points of Σ (usually called "marked points")
- and $u:\Sigma\to X$ is a holomorphic map.

The genus used here is the arithmetic genus. Here is a definition for topologists. Associate to Σ a graph Γ_Σ:

- any component of Σ gives a vertex,
- two vertices are connected by an edge when the two corresponding components intersect.

The *arithmetic genus* of Σ is, by definition, the sum of the genera of the components plus the dimension of $H_1(\Gamma_\Sigma)$.

I have not explained yet what the word "stable" means. It comes from the (simpler) theory of Deligne-Mumford moduli spaces of stable marked curves. Let us recall this very briefly.

Stable marked curves. A stable marked curve is simply a pacckage $(\Sigma,\vec{z})$ exactly as above (we just forget to mention a manifold X and a map u). The stability condition arises when we want to consider the space $\overline{\mathcal{M}}_{g,m}$ of isomorphism classes of marked curves: it is said that $(\Sigma,\vec{z})$ is *stable* if it has no infinitesimal autmorphisms, in other words if its automorphim group is discrete.

Examples 9.1.

- In genus 0. An automorphism of $\mathbb{P}^1$ is determined by the images of three distinct points. The automorphisms of $\mathbb{P}^1$ which fix two points or less form a subgroup of strictly positive dimension in $PGL(2;\mathbb{C})$. Thus
  - The marked curve $(\mathbb{P}^1,z_1,\dots,z_m)$ is stable if and only if $m\geq 3$.
  - The space $\overline{\mathcal{M}}_{0,3}$ is a point. See a description of $\overline{\mathcal{M}}_{0,4}$ in § 14.
- In genus 1. A smooth genus-1 curve is, up to the choice of an origin, an elliptic curve. This means that it has a 1-dimensional automorphism group ... and thus that, if Σ has genus 1, $(\Sigma,z_1,\dots,z_m)$ is stable if and only if $m\geq 1$. For genus-1 curves, see also Appendix A.
- In higher genus. A smooth curve of genus $g\geq 2$ has a finite group of automorphisms, so that a marked curve $(\Sigma,\vec{z})$ is always stable.

[3] In general, one gets the structure of a Frobenius *super*-manifold, see[KM94, Man96].

Notice that the shortest way to express the stability condition here is to say that $\overline{\mathcal{M}}_{g,m}$ exists only if $m+2g \geq 3$.

Remark 9.2. Except for $\overline{\mathcal{M}}_{1,1}$, I will only use $\overline{\mathcal{M}}_{0,m}$, the Deligne-Mumford-Knudsen compactification of the space of isomorphism classes of m ordered distinct points in $\mathbb{P}^1$. For the properties of these spaces, see [Knu83, Kee92].

Stable maps. We impose an analogous stability condition to our "maps" $(\Sigma, \vec{z}, u)$. Let us first precise what the isomorphisms are. Two stable maps $(\Sigma, \vec{z}, u)$ and $(\Sigma', \vec{z}', u')$ are *isomorphic* if there exists an isomorphism $\varphi : \Sigma \to \Sigma'$ such that $\varphi(z_i) = z_i'$ for all i and $u' \circ \varphi = u$.

We require that the data above have no infinitesimal automorphism. This amounts to requiring that the components of Σ on which u is constant be *stable curves* in the sense above.

More concretely, relying on the examples above

- if u is constant on a genus-0 component, there must be at least three special points (singular or marked) on this component,
- analogously, if u is constant on a genus-1 component, it must contain at least one special point.

The "moduli space" of stable maps. The set of isomorphism classes of stable maps from genus-g curves with m marked points representing the class A in the manifold X will be denoted $\overline{\mathcal{M}}_{g,m}(X, A)$.

Remark 9.3. We are not considering the image curve, but notice also that we are not really considering the mappings: to quotient by isomorphisms destroys the parametrization.

When the isomorphism class of the marked curve $(\Sigma, \vec{z})$ cannot vary (*e.g.* if $g = 0$ and $m \leq 3$), it is possible to keep track of the parametrisation by looking at graphs (see Appendix B).

Remark 9.4. Notice that the singularities of the source curve are ordinary double points but that the image curve may be very singular. Think of the singular examples depicted in Figures 2 and 4.

10. Examples of spaces of stable maps

The case of a point. When X is a point, the map u is constant and we get the space $\overline{\mathcal{M}}_{g,m}$ of stable marked curves. Recall that the stability condition here is simply $m + 2g \geq 3$.

Constant maps. Let us look now at (more general) constant maps. We begin by stating and proving an elementary but crucial property.

Proposition 10.1. *Let ω be a Kähler form on X and A be a homology class. If A can be represented by a holomorphic map from a complex curve to X, it satisfies $\langle \omega, A\rangle \geq 0$, with equality if and only if the holomorphic map is constant.*

In particular, the class $A = 0$ can only be represented by *constant* holomorphic maps.

Proof. Let $u : \Sigma \to X$ be a holomorphic map. Since its derivative is complex linear, for any $z \in \Sigma$, any $\alpha \in T_z\Sigma$,

$$(u^\star\omega)_z(\alpha, i\alpha) = \omega_{u(z)}(T_z u(\alpha), iT_z u(\alpha)) \geq 0$$

as ω is a Kähler form: $X \mapsto \omega(X, iX)$ is a Riemannian metric. Thus

$$\langle \omega, A\rangle = \langle u^\star\omega, [\Sigma]\rangle \geq 0.$$

Moreover, if u is not constant, there is an open subset $U \subset \Sigma$ and a non zero vector field α on U such that $(u^\star\omega)_z(\alpha, i\alpha) > 0$ for $z \in U$. Thus $\langle \omega, A\rangle > 0$. □

If $A = 0$, we thus see that the map u is constant and hence the moduli space $\overline{\mathcal{M}}_{g,m}(X, 0)$ is isomorphic to $\overline{\mathcal{M}}_{g,m} \times X$.

Degree-1 maps. Another simple case is that of the space $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, L)$: we are dealing with degree-1 maps

$$u : \mathbb{P}^1 \longrightarrow \mathbb{P}^1.$$

Up to isomorphism, there is only one such map, so that the set $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, L)$ is a point. See a generalisation of this example in Appendix B.

Remark 10.2. More generally, one can replace L by dL: the moduli space $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, dL)$ is a compactification of the Hurwitz space of degree-d polynomials. One can also replace $\mathbb{P}^1$ by $\mathbb{P}^n$, it is clear that $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, L)$ is the Grassmannian $G(2, \mathbb{C}^{n+1})$ of lines in $\mathbb{P}^n$.

Plane conics. Consider the space $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 2L)$ of degree-2 curves in $\mathbb{P}^2$. It consists of four pieces:

(1) The space of smooth conics. Any conic is parametrized by a degree-2 map

$$u : \mathbb{P}^1 \longrightarrow \mathbb{P}^2.$$

This space has dimension 5 (three homogeneous polynomials of degree 2 up to rescaling give $9 - 1 = 8$, modulo automorphisms of $\mathbb{P}^1$ gives $8 - 3 = 5$).

(2) The space of degree-2 maps $\mathbb{P}^1 \to \mathbb{P}^2$ sending $\mathbb{P}^1$ to a line. It has dimension 4 (the line and the two branch points on it give $2 + 2 = 4$).

(3) The space of maps $C \to \mathbb{P}^2$, where C is the union of two secant lines and the image consists of two secant lines. It also has dimension 4 (two lines in $\mathbb{P}^2$ give $2 \times 2 = 4$).

(4) The space of maps $C \to \mathbb{P}^2$, where C is (again) the union of two secant lines and the image is a (double) line. It has dimension 3 (one line in $\mathbb{P}^2$ gives 2, adding a point on this line gives $2 + 1 = 3$).

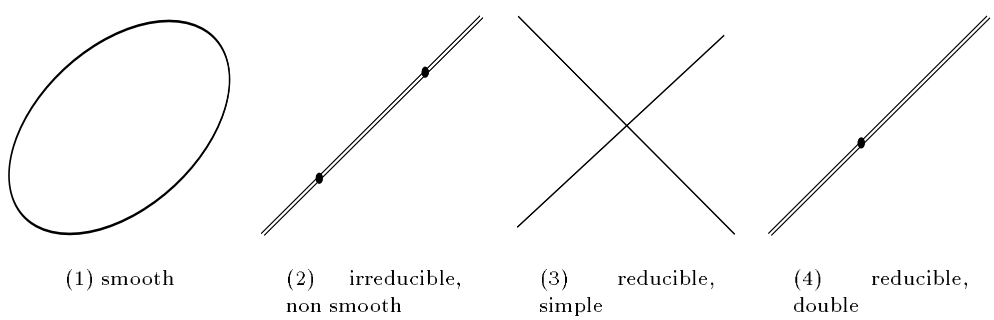


FIGURE 4. Conics

Rational degree-d plane curves. A description of the space of higher degree curves would be very complicated. Let us mention *rational* degree-d curves, as they appear below (in § 13 and in Theorem 14.10).

A smooth degree-d plane curve is never rational if $d \geq 3$. Its genus is $(d-1)(d-2)/2$. However there are rational plane curves in all degrees, but they are singular. For instance, a plane cubic is rational if and only if it is singular (see the example of the cuspidal cubic in Figure 2): to decrease the genus increases the singularities.

11. Evaluation and contraction

It is in the nature of the space $\overline{\mathcal{M}}_{g,m}(X, A)$ that it is related both to the manifold X and the space of stable curves $\overline{\mathcal{M}}_{g,m}$ respectively by

- evaluating the map at the marked points
- forgetting the map.

To be useful, these maps must be, at least, continuous: recall from § 8 that we plan to do some cohomology computations. Up to now the "space" $\overline{\mathcal{M}}_{g,m}(X,A)$ I have depicted is merely a set. Of course it has much more structure and we shall see below (Theorem 12.3) that, under certain assumptions, this is a projective variety. In general, it is, at least, a topological space: it can be endowed, for example with a variant of the $\mathcal{C}^k$-topology (for any k).

The topology is rather intricate but there are a few good reasons (besides fun) why it is interesting to describe it here:

- Although everything in this paper is in the algebraic category, all this theory originates in symplectic geometry and specifically in Gromov's compactness Theorem (see [Gro85] and [Pan94]) which uses the topology depicted below.
- I feel that the definition of this topology shows how, dealing with (convergence of) isomorphism classes of stable maps, we are somewhere between the convergence of the image curves and the convergence of the maps.
- It is very instructive to understand the attendence of singular curves in the "boundary" of the moduli space in terms of the birth of bubbles.

The purely algebraic readers can skip the definition. I will only use the continuity of the evaluation map (Proposition 11.2) and the compactness of the moduli space (Proposition 11.3).

Topology on $\overline{\mathcal{M}}_{g,m}(X,A)$. Here is a description of a basis of neighbourhoods of the class of a stable map $(\Sigma,\vec{z},u)$. Fix a real number $\varepsilon>0$, a neighbourhood U of the singular points of Σ, a neighbourhood V of the set of marked points and a metric μ on every component of Σ. The neighbourhood defined by these data consists of the stable maps $(\Sigma',\vec{z}',u')$ such that:

- there exists a continuous map $\sigma:\Sigma'\to\Sigma$ which is a diffeomorphism outside singular points and such that the inverse image of every double point is either a double point of Σ' or an annulus of modulus $\geq 1/\varepsilon$, containing no point of $\vec{z}'$,
- $\left\|u-u'\circ\sigma^{-1}\right\|_{\mathcal{C}^k}<\varepsilon$ outside U,
- $\left\|J_\Sigma-\sigma_\star^{-1}J_{\Sigma'}\right\|_{\mathcal{C}^k}<\varepsilon$,
- $\sigma(z_i')\in V$.

Remarks 11.1. (1) Recall that a complex annulus is conformally equivalent to $S^1\times]0,L[$ for a unique $L>0$, its *modulus*: conformally, to be very long means to be very thin.
(2) J_Σ is the complex structure of Σ, considered as an automorphism of $T\Sigma$, multiplication by i. The condition required on J_Σ and J'_Σ expresses the fact that Σ and Σ' shall be close to each other in the *ad hoc* moduli space. Norms are computed with the metric μ we have fixed on Σ and with the Kähler metric of the projective variety X.

There are two natural maps:

- the evaluation mapping
$$\begin{array}{rcl}\operatorname{ev}_m:\overline{\mathcal{M}}_{g,m}(X,A) & \longrightarrow & X^m\\ (\Sigma,\vec{z},u) & \longmapsto & (u(z_1),\dots,u(z_m))\end{array}$$
(for $m\geq 1$),
- the contraction mapping
$$\overline{\mathcal{M}}_{g,m}(X,A)\longrightarrow\overline{\mathcal{M}}_{g,m}$$
(for $m+2g\geq 3$). This is the map which forgets u, with a small technicality that I explain now. Trying to send the isomorphism class of $(\Sigma,\vec{z},u)$ to that of $(\Sigma,\vec{z})$, you face the difficulty that $(\Sigma,\vec{z})$ may have unstable components. They were allowed if u was not constant on them. The solution is to contract these components to points, *e.g.* as on Figure 5.

The topology is such that:

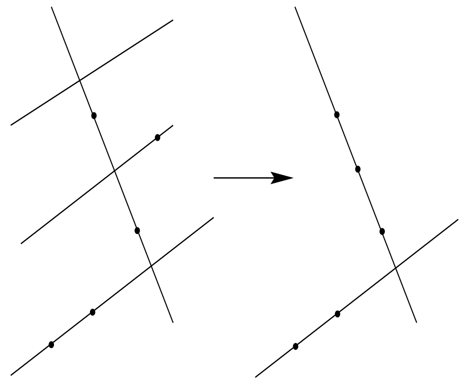


FIGURE 5

Proposition 11.2. *The evaluation at the marked points* $\mathrm{ev}_m : \overline{\mathcal{M}}_{g,m}(X,A) \to X^m$ *for* $m \geq 1$ *and the contraction mapping* $\overline{\mathcal{M}}_{g,m}(X,A) \longrightarrow \overline{\mathcal{M}}_{g,m}$ *for* $m + 2g \geq 3$ *are continuous.* □

Compactness. The main property is:

Proposition 11.3. *The space* $\overline{\mathcal{M}}_{g,m}(X,A)$ *is compact.*

Sketch of the proof. The contraction being continuous, this is a consequence of the compactness of the base space $\overline{\mathcal{M}}_{g,m}$ (see [Knu83]) and of the compactness of the fibers. The latter is a consequence of Theorem 11.4 below and can be viewed as a variant, either of the completeness of the *ad hoc* Hilbert scheme, or of Gromov compactness Theorem. □

Let us recall and comment the *ad hoc* version of this beautiful theorem.

Theorem 11.4. *Let* (X,ω) *be a Kähler manifold and* Σ *be a complex curve. Let* $u_n : \Sigma \to X$ *be a sequence of holomorphic maps. Assume there exists a constant* M *such that*

$$\langle u_n^\star \omega, [\Sigma] \rangle \leq M$$

for all n. *There exists a finite subset* $\Gamma \subset \Sigma$ *such that*

- *a subsequence of the sequence of graphs* $\widetilde{u}_n : \Sigma \to \Sigma \times X$ *converges to the graph* $\widetilde{u}$ *of a holomorphic map* $u : \Sigma \to X$ *outside* Γ *(for the* $\mathcal{C}^k$ *topology).*
- *If for* $z \in \Gamma$, $\widetilde{u}_n(z)$ *does not converge to* $\widetilde{u}(z)$, *then there is a non trivial rational curve*

$$\varphi_z : C_z \longrightarrow \{z\} \times X$$

passing through $u(z)$.
- *For* n *large enough,* $u_n^\star[\Sigma] = u_\star[\Sigma] + \sum_{z\in\Gamma} (\varphi_z)_\star [C_z] \in H_2(X;\mathbb{Z})$.

□

Remarks 11.5.
(1) In other words, a limit of graphs is again a graph except, maybe, for a few vertical bubbles (see Figure 6).
(2) In the algebro-geometric context where we have stated Theorem 11.4, this is actually a theorem on families and not only a theorem on sequences.
(3) The Kähler manifold X can be replaced by a Riemannian almost complex manifold here. Holomorphic maps must then be replaced by pseudo-holomorphic maps. Notice also that the almost complex structures

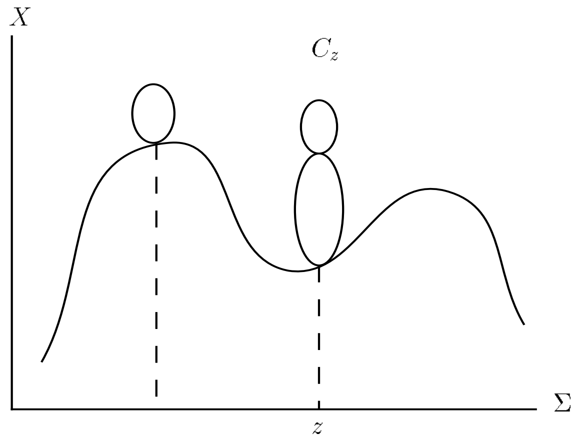


FIGURE 6

used may depend on n (see [Gro85, Pan94]). Even with these modifications, this is still a special case of Gromov's compactness theorem for the curve Σ is fixed.

(4) The rational curves C_z (the bubbles) may themselves have nodes, as is shown by the following example, which was given to me by Jean-Claude Sikorav.

Let Σ_1 be the plane cubic of homogeneous equation $y^3 = x^3 - xz^2$ and $u_\alpha : \Sigma_1 \to \mathbb{P}^2$ be the map $[x, y, z] \mapsto [x, \alpha y, \alpha^2 z]$, whose image is the curve Σ_α of equation $\alpha y^3 = \alpha^6 x^3 - xz^2$.

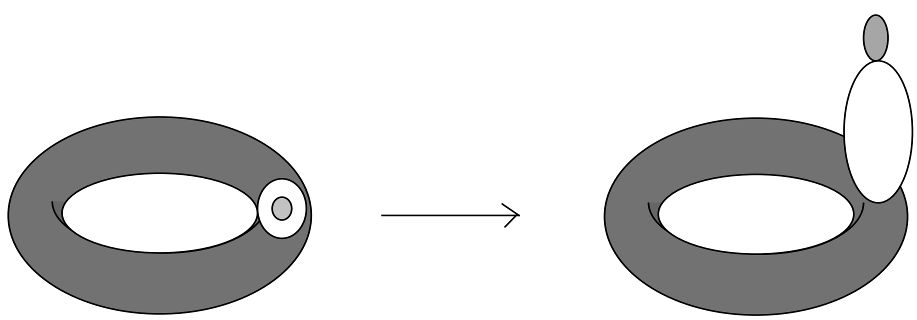


FIGURE 7

Let α tend to 0. The limit curve $xz^2 = 0$ consists of two lines, one of them being double. The map u_α converges to the constant map $u([x, y, z]) = [1, 0, 0]$ outside the inflexion point $[0, 0, 1]$. Thus the finite set Γ consists of a single point $z = [0, 0, 1]$ and the corresponding rational curve C_z is reducible and consists of two components, one of them being a double cover.

Viewed from the point of view of Gromov's theorem, one of the bubbles comes from a disc and the other one from an annulus (Figure 7).

Density of irreducible curves. Notice also that Kontsevich's space can often be considered as being too big in the sense that $\overline{\mathcal{M}}_{g,m}(X,A)$ is, in general bigger than the closure of the set of stable maps $\Sigma \to X$ where the curve Σ is irreducible.

This can be understood quite easily already in the case of plane genus-1 cubics. The space $\overline{\mathcal{M}}_{1,1}(\mathbb{P}^2, 3L)$ has dimension 11 (in general, $\dim \overline{\mathcal{M}}_{1,m}(\mathbb{P}^2, dL) = m + 3d + 1$). It contains all the maps $u : C \cup \mathbb{P}^1 \to \mathbb{P}^2$, where C has genus 1, C and $\mathbb{P}^1$ meet at some point, u is constant on C and has degree 3 on $\mathbb{P}^1$. In other words, using the excedentary marked points to glue curves together, it contains a copy of $\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,2}(\mathbb{P}^2, 3L)$. But this space, too, has dimension 11.

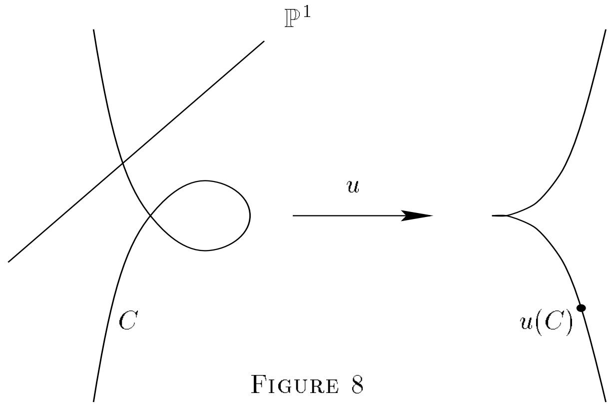


FIGURE 8

It is easy to understand that a map $u : C \cup \mathbb{P}^1 \to \mathbb{P}^2$ can only be in the closure of the set of maps $\Sigma \to \mathbb{P}^2$ (with Σ irreducible) if u has a singularity at the point where the two components meet: decreasing the genus adds singularities (as we have already mentioned it). See a counter-example on Figure 8: the rational component $\mathbb{P}^1$ is sent to a cuspidal cubic by a degree-3 map which sends the node to a regular point, the map u is constant on the genus-1 component C.

This cannot happen for rational (genus-0) curves in *convex* manifolds (see below): here maps from irreducible curves are dense in $\overline{\mathcal{M}}_{0,m}(X,A)$.

12. Convex manifolds and smoothness properties

The spaces $\overline{\mathcal{M}}_{g,m}(X,A)$ are often projective varieties (see [FP96], [Cor95]) and, as such, according to Borel and Haefliger [BH61], carry fundamental classes. However, we need much more, since we want to interpret cohomology computations in terms of intersection numbers (*e.g.* to get enumerative conclusions). That is, we need Poincaré duality (in other words a smoothness property) at least over $\mathbb{Q}$ (an orbifold structure would be enough). Also, if there were excedentary components, they could cause some difficulties dealing with enumerative questions.

For these reasons, we will now concentrate on rational curves (spaces $\overline{\mathcal{M}}_{0,m}(X,A)$). I will not try to obtain the best possible results and I will impose some restrictions on the projective manifold X (see below § 15 for generalisations and references).

Definition 12.1. The manifold X is convex for the class A, or A-convex, if, for any decomposition

$$A = A_1 + \cdots + A_k$$

where each A_j may be represented by a rational curve, and for each $u:\mathbb{P}^1\to X$ such that $u_\star[\mathbb{P}^1]=A_j$ for some j,

$$u^\star TX=\bigoplus_{i=1}^{n}\mathcal{O}(m_i)$$

with $m_i\geq 0$ for all i.

The convexity condition ensures that the various strata of $\overline{\mathcal{M}}_{0,m}(X,A)$ are smooth:

- the positivity condition on the m_i's implies that $H^1(\mathbb{P}^1,u^\star TX)=0$ for all maps u representing one of the classes A_j, or — in terms familiar to symplectic topologists — that the linearisation of the $\overline{\partial}$ operator associated with the complex structure on X is surjective (this is explained *e.g.* in § 2.1 of [Aud97]).
- But it also contains a positivity condition on the first Chern class expressed in the next lemma, that is related to compactness properties (again, this can be found *e.g.* in § 2.2 of [Aud97]).

Lemma 12.2. *If X is A-convex, then for any decomposition $A=A_1+\cdots+A_k$ where each A_j is non-zero and representable by a rational curve, $\langle c_1,A_j\rangle\geq 2$.*

Proof. By definition of A-convexity, for any map $u:\mathbb{P}^1\to X$ representing a class A_j, we have

$$u^\star TX=\bigoplus_{i=1}^{n}\mathcal{O}(m_i)\text{ with }m_i\geq 0\text{ for all }i.$$

Now, if A_j is non zero, u is non constant, hence its differential is injective (as a sheaf map) and carries a vector field with two zeroes on $\mathbb{P}^1$ to a non-zero section of $u^\star TX$ with at least two zeroes. Hence one of the m_i's must be at least 2. □

Now the A-convexity property gives everything we want:

Theorem 12.3. *If X is A-convex, $\overline{\mathcal{M}}_{0,m}(X,A)$ is a locally normal projective variety and locally the quotient of a nonsingular variety by a finite group. It has complex dimension* $\dim X+m-3+\langle c_1,A\rangle$.

There is a beautiful proof in [FP96]. Let us sketch it very briefly. The authors consider the case of the projective space ($X=\mathbb{P}^n$). Then they systematically add marked points to the curves in the space of stable maps, the points where the curve meets the coordinate hyperplanes, assuming these intersection points are distinct from the special (marked or singular) points already present on the curve. The choice of an ordering[4] for the new marked points gives an abstract stable marked curve. Notice that the data of the intersection points of the curve with the coordinate hyperplanes allows us to reconstruct the stable map, up to the action of the big torus $(\mathbb{C}^\star)^n$ on $\mathbb{P}^n$. In other words, we have established a close relation between some big open subset of $\overline{\mathcal{M}}_{0,m}(\mathbb{P}^n,A)$ and some subset of some $\overline{\mathcal{M}}_{0,N}$ — which is open in this case (genus 0).

The consideration of different choices of coordinates in $\mathbb{P}^n$ leads to a covering of $\overline{\mathcal{M}}_{0,m}(\mathbb{P}^n,A)$ with such open subsets. Then we can use the properties of $\overline{\mathcal{M}}_{0,N}$ to conclude the proof in the case of $\mathbb{P}^n$. The authors of [FP96] are then able to derive the property for any projective variety X. □

Remark 12.4. It is also shown in [FP96] that $\overline{\mathcal{M}}_{0,m}(X,A)$ is a moduli space in the sense of algebraic geometry (namely, it represents the *ad hoc* functor). There is also a universal stable map[5], that is, a space

$$\mathcal{U}_{0,m}\longrightarrow\overline{\mathcal{M}}_{0,m}(X,A)$$

which looks very much like $\overline{\mathcal{M}}_{0,m+1}(X,A)$ (the fiber at a point $(\Sigma,\vec{z},u)$ is the curve Σ itself) and which is endowed with a "universal map" to X (the evaluation at the $(m+1)^{\text{st}}$ point)

$$(\Sigma,\vec{z},u,z)\longmapsto u(z).$$

[4] This is a place where the "quotient by a finite group" comes from.

[5] at least over the subspace consisting of stable maps that have no automorphism at all ...

Definition 12.5. A projective manifold X is *convex* if it is convex for all classes containing rational curves.

Remark 12.6. Notice that this property is equivalent to the following:

$$\forall u : \mathbb{P}^1 \to X, \quad H^1(\mathbb{P}^1; u^\star TX) = 0.$$

Notice also that the property is true for *all* maps $u : \mathbb{P}^1 \to X$. This implies that the degrees of the summands of $u^\star TX$ are ≥ 0 (if one of them was ≤ -1, compose with a degree-2 map $\mathbb{P}^1 \to \mathbb{P}^1$ to get a degree-< -1 bundle and a contradiction).

Examples 12.7. (1) Convex manifolds include homogeneous spaces G/P where G is a linear algebraic group and P a parabolic subgroup, and in particular
- projective spaces $GL(n;\mathbb{C})/GL(1;\mathbb{C}) \times GL(n-1;\mathbb{C})$,
- Grassmannians $GL(n;\mathbb{C})/GL(k;\mathbb{C}) \times GL(n-k;\mathbb{C})$,
- and flag manifolds, complete (namely $GL(n;\mathbb{C})/GL(1;\mathbb{C}) \times \cdots \times GL(1;\mathbb{C})$) or not.

(2) There are very simple examples of *non*-convex manifolds. Consider for instance a Hirzebruch surface, that is, the total space of the bundle $\mathbb{P}(\mathcal{O}(k) \oplus \mathbf{1}) \to \mathbb{P}^1$. It is obvious that this surface contains an embedded rational curve C of self-intersection $-k$ (the section at infinity). Then, for the embedding $u : \mathbb{P}^1 \to \mathbb{P}(\mathcal{O}(k) \oplus \mathbf{1})$ whose image is C,

$$\langle c_1(u^\star TX), [\mathbb{P}^1] \rangle = -k + 2$$

so that, if $u^\star TX = \mathcal{O}(m_1) \oplus \mathcal{O}(m_2)$, $m_1 + m_2 = -k + 2$. If $k \geq 2$, at least one of the m_i's is ≤ -2. If $k = 1$, consider the composition of u with a self-map of degree 2 of $\mathbb{P}^1$ to get the same conclusion. Thus, the Hirzebruch surface $\mathbb{P}(\mathcal{O}(k) \oplus \mathbf{1})$ is convex if and only if $k = 0$.

(3) For instance, for $k = 1$, the Hirzebruch surface is $\widetilde{\mathbb{P}^2}$, the plane blown-up at a point. The homology class of the exceptional divisor is denoted E. It is represented by an isolated embedded rational curve. The composition with any self-map of degree 2 in $\mathbb{P}^1$ gives a rational curve in the class $2E$. Now, these degree-2 self-maps of $\mathbb{P}^1$ form a 5-dimensional space, so that

$$\dim \overline{\mathcal{M}}_{0,3}(\widetilde{\mathbb{P}^2}, 2E) \geq 5.$$

The dimension we could expect from Theorem 12.3 is 4: this confirms that $\widetilde{\mathbb{P}^2}$ is not convex.

Part 3. Quantum cohomology

13. Gromov-Witten invariants and potential

For convex manifolds, we are now able to achieve the program sketched in § 8 and to define the Gromov-Witten invariants. Let us use rational cohomology (so that $H^\star(X^m) = (H^\star(X))^{\otimes m}$). Recall that we concentrate on even-degree classes.

Definition 13.1. Let A be a homology class in $H_2(X;\mathbb{Z})$. Assume that X is A-convex. For classes $\xi_1, \ldots, \xi_m$ in $H^\star(X)$, the *Gromov-Witten invariant* $\Psi_A(\xi_1 \otimes \cdots \otimes \xi_m)$ is the rational number

$$\Psi_A(\xi_1 \otimes \cdots \otimes \xi_m) = \left\langle (\mathrm{ev}_m)^\star(\xi_1 \otimes \cdots \otimes \xi_m), \left[\overline{\mathcal{M}}_{0,m}(X,A)\right] \right\rangle.$$

Notice that $\Psi_A(\xi_1 \otimes \cdots \otimes \xi_m)$ can be non zero only if

$$\frac{1}{2}\sum_{i=1}^{m} \deg \xi_i = \dim \overline{\mathcal{M}}_{0,m}(X,A) = \dim X + m - 3 + \langle c_1, A \rangle.$$

It is also obvious that the number $\Psi_A(\xi_1 \otimes \cdots \otimes \xi_m)$ does not depend on the ordering of the cohomology classes inside. The first proposition states an important first property of the invariants: they contain (and thus generalise) the cup-product on X.

Proposition 13.2. *The Gromov-Witten invariants Ψ_A satisfy the relations*

$$\Psi_A(\xi_1 \otimes \cdots \otimes \xi_{m-1} \otimes 1) = 0 \text{ if } m \geq 4 \text{ or } A \neq 0$$

and

$$\Psi_0(\xi_1 \otimes \cdots \otimes \xi_m) = \begin{cases} 0 \text{ for } m > 3 \\ \langle \xi_1 \smile \xi_2 \smile \xi_3, [X] \rangle \text{ for } m = 3. \end{cases}$$

Proof. Assume $m \geq 3$ and consider the $A = 0$ case. Recall from § 10 that for the zero class, we have

$$\overline{\mathcal{M}}_{0,m}(X, 0) = \overline{\mathcal{M}}_{0,m} \times X$$

and that in this case the evaluation mapping is

$$\begin{array}{rcl} \overline{\mathcal{M}}_{0,m} \times X & \longrightarrow & X^m \\ (\Sigma, \vec{z}, x) & \longmapsto & (x, x, \ldots, x) \end{array}$$

therefore, if $p : \overline{\mathcal{M}}_{0,m} \times X \to X$ is the projection,

$$\begin{aligned} \left\langle (\mathrm{ev}_m)^\star(\xi_1 \otimes \cdots \otimes \xi_m), [\overline{\mathcal{M}}_{0,m} \times X] \right\rangle &= \left\langle p^\star \xi_1 \smile \cdots \smile p^\star \xi_m, [\overline{\mathcal{M}}_{0,m} \times X] \right\rangle \\ &= \begin{cases} 0 \text{ if } m > 3 \\ \langle \xi_1 \smile \xi_2 \smile \xi_3, [X] \rangle \text{ if } m = 3 \end{cases} \end{aligned}$$

as $\overline{\mathcal{M}}_{0,m}$ has positive dimension for $m > 3$.

Consider now the unit $1 \in H^0(X)$ of the cup-product. Recall this is the class dual to the submanifold $X \subset X$ itself and notice that to say that a marked point is sent to X says nothing. This is the remark that is formalized in the proof. By definition,

$$\Psi_A(\xi_1 \otimes \cdots \otimes \xi_{m-1} \otimes 1) = \left\langle (\mathrm{ev}_m)^\star(\xi_1 \otimes \cdots \otimes \xi_{m-1} \otimes 1), [\overline{\mathcal{M}}_{0,m}(X, A)] \right\rangle.$$

Assume $m \geq 4$ or $A \neq 0$, so that there is a well-defined map $\theta : \overline{\mathcal{M}}_{0,m}(X, A) \to \overline{\mathcal{M}}_{0,m-1}(X, A)$, forgetting the last marked point (and contracting the components that became unstable). Consider the commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,m}(X, A) & \xrightarrow{\mathrm{ev}_m} & X^m \\ \Big\downarrow \theta & & \Big\downarrow p \\ \overline{\mathcal{M}}_{0,m-1}(X, A) & \xrightarrow{\mathrm{ev}_{m-1}} & X^{m-1} \end{array}$$

where the projection p forgets the last factor. As $\xi_1 \otimes \cdots \otimes \xi_{m-1} \otimes 1 = p^\star(\xi_1 \otimes \cdots \otimes \xi_{m-1})$,

$$\begin{aligned} \left\langle (\mathrm{ev}_m)^\star(\xi_1 \otimes \cdots \otimes \xi_{m-1} \otimes 1), [\overline{\mathcal{M}}_{0,m}(X, A)] \right\rangle &= \left\langle \theta^\star(\mathrm{ev}_{m-1})^\star(\xi_1 \otimes \cdots \otimes \xi_{m-1}), [\overline{\mathcal{M}}_{0,m}(X, A)] \right\rangle \\ &= \left\langle (\mathrm{ev}_{m-1})^\star(\xi_1 \otimes \cdots \otimes \xi_{m-1}), \theta_\star[\overline{\mathcal{M}}_{0,m}(X, A)] \right\rangle. \end{aligned}$$

But, as $\dim \overline{\mathcal{M}}_{0,m-1}(X, A) < \dim \overline{\mathcal{M}}_{0,m}(X, A)$, the class $\theta_\star[\overline{\mathcal{M}}_{0,m}(X, A)]$ is in a zero homology group and thus is zero. □

Examples 13.3. (1) Let X be, once again, the complex projective space $\mathbb{P}^n$. Consider the space $\overline{\mathcal{M}}_{0,2}(\mathbb{P}^n, L)$. It has dimension $2n$ and its generic points are parametrised lines in $\mathbb{P}^n$ with two distinct marked points. It has an evaluation mapping to the $2n$ dimensional space $\mathbb{P}^n \times \mathbb{P}^n$. Recall that p is the generator of $H^2(\mathbb{P}^n; \mathbb{Z})$, so that $\Psi_L(p^n \otimes p^n)$ is a well-defined Gromov-Witten invariant. From the enumerative point of view, this is the number of lines through two distinct points (p^n is the cohomology class dual to a point):

$$\Psi_L(p^n \otimes p^n) = 1.$$

(2) The next example is that of degree-d rational plane curves, namely that of the space $\overline{\mathcal{M}}_{0,m}(\mathbb{P}^2, dL)$, a space of dimension $m-1+3d$. The evaluation map sends it to $(\mathbb{P}^2)^m$, a $2m$ dimensional space. When $m = 3d-1$, the two dimensions coincide and the evaluation map

$$\overline{\mathcal{M}}_{0,3d-1}(\mathbb{P}^2, dL) \longrightarrow (\mathbb{P}^2)^{3d-1}$$

has a *degree*, traditionally denoted by N_d. As the class dual to a point in $\mathbb{P}^2$ is p^2, this degree is the Gromov-Witten invariant $\Psi_{dL}\left((p^2)^{\otimes(3d-1)}\right)$. From the enumerative point of view, N_d is the number of rational plane curves of degree d through $3d-1$ general points. One of the most popular applications of the theory presented here was the computation of these numbers by Kontsevich [Kon95] that we will derive in § 14 (Theorem 14.10).

Degree-2 classes. It is in the nature of the Gromov-Witten invariants that degree-2 cohomology classes play a special role. Here is a property that we will use later and that illustrates this assertion.

Proposition 13.4. *Assume X is A-convex. Let ξ be a degree-2 cohomology class. Then, for any cohomology class α in $H^\star(X^{\otimes m})$,*

$$\Psi_A(\alpha \otimes \xi) = \langle \xi, A\rangle \Psi_A(\alpha).$$

This is the "divisor axiom" of Kontsevich and Manin. Assume that the cohomology class ξ is dual to a divisor (or a hypersurface) Z in X. Let us compare the two invariants appearing in this equality:

- On the one hand, the invariant $\Psi_A(\alpha)$ counts the rational curves with m marked points in the class A, with the marked points sent to the cycles determined by α,
- on the other hand, the invariant $\Psi_A(\alpha \otimes \xi)$ counts the rational curves in the same class A, which satisfy the same incidence conditions (the first m marked points going to the duals of the classes α_i in α) plus the fact that an additional marked point is sent to Z.

Of course, the actual rational curves are the same in both counts, but we have to choose the $(m+1)$st marked point at one of the intersection points of the curve and Z ... and there are $\langle \xi, A\rangle$ such points. This explains the formula. Here is a formal proof.

Proof of Proposition 13.4. Consider the map

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,m+1}(X, A) & \xrightarrow{\widetilde{\theta}} & \overline{\mathcal{M}}_{0,m}(X, A) \times X \\ (\Sigma, \vec{z}, u) & \longmapsto & (\theta(\Sigma, \vec{z}, u), u(z_{m+1})) \end{array}$$

(as above, θ forgets the last point and contracts the unstable components). As the dimension of $\overline{\mathcal{M}}_{0,m+1}(X, A)$ is $\dim \overline{\mathcal{M}}_{0,m}(X, A) + 1$,

$$\widetilde{\theta}_\star[\overline{\mathcal{M}}_{0,m+1}(X, A)] = [\overline{\mathcal{M}}_{0,m}(X, A)] \otimes A' + b$$

where $A' \in H_2(X)$ and the components of b on $H_\star(\overline{\mathcal{M}}_{0,m}(X, A))$ have dimension strictly less than the fundamental class. Obviously then,

$$\Psi_A(\alpha \otimes \xi) = \langle \xi, A'\rangle \Psi_A(\alpha),$$

so that we only need to check that $A' = A$. But this is obvious from the definition of $\widetilde{\theta}$, for if $(\mathbb{P}^1, \vec{z}, u)$ represents an element of $\overline{\mathcal{M}}_{0,m+1}(X, A)$, $u_\star[\mathbb{P}^1] = A$ and this is the class we see on the X factor of $\overline{\mathcal{M}}_{0,m}(X, A) \times X$. □

Example 13.5. The Gromov-Witten invariant $\Psi_L(p^n \otimes p^n \otimes p)$ in $\mathbb{P}^n$ counts the lines through two generic points which meet a generic hyperplane somewhere. Any line meets a generic hyperplane in exactly one point, so that the last condition adds nothing. Therefore

$$\Psi_L(p^n \otimes p^n \otimes p) = \langle p, L\rangle \Psi_L(p^n \otimes p^n) = \Psi_L(p^n \otimes p^n) \quad (= 1).$$

The potential. We want to define a function on $H^\star(X)$:

$$\xi \longmapsto \sum_A \left(\sum_{m\geq 3} \frac{1}{m!} \Psi_A(\xi^{\otimes m}) \right)$$

(to start the summation at $m = 3$ is enough, the potentials being defined only up to quadratic terms).

Fix a class $\xi \in H^\star(X)$ and a class $A \in H_2(X;\mathbb{Z})$. Assume first that ξ is homogeneous (that is, $\xi \in H^{2\ell}(X)$ for some ℓ, $\deg \xi = 2\ell$). The class $\Psi_A(\xi^{\otimes m})$ can only be non zero if

$$m \deg \xi = 2(\dim X + m - 3 + \langle c_1, A\rangle)$$

so that

- if $\deg \xi \neq 2$, $\Psi_A(\xi^{\otimes m}) \neq 0$ for only one value of m,
- if $\deg \xi = 2$, using Proposition 13.4,

$$\begin{aligned}\sum_{m\geq 3} \frac{1}{m!} \Psi_A(\xi^{\otimes m}) &= \sum_{m\geq 3} \frac{1}{m!} \langle \xi, A\rangle^{m-1} \Psi_A(\xi) \\ &= \Psi_A(\xi) \sum_{m\geq 3} \frac{1}{m!} \langle \xi, A\rangle^{m-1},\end{aligned}$$

  a convergent series.

Thus, in both cases, the sum

$$\sum_{m\geq 3} \frac{1}{m!} \Psi_A(\xi^{\otimes m})$$

defines a number[6]. Taking sums of homogeneous elements, we get

Lemma 13.6. *For any class $A \in H_2(X;\mathbb{Z})$, the formula*

$$\Phi_A(\xi) = \sum_{m\geq 3} \frac{1}{m!} \Psi_A(\xi^{\otimes m})$$

defines a function $\Phi_A : H^\star(X;\mathbb{C}) \to \mathbb{C}$. □

We still must sum over all classes A. To this aim, let us add a formal (multi-)variable q. Recall that we assume the second homology group of X to be free abelian. Let Λ be its group ring:

$$\Lambda = \mathbb{Z}[H_2(X;\mathbb{Z})].$$

Denote by q^A the (multiplicative) counterpart in Λ of $A \in H_2(X;\mathbb{Z})$. Thus q is a multi-variable of dimension $k = \operatorname{rk} H_2(X;\mathbb{Z})$.

Equivalently (but in a less intrinsic way), let $A_1, \ldots, A_k$ be a basis of the free abelian group $H_2(X;\mathbb{Z})$. Then

$$\Lambda = \mathbb{Z}[q_1, \ldots, q_k, q_1^{-1}, \ldots, q_k^{-1}]$$

and if $A = d_1 A_1 + \cdots + d_k A_k$, q^A denotes the monomial $q_1^{d_1} \cdots q_k^{d_k}$.

Define now

$$\Psi_q(\xi) = \sum_A \Phi_A(\xi) q^A = \sum_A \left(\sum_{m\geq 3} \frac{1}{m!} \Psi_A(\xi^{\otimes m}) \right) q^A.$$

This is a formal series in q whose coefficients are functions of ξ (involving only monomials and exponentials). To be more specific, let us choose a basis of $H^{2\star}(X;\mathbb{Z})$ (considered modulo torsion) consisting of homogeneous elements. It will be convenient to use different names for the coordinates according to the degrees, ξ_0 in degree

[6] A real number if we started with rational coefficients.

0, $\xi_1, \dots, \xi_m$ in degrees ≥ 4 and $\eta_1, \dots, \eta_k$ in degree 2. The basis, in the same order, is of course, $\partial/\partial\xi_i$ $(0 \leq i \leq m)$, $\partial/\partial\eta_j$ $(1 \leq j \leq k)$. Let us write

$$N_A(k_1, \dots, k_m) = \Psi_A \left[\left(\frac{\partial}{\partial\xi_1}\right)^{\otimes k_1} \otimes \cdots \otimes \left(\frac{\partial}{\partial\xi_m}\right)^{\otimes k_m} \right]$$

so that

$$\Psi_q(\xi, \eta) = \sum_A \left(\sum_{k_1, \dots, k_m} N_A(k_1, \dots, k_m) \frac{\xi_1^{k_1} \dots \xi_m^{k_m}}{k_1! \dots k_m!} \right) \exp\langle \eta, A \rangle q^A.$$

Remark 13.7. All this is completely formal, I do not discuss convergence problems here.

14. Quantum cohomology

We aim to show now that the Gromov-Witten potential is the potential of a formal Frobenius structure on $H^\star(X)$. We have a metric

$$g(\alpha, \beta) = \langle \alpha \smile \beta, [X] \rangle$$

which is constant in linear coordinates and thus flat. We need to define products $\star_\xi$ depending on $\xi \in H^\star(X)$, and we hope that they can be defined by the third derivative of Ψ. We begin by noticing that

$$g(\alpha \star_\xi \beta, \gamma) = \left(d^3\Psi\right)_\xi (\alpha, \beta, \gamma)$$

would define a commutative (by symmetry) and unital (using Proposition 13.2) product if there was no convergence problem. So, let us use the formal potential Ψ_q as above. Call $\widetilde{\Lambda}$ the module of formal series in the variables q, q^{-1}.

Proposition 14.1. *The formula*

$$g(\alpha \star_{\xi,q} \beta, \gamma) = \left(d^3\Psi_q\right)_\xi (\alpha, \beta, \gamma)$$

defines a commutative product $\star_{\xi,q}$ on $H^\star(X)$ with values in $H^\star(X) \otimes \widetilde{\Lambda}$, graded by the natural graduation of the cohomology and by $\deg q^A = 2\langle c_1, A \rangle$, and having the same unit as the cup product.

Remark 14.2. The module $\widetilde{\Lambda}$ has no ring structure, so that it is not possible just to extend the product by linearity to a ring structure on $H^\star(X) \otimes \widetilde{\Lambda}$.

Proof. The right hand side is indeed an element of $\widetilde{\Lambda}$. Notice that

$$\left(d^3\Psi_q\right)_\xi (\alpha, \beta, \gamma) = \sum_A \left(\sum_{m \geq 0} \frac{1}{m!} \Psi_A(\xi^{\otimes m} \otimes \alpha \otimes \beta \otimes \gamma) \right) q^A.$$

As above, the sum

$$\sum_{m \geq 0} \frac{1}{m!} \Psi_A(\xi^{\otimes m} \otimes \alpha \otimes \beta \otimes \gamma)$$

defines a complex number for all A, α, β, γ. The product $\star_{\xi,q}$ is obviously commutative and the assertion on the unit follows from Proposition 13.2. □

Remark 14.3. There are various definitions of "the quantum cup-product". This is the most general version, as it depends on the point ξ, and, because of the variables q (and the use of $\widetilde{\Lambda}$), there is no convergence problem in the definition.

Specialization and convergence assumptions. There are basically two ways to get a less general structure: to specialize, either at a value of ξ, or at a value of q, the latter leading to the mentioned convergence problems.

- To specialise at $q = 1$, allowing ξ to be any cohomology class, but assuming the series defining $\alpha \star_{\xi,1} \beta$, that we will denote $\alpha \star_\xi \beta$ to converge. This is the *global* quantum product.
- To specialise at $\xi = 0$ would give the "usual" *small* quantum product, a formal series in q, denoted $\alpha \star \beta$:

$$g(\alpha \star \beta, \gamma) = \sum_A \Psi_A(\alpha \otimes \beta \otimes \gamma) q^A.$$

As we shall see in § 16, these two apparently different specialisations are deeply related.

There are finiteness asumptions on the manifold X under which it is possible to use a sub-*ring* $\widehat{\Lambda}$ of $\widetilde{\Lambda}$ such that the product $\star$ defines a ring structure on $H^\star(X) \otimes \widehat{\Lambda}$.

This is the case if $H_2(X;\mathbb{Z})$ has a basis $A_1, \dots, A_k$ such that all the classes that can be represented by rational curves have *positive* coefficients in this basis, in other words lie in the first octant (Figure 9).

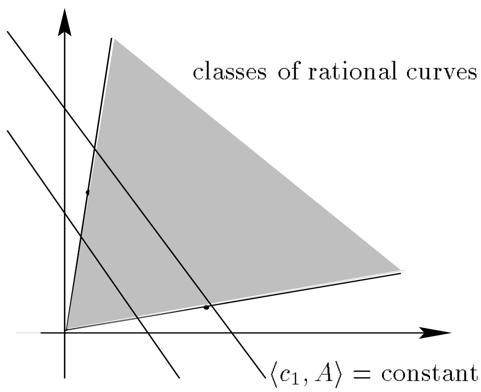


FIGURE 9

Notice that, if α, β and γ are elements of $H^\star(X)$, the invariant $\Psi_A(\alpha \otimes \beta \otimes \gamma)$ can be non zero only if

$$\frac{1}{2}(\deg \alpha + \deg \beta + \deg \gamma) = \dim X + \langle c_1, A \rangle,$$

which bounds the possible numbers $\langle c_1, A \rangle$:

$$-\dim X \leq \langle c_1, A \rangle \leq 2 \dim X.$$

- If we know that there are only finitely many classes A satisfying these inequalities, we can use the ring Λ itself. Let $(p_1, \dots, p_k)$ be the basis of $H^2(X)$ dual to $(A_1, \dots, A_k)$, so that $c_1 = \sum r_i p_i$ for some integers $r_1, \dots, r_k$. If all the r_i are positive, the classes form a finite set (Figure 9) and the product $\star$ defines a ring structure on $H^\star(X) \otimes \Lambda$ (we can even use the ring of the semi-group, namely $\mathbb{Z}[q_1, \dots, q_k]$ in this case). The manifold X is a *Fano* manifold. This is the case of $\mathbb{P}^n$ and of the other flag manifolds for instance.
- More generally, assume that X is convex and that, for all K, the set of homology classes

$$\{A \in H_2(X;\mathbb{Z}) \mid A \text{ can be represented by a rational curve and } \langle c_1, A \rangle \leq K\}$$

be finite. Recall that $\langle c_1, A \rangle$ is bounded from below (due to Lemma 12.2). It is possible to use the Novikov ring $\widehat{\Lambda}$ (see [HS95]) and our formulas give a ring structure to $H^\star(X) \otimes \widehat{\Lambda}$.

Composition rule and associativity. Let us prove now the non trivial result, related to what physicists call the "composition rule":

Theorem 14.4. *The product $\star_{\xi,q}$ is associative.*

All the proofs of the associativity of the quantum product are based on a degeneracy argument: we must understand what happens when the class A decomposes as a sum $A = A_1 + A_2$ (see [MS94] and [RT95] for pseudo-holomorphic approaches). The proof I give below comes from [FP96], and is based on a simple geometric argument:

- We will describe what happens in the homology of $\overline{\mathcal{M}}_{0,m}(X, A)$ when the homology class A decomposes as a sum. We will thus look at curves that split into two parts, each of them containing some of the marked points.
- Eventually, we reduce the homology computation to the investigation of the same problem in the moduli space $\overline{\mathcal{M}}_{0,4}$ of "four points on the line".

Four points on a line. Let us look first at the Deligne-Mumford space $\overline{\mathcal{M}}_{0,4}$. This is a compactification of the space $\mathcal{M}_{0,4}$ of (isomorphism classes of) sets of four ordered distinct points on $\mathbb{P}^1$. Now, four ordered distinct points on $\mathbb{P}^1$, up to isomorphism, this is a point in $\mathbb{P}^1 - \{\infty, 0, 1\}$, their cross ratio: send z_1 to ∞, z_2 to 0 and z_3 to 1 by a (unique) automorphism of $\mathbb{P}^1$, which sends z_4 to the cross ratio, so that

$$\mathcal{M}_{0,4} \cong \mathbb{P}^1 - \{\infty, 0, 1\}.$$

The Deligne-Mumford compactification is something very clever: when two of the marked points tend to the same point, they transform into a new rational component (figure 10).

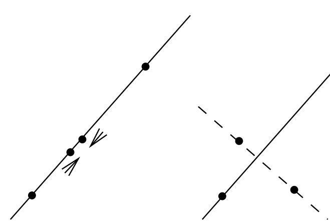


FIGURE 10

Of course the compactification $\overline{\mathcal{M}}_{0,4}$ of $\mathbb{P}^1 - \{\infty, 0, 1\}$ is isomorphic to $\mathbb{P}^1$. We have to add reducible rational curves consisting of two secant lines with two of the marked points on each. There are three isomorphism classes: up to isomorphism, the only thing you see is the labels of the points on the components (which are indistinguishable). Using the cross ratio, the three added points $(14 \mid 23)$, $(24 \mid 13)$ and $(34 \mid 12)$ correspond to ∞, 0 and 1 respectively.

Proof of Theorem 14.4. Associativity amounts to the fact that Ψ_q satisfies the WDVV equation (see § 4). Writing (∂_i) for a basis of the cohomology of X and often forgetting $\otimes$'s in the notation, this is:

$$\sum \frac{1}{m_1!m_2!} \Psi_{A_1}(\xi^{m_1}\partial_i\partial_j\partial_k) g^{k,n} \Psi_{A_2}(\xi^{m_2}\partial_r\partial_s\partial_n) q^{A_1+A_2}$$
$$= \sum \frac{1}{m_1!m_2!} \Psi_{A_1}(\xi^{m_1}\partial_r\partial_j\partial_k) g^{k,n} \Psi_{A_2}(\xi^{m_2}\partial_i\partial_s\partial_n) q^{A_1+A_2}.$$

Equating coefficients of powers of q, we see that we are discussing the decomposition of a homology class A as a sum $A_1 + A_2$ and the ways to represent this decomposition by stable maps. We thus need to understand what happens — homologically in $\overline{\mathcal{M}}_{0,m}(X, A)$ — when the curves we are looking at split into two components, each

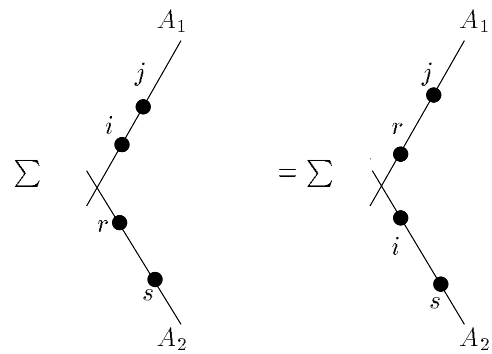


FIGURE 11

of them containing some of the marked points. (See Figure 11.) Let us write $\{1,\dots,m\} = E_1 \cup E_2$ (and require $\# E_i \geq 2$).

Fixing the decompositions of A and $\{1,\dots,m\}$, we get an embedding

$$j : \overline{\mathcal{M}}_{0,E_1\cup\{z_0\}}(X, A_1) \times_X \overline{\mathcal{M}}_{0,E_2\cup\{z_0\}}(X, A_2) \longrightarrow \overline{\mathcal{M}}_{0,m}(X, A)$$

(using obvious notation). For simplicity, let us write

$$M_i = \overline{\mathcal{M}}_{0,E_i\cup\{z_0\}}(X, A_i), \qquad M = \overline{\mathcal{M}}_{0,m}(X, A) \quad \text{and } D_{E_1,E_2}(A_1, A_2) = M_1 \times_X M_2.$$

As $\dim M_i = n + \# E_i - 2 + \langle c_1, A_i\rangle$, the dimension of $D_{E_1,E_2}(A_1, A_2)$ is $\dim M - 1$, so that we are considering a divisor in our moduli space. Call i the inclusion $D_{E_1,E_2}(A_1, A_2) \subset M_1 \times M_2$ and, if a marked point is labelled by some element p, call e_p the evaluation mapping at this point.

Lemma 14.5. *For any cohomology class $\eta = \eta_1 \otimes \cdots \otimes \eta_m$ in $H^\star(X)$,*

$$i_\star j^\star(\mathrm{ev}_m)^\star(\eta) = \sum_{k,n} g^{k,n} \left[\left(\prod_{p\in E_1} e_p^\star(\eta_p)\right) e_{z_0}^\star(\partial_k)\right] \times \left[\left(\prod_{q\in E_2} e_q^\star(\eta_q)\right) e_{z_0}^\star(\partial_n)\right].$$

Postponing the proof of the lemma, which is a direct computation, we get the equality

$$\sum_{\substack{a,b\in E_1\\ c,d\in E_2}} g^{k,n}\Psi_{A_1}\left(\left(\bigotimes_{p\in E_1}\eta_p\right)\otimes\partial_k\right)\Psi_{A_2}\left(\left(\bigotimes_{q\in E_2}\eta_q\right)\otimes\partial_n\right) = \sum_{\substack{a,b\in E_1\\ c,d\in E_2}} \langle \mathrm{ev}_m^\star(\eta), [D_{E_1,E_2}(A_1, A_2)]\rangle.$$

Put

$$\begin{cases} \eta_1 = \cdots = \eta_{m-4} = \xi \\ \eta_{m-3} = \partial_i, \eta_{m-2} = \partial_j, \eta_{m-1} = \partial_r, \eta_m = \partial_s \\ a = m-3, b = m-2, c = m-1, d = m \end{cases}$$

in the LHS to get

$$\mathrm{LHS} = \sum C(m_1, m_2) g^{k,n}\Psi_{A_1}\left(\xi^{m_1-2}\partial_i\partial_j\partial_k\right)\Psi_{A_2}\left(\xi^{m_2-2}\partial_r\partial_s\partial_n\right)$$

where $C(m_1, m_2)$ is the number of decompositions of $\{1,\dots,m\}$ as $E_1 \cup E_2$ with $\# E_i = m_i$ and the sum is over all decompositions with $m_1 \geq 2$ and $m_2 \geq 2$, so that

$$\mathrm{LHS} = m! \sum_{\substack{m_1,m_2\geq 0\\ m_1+m_2=m-4}} \frac{1}{m_1! m_2!} g^{k,n}\Psi_{A_1}\left(\xi^{m_1}\partial_i\partial_j\partial_k\right)\Psi_{A_2}\left(\xi^{m_2}\partial_r\partial_s\partial_n\right).$$

Lemma 14.6. *In $H_\star\left(\overline{\mathcal{M}}_{0,m}(X,A)\right)$, if a, b, c, d are four distinct elements of $\{1,\dots,m\}$,*

$$\sum_{\substack{a,b\in E_1\\ c,d\in E_2}} [D_{E_1,E_2}(A_1,A_2)] = \sum_{\substack{c,b\in E_1\\ a,d\in E_2}} [D_{E_1,E_2}(A_1,A_2)]\,.$$

This is precisely the symmetry argument needed to conclude the proof of Theorem 14.4. □

Proof of Lemma 14.6. This is the geometric heart and most beautiful part of the proof. The idea is that $\overline{\mathcal{M}}_{0,4}$ is isomorphic to $\mathbb{P}^1$ (see the description of $\overline{\mathcal{M}}_{0,4}$ above) and that any two points of $\mathbb{P}^1$ represent the same homology class (actually they are linearly equivalent — and this gives a linear equivalence of divisors rather than an equality of homology classes).

Look now at the composed contraction:

$$\overline{\mathcal{M}}_{0,m}(X,A) \to \overline{\mathcal{M}}_{0,m} \to \overline{\mathcal{M}}_{0,\{a,b,c,d\}}.$$

The divisor $\sum_{\substack{a,b\in E_1\\ c,d\in E_2}} D_{E_1,E_2}(A_1,A_2)$ is the inverse image of the point $(a,b\mid c,d)$, while the divisor

$$\sum_{\substack{c,b\in E_1\\ a,d\in E_2}} D_{E_1,E_2}(A_1,A_2)$$

is that of $(c,b\mid a,d)$. As the two points of $\mathbb{P}^1$ are linearly equivalent, so are the divisors in $\overline{\mathcal{M}}_{0,m}(X,A)$. □

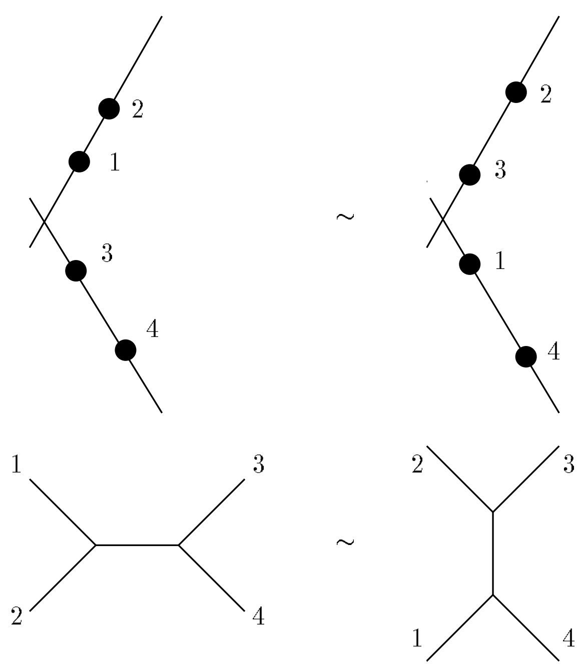


FIGURE 12

Remark 14.7. This can be easily expressed by pictures that are familiar to physicists (Figure 12). See [FI95] for related discussions.

Proof of Lemma 14.5. This is a straightforward computation, starting from the commutative diagram

$$\begin{array}{ccccc} M & \xleftarrow{\ j\ } & D_{E_1,E_2}(A_1,A_2) & \xrightarrow{\ i\ } & M_1 \times M_2 \\ {\scriptstyle \mathrm{ev}_m}\big\downarrow & & {\scriptstyle \varepsilon}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ev}_{m_1+1}\times \mathrm{ev}_{m_2+1}} \\ X^m & \xleftarrow{\ \mathrm{proj}\ } & X^{m+1} & \xrightarrow{\ \Delta\ } & X^{m+2} \end{array}$$

in which it is seen that

$$\begin{aligned} i_\star j^\star(\mathrm{ev}_m)^\star(\eta) &= i_\star \varepsilon^\star \mathrm{proj}^\star(\eta) \\ &= i_\star \varepsilon^\star\left(\eta \otimes D[X]\right) \\ &= \left(\mathrm{ev}_{m_1+1}\times \mathrm{ev}_{m_2+1}\right)^\star \Delta_\star\left(\eta\otimes D[X]\right) \\ &= \left(\mathrm{ev}_{m_1+1}\times \mathrm{ev}_{m_2+1}\right)^\star \left(\eta\otimes D[\Delta]\right). \end{aligned}$$

This is the place where one has to realise that the Poincaré dual of the diagonal in $X \times X$ is

$$D[\Delta] = \sum_{k,n} g^{k,n}\,\partial_k \otimes \partial_n .$$

Then,

$$\begin{aligned} i_\star j^\star(\mathrm{ev}_m)^\star(\eta) &= \textstyle\sum_{k,n} g^{k,n}\left(\mathrm{ev}_{m_1+1}\times \mathrm{ev}_{m_2+1}\right)^\star\left(\eta\otimes\partial_k\otimes\partial_n\right) \\ &= \textstyle\sum_{k,n} g^{k,n}\left[\left(\prod_{p\in E_1} e_p^\star(\eta_p)\right) e_{z_0}^\star(\partial_k)\right] \times \left[\left(\prod_{q\in E_2} e_q^\star(\eta_q)\right) e_{z_0}^\star(\partial_n)\right], \end{aligned}$$

which is the relation we wanted to prove. □

The Frobenius structure. In this paragraph, we collect all the information above to show that the quantum products $\star_\xi$ together with the flat metric defined by Poincaré duality, are pieces of a formal Frobenius structure on the cohomology of X. We still need an Euler vector field.

Let us first use a basis of $H^\star(X;\mathbb{Z})$ (considered modulo torsion) as above (end of § 13), express the first Chern class of X in this basis:

$$c_1 = \sum_{j=1}^{k} r_j \frac{\partial}{\partial \eta_j} \text{ for some integers } r_1, \ldots, r_k$$

and define the vector field

$$\mathcal{E} = \sum_{i=0}^{m} (1-\delta_i)\xi_i \frac{\partial}{\partial \xi_i} + \sum_{j=1}^{k} r_j \frac{\partial}{\partial \eta_j}.$$

Theorem 14.8. *Let X be a convex projective manifold. Poincaré duality, the quantum product and the Euler vector field $\mathcal{E}$ define the structure of a (formal) Frobenius manifold on the even part of the complex cohomology of X.*

Proof. Still using the notation of the end of § 13, recall that

$$\Psi(\xi,\eta) = \sum_A \exp\langle \eta, A\rangle \sum_{k_1,\ldots,k_m} N_A(k_1,\ldots,k_m)\frac{\xi_1^{k_1}\cdots\xi_m^{k_m}}{k_1!\ldots k_m!}.$$

Notice that $N_A(k_1,\ldots,k_m)$ can be nonzero only if

$$k_1(\delta_1 - 1) + \cdots + k_m(\delta_m - 1) = \dim X + \langle c_1, A\rangle - 3.$$

Now, the computation is straightforward:

$$\left(\sum r_j \frac{\partial}{\partial \eta_j}\Psi\right)(\xi,\eta) = \sum_A \langle c_1, A\rangle \exp\langle \eta, A\rangle \sum N_A(k_1,\ldots,k_m)\frac{\xi_1^{k_1}\cdots\xi_m^{k_m}}{k_1!\ldots k_m!}$$

and

$$\left(\sum(1-\delta_i)\xi_i\frac{\partial}{\partial\xi_i}\Psi\right)(\xi,\eta)=\sum_A\exp\langle\eta,A\rangle\sum_i(1-\delta_i)k_i\sum N_A(k_1,\dots,k_m)\frac{\xi_1^{k_1}\dots\xi_m^{k_m}}{k_1!\dots k_m!}.$$

Thus,

$$(\mathcal{E}\cdot\Psi)(\xi,\eta)=\sum_A\exp\langle\eta,A\rangle\sum\left(\sum(1-\delta_i)k_i+\langle c_1,A\rangle\right)N_A(k_1,\dots,k_m)\frac{\xi_1^{k_1}\dots\xi_m^{k_m}}{k_1!\dots k_m!}.$$

According to the remark on dimensions,

$$(\mathcal{E}\cdot\Psi)(\xi,\eta)=(3-\dim X)\Psi(\xi,\eta)$$

so that Ψ is indeed $\mathcal{E}$-homogeneous of degree $(3-\dim X)$. □

Remarks 14.9. (1) This is a place where 3-dimensional manifolds obviously play a special role.

(2) That there are exponentials in the potential Ψ is related to the terms $r_j\partial/\partial\eta_j$ in the expression of the Euler vector field. The potentials of the Frobenius manifolds obtained this way seem to be rather different from the ones coming from unfoldings: we have noticed at the beginning of § 7 that these potentials are polynomials, see also the baby-example in § 1, where the case $a(t_1)=t_1$ corresponds to the unfolding of z^3.

(3) The existence of the Gromov-Witten potential contains the symmetry of the tensor ∇c, or the closedness of the form Ω as in § 4. Hence the quantum products define a spectral cover. We will come back to this in § 16.

(4) By integration along the fibers of the contraction map

$$\pi_A:\overline{\mathcal{M}}_{0,m+1}(X,A)\longrightarrow\overline{\mathcal{M}}_{0,m+1}$$

we get morphisms

$$\begin{array}{ccc}H^\star(X)^{\otimes m+1} & \longrightarrow & H^\star(\overline{\mathcal{M}}_{0,m+1})\\ \xi & \longmapsto & (\pi_A)_\star\mathrm{ev}^\star_{m+1}(\xi)\end{array}$$

the tree-level system of Gromov-Witten invariants of Kontsevich and Manin [KM94].

Note that, dualising these morphisms, we get a family of morphisms

$$I_m^A:H_\star(\overline{\mathcal{M}}_{0,m+1})\longrightarrow\mathrm{Hom}\left(H^\star(X)^{\otimes m},H^\star(X)\right)$$

and the associativity property (§ 14) says that this is a morphism of operads. According to Manin [Man96], this is the essence of a Frobenius structure.

(5) It is possible to define more general invariants using more of the cohomology of $\overline{\mathcal{M}}_{0,m}(X,A)$ than the fundamental class. If $\zeta\in H^\star(\overline{\mathcal{M}}_{0,m}(X,A))$, one considers

$$\langle\mathrm{ev}^\star_m(\xi)\smile\zeta,[\overline{\mathcal{M}}_{0,m}(X,A)]\rangle.$$

This leads to the "gravitational descendents" of [KM94]. It can be shown (for the genus-0 invariants considered here) that they do not give more information than the invariants we have considered.

(6) A very interesting question is to know which projective varieties have a *massive* quantum cohomology. The question is whether there is a point ξ in $H^\star(X;\mathbb{C})$ for which $\star_\xi$ is the structure of a semi-simple ring. I will come back to this question in Appendix C.

The case of $\mathbb{P}^1$. Recall that in our notation,

$$H^\star(\mathbb{P}^1;\mathbb{Z})=\mathbb{Z}[p]/p^2.$$

Writing $\xi=\xi_0+\xi_1 p$, one gets easily

$$p\star_{\xi,q}p=e^{\xi_1}q.$$

Specialising at $q = 1$ gives a Frobenius structure on $\mathbb{C}^2$, which was one of our baby-examples (in § 1) and is everywhere semi-simple as e^{ξ_1} never vanishes. The potential is

$$\Psi(\xi) = \frac{1}{2}\xi_0^2\xi_1 + e^{\xi_1}$$

and the Euler vector field is indeed

$$\mathcal{E} = \xi_0\frac{\partial}{\partial\xi_0} + 2\frac{\partial}{\partial\xi_1}.$$

The potential for $\mathbb{P}^2$ and numbers of rational plane curves. Write $\xi = \xi_0 + \xi_1 p + \xi_2 p^2$ for the elements of $H^\star(\mathbb{P}^2)$, so that the Gromov-Witten potential

$$\Psi(\xi) = \sum_{m\geq 3}\frac{1}{m!}\sum_{d\geq 0}\Psi_{dL}(\xi^{\otimes m})$$

can be written

$$\Psi(\xi) = \frac{1}{2}\left(\xi_0^2\xi_2 + \xi_0\xi_1^2\right) + \sum_{d>0}\sum_{m\geq 3}\frac{1}{m!}\Psi_{dL}(\xi^{\otimes m}).$$

The first term is usually called "classical" (this is the cubic form corresponding to the cup-product, see Remark 4.2) and the second is the "quantum" contribution. This can be rewritten, using the fact that p has degree 2 and Proposition 13.4,

$$\sum_{m\geq 3}\sum_{d>0}\Psi_{dL}\left((p^2)^{\otimes m}\right)\frac{\xi_2^m}{m!}\exp\langle\xi_1 p, dL\rangle.$$

We have already noticed (in § 13) that the invariant $\Psi_{dL}((p^2)^{\otimes m})$ can be non zero only if $m = 3d-1$, in which case it is the number N_d of rational degree-d plane curves through $3d-1$ general points. Thus the Gromov-Witten potential for $\mathbb{P}^2$ is

$$\Psi(\xi) = \frac{1}{2}\left(\xi_0^2\xi_2 + \xi_0\xi_1^2\right) + \sum_{d\geq 1}N_d e^{d\xi_1}\frac{\xi_2^{3d-1}}{(3d-1)!}.$$

We have also noticed in § 4 that, in dimension 3, there is only one associativity equation, namely

$$\frac{\partial^3\Psi}{\partial\xi_2^3} = \left(\frac{\partial^3\Psi}{\partial\xi_1^2\partial\xi_2}\right)^2 - \frac{\partial^3\Psi}{\partial\xi_1^3}\frac{\partial^3\Psi}{\partial\xi_1\partial\xi_2^2}.$$

Equating the coefficients of $e^{d\xi_1}\xi_2^{3d-4}$ in the two sides of this equation gives:

Theorem 14.10 (Kontsevich [Kon95])**.** *The number N_d of degree d rational plane curves through $3d-1$ generic points is given by the recursion formula*

$$N_d + \sum_{\substack{d_1+d_2=d\\ d_1,d_2\geq 1}} d_1^3 d_2\binom{3d-4}{3d_1-1}N_{d_1}N_{d_2} = \sum_{\substack{d_1+d_2=d\\ d_1,d_2\geq 1}} d_1^2 d_2^2\binom{3d-4}{3d_1-2}N_{d_1}N_{d_2}$$

and the initial value $N_1 = 1$. □

This relation theoretically allows you (or your computer) to compute N_d starting from $N_1 = 1$.

15. More general Gromov-Witten invariants

There are Gromov-Witten invariants for all symplectic manifolds. They can be defined using "virtual fundamental classes": the space $\overline{\mathcal{M}}_{g,m}(X,A)$ might be very bad, various authors (see *e.g.* [Sie96, FO96, Beh97]) are nevertheless able to show that it carries a homology class of the right (expected) dimension, which can be used as a fundamental class.

In this section, I explain a simple case of such a construction, that can be used to define Gromov-Witten invariants for certain (non convex) submanifolds of convex manifolds, including *e.g.* complete intersections in projective spaces. This generalisation comes from [Kon95] and was used *e.g.* in [Giv96]. I also mention, very briefly, the relations between the Gromov-Witten invariants defined so far and other avatars available in the literature.

Convex vector bundles on convex manifolds. Let X be, as above, a convex projective manifold and let $V \to X$ be a vector bundle. It is said to be *convex* if it is generated by its (global) sections, that is, if $H^1(X;V)=0$. For instance, the tangent bundle TX of the convex manifold X is a convex vector bundle.

Let us look now at the zero set Y of a holomorphic section of V transverse to the zero section: this is a submanifold of X, whose normal bundle is the pull-back of V on Y:

$$j : Y \subset X, \qquad \nu_j = j^\star V.$$

There is no reason why Y should be convex. However, we aim to define some Gromov-Witten invariants for Y. The moduli space $\overline{\mathcal{M}}_{0,m}(Y,A)$ itself can be very bad, it may not be smooth, or not have the expected dimension.

Topologists are well-trained to solve such problems: find a vector bundle $\mathcal{V}$ on some bigger space such that $\overline{\mathcal{M}}_{0,m}(Y,A)$ is the zero set of some section of $\mathcal{V}$, and use the Euler class of $\mathcal{V}$ (the zero set of a section *transverse to the zero section*) in place of your moduli space.

In the case at hand, this works as follows:

- Show that there exists a vector bundle $\mathcal{V}$ on $\overline{\mathcal{M}}_{0,m}(X,j_\star A)$ (j is of course the inclusion $Y \to X$) whose fiber at a point represented by a map $u : C \to X$ is $H^0(C, u^\star V)$ (this can be achieved using the fact that $\overline{\mathcal{M}}_{0,m}(X,j_\star A)$ is a moduli space in the sense of algebraic geometry, endowed with a universal stable map — see § 12).
- For $\alpha \in H^\star(X)^{\otimes m}$, define

$$\Psi_A\left((j^m)^\star\alpha\right) = \left\langle (\mathrm{ev}_m)^\star\, \alpha \smile e(\mathcal{V}), \left[\overline{\mathcal{M}}_{0,m}(X,j_\star A)\right]\right\rangle.$$

  I insist that these invariants are only defined on cohomology classes coming from the ambient space X.

Remarks 15.1. (1) One easily checks that, if Y is A-convex, the definition coincides with the previous definition of $\Psi_A((j_m)^\star\alpha)$.

(2) One can also check that the result does not depend on the choice of the embedding in the convex manifold used.

(3) Because the Euler class is multiplicative, it can be shown that these invariants satisfy the composition rule as in § 14.

The example of the plane blown-up at a point. Let Y be the manifold obtained by blowing up $\mathbb{P}^2$ at a point. This is the Hirzebruch surface $\mathbb{P}(\mathcal{O}(1)\oplus \mathbf{1})$ and, as we have mentioned in the examples of § 12, if E is the class of the exceptional divisor, Y is not $2E$-convex.

Embed Y as the bidegree-$(1,1)$ hypersurface in $\mathbb{P}^1\times\mathbb{P}^2$, so that it is the zero set of a section of the convex line bundle $V=\mathcal{O}(1)\otimes\mathcal{O}(1)$:

$$Y = \left\{(\ell,d)\in\mathbb{P}^1\times\mathbb{P}^2 \mid d\subset \ell\oplus\mathbb{C}\right\} = \left\{([a,b],[x,y,z]) \mid ay-bx=0\right\}.$$

As the cohomology of Y is generated by ambient classes, the recipe above allows to compute all the Gromov-Witten invariants of Y. Let us concentrate on the invariant Ψ_{2E}.

Notice first that the image class $j_\star E$ is the class F of a fiber $\mathbb{P}^1 \times \{\mathrm{pt}\}$ in $\mathbb{P}^1 \times \mathbb{P}^2$. We must thus look at the moduli space $\overline{\mathcal{M}}_{0,m}(\mathbb{P}^1 \times \mathbb{P}^2, 2F)$. Notice that the maps in the class $2F$ have a constant projection on $\mathbb{P}^2$, so that

$$\overline{\mathcal{M}}_{0,m}(\mathbb{P}^1 \times \mathbb{P}^2, 2F) \cong \overline{\mathcal{M}}_{0,m}(\mathbb{P}^1, 2L) \times \mathbb{P}^2.$$

For the same reason, the bundle $\mathcal{V}$ we are looking at has the form

$$\mathcal{V} \cong \mathcal{V}' \otimes \mathcal{O}_{\mathbb{P}^2}(1) \longrightarrow \overline{\mathcal{M}}_{0,m}(\mathbb{P}^1, 2L) \times \mathbb{P}^2$$

where $\mathcal{V}' \to \overline{\mathcal{M}}_{0,m}(\mathbb{P}^1, 2L)$ is a bundle whose fiber at $u' : C \to \mathbb{P}^1$ is $H^0(C, u'^\star \mathcal{O}(1))$. The vector bundle $\mathcal{V}'$ has rank 3 and

$$e(\mathcal{V}) = c_3(\mathcal{V}') \otimes 1 + c_2(\mathcal{V}') \otimes t + c_1(\mathcal{V}') \otimes t^2 \in H^6\left(\overline{\mathcal{M}}_{0,m}(\mathbb{P}^1 \times \mathbb{P}^2, 2F)\right)$$

(we have denoted $t = c_1(\mathcal{O}_{\mathbb{P}^2}(1))$ the generator of $H^2(\mathbb{P}^2)$).

The next remark is that $\mathcal{V}'$ has two everywhere independent sections, so that $c_3(\mathcal{V}') = 0$, $c_2(\mathcal{V}') = 0$ and $e(\mathcal{V}) = c_1(\mathcal{V}') \otimes t^2$. To show this, choose two independent elements s_1 and s_2 in the dimension-2 vector space $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. Define sections $\widetilde{s}_1$, $\widetilde{s}_2$ of $\mathcal{V}'$ by

$$\widetilde{s}_i(C, \vec{z}, u) = s_i \circ u.$$

Now, if $\lambda\widetilde{s}_1 + \mu\widetilde{s}_2$ vanishes at some point of $\overline{\mathcal{M}}_{0,m}(\mathbb{P}^1, 2L)$ represented by $(C, \vec{z}, u)$,

$$(\lambda s_1 + \mu s_2) \circ u \equiv 0.$$

Thus the image of u must be contained in the zero set of $\lambda s_1 + \mu s_2$, which is a point of $\mathbb{P}^1$. Thus, u must be constant, a contradiction since it has degree 2. Thus the two sections $\widetilde{s}_1$ and $\widetilde{s}_2$ are indeed independent everywhere.

From this remark, it is easy to deduce the following result, which will be useful in the computation of § 16.

Proposition 15.2. *If $\alpha \in H^\star\left((\mathbb{P}^1 \times \mathbb{P}^2)^m\right)$ is divisible by t, then $\Psi_{2E}\left((j^m)^\star \alpha\right) = 0$.*

Proof. The assumption on the cohomology class α means that, if one writes

$$\alpha = \sum_i \alpha_1^i \otimes \cdots \otimes \alpha_m^i,$$

then, for every i, one of the α_j^i's has the form $a \otimes bt$ for some $a \in H^\star(\mathbb{P}^1)$, $b \in H^\star(\mathbb{P}^2)$. What we want to prove is that

$$\Psi_{2E}\left((j^m)^\star\left(\alpha_1 \otimes \cdots \otimes \alpha_{m-1} \otimes (a \otimes bt)\right)\right) = 0$$

for all $\alpha_i \in H^\star(\mathbb{P}^1 \times \mathbb{P}^2)$, $a \in H^\star(\mathbb{P}^1)$, $b \in H^\star(\mathbb{P}^2)$. But the latter is obtained as

$$\left\langle \mathrm{ev}_m^\star\left(\alpha_1 \otimes \cdots \otimes \alpha_{m-1} \otimes (a \otimes bt)\right) \smile \left(c_1(\mathcal{V}') \otimes t^2\right), \left[\overline{\mathcal{M}}_{0,m}(\mathbb{P}^1, 2L) \times \mathbb{P}^2\right]\right\rangle.$$

Now, the map

$$\mathrm{ev}_m : \overline{\mathcal{M}}_{0,m}(\mathbb{P}^1 \times \mathbb{P}^2, 2F) \longrightarrow (\mathbb{P}^1 \times \mathbb{P}^2)^m \cong (\mathbb{P}^1)^m \times (\mathbb{P}^2)^m$$

coincides with

$$\mathrm{ev}'_m \times \Delta : \overline{\mathcal{M}}_{0,m}(\mathbb{P}^1, 2L) \times \mathbb{P}^2 \longrightarrow (\mathbb{P}^1)^m \times (\mathbb{P}^2)^m$$

so that, decomposing the α_i's, the invariant is a sum of terms of the form

$$\left\langle \left[\mathrm{ev}'^\star_m(a' \otimes a) \otimes (b' \smile bt)\right] \smile \left(c_1(\mathcal{V}') \otimes t^2\right), \left[\overline{\mathcal{M}}_{0,m}(\mathbb{P}^1, 2L)\right] \otimes \left[\mathbb{P}^2\right]\right\rangle$$

which are zero since $t^3 = 0$ in $H^\star(\mathbb{P}^2)$. □

Relations with symplectic "Gromov-Witten invariants". Let us end this section by a few remarks on the other "Gromov-Witten invariants" the readers may have met in different contexts.

Gromov-Witten invariants come from symplectic topology.Various descriptions and alternative definitions for a (restricted) class of symplectic manifolds can be found in *e.g.* [RT95, MS94] (see also the survey [Aud97]).

There exist now symplectic and algebraic definitions available in greater generality — some using spaces of stable pseudo-holomorphic maps. See *e.g.* [Sie96, FO96, Beh97]. Without entering in that generality, let us make a few remarks on the symplectic approach.

Remarks 15.3. (1) In the symplectic context, Gromov-Witten invariants are invariants of the deformation class of the symplectic structure. This is where their qualification as "invariants" come from.

(2) The Ruan and Tian invariants were defined using moduli spaces of rational curves that are *almost* holomorphic with respect to some *almost* complex structure on the symplectic manifold X: the almost complex structure J defines a $\overline{\partial}$-operator $\overline{\partial}_J$, holomorphic curves satisfy $\overline{\partial}_J u = 0$ and you perturb the equation as $\overline{\partial}_J u = \nu$. This allows you to use transversality arguments to prove *e.g.* smoothness properties (but of course, you may lose the enumerative interpretation of your invariants).

(3) This approach by pertubations should be compared with the use of the Euler class above.

(4) It seems to me that one should be able to use both approaches at the same time: the symplectic approach is more flexible — but, dealing with enumerative questions, one does not like to vary the complex structure.

16. Degree-2 classes and the small quantum product

Notation. In this section, I prefer to use the pairing $\langle \alpha, b\rangle$ between homology and cohomology and Poincaré duality as an isomorphism between homology and cohomology instead of the metric g. To save on notation, I use greek characters for cohomology classes and latin for homology classes, Poincaré duality exchanging α and a etc. With this convention,

$$\langle \alpha, b\rangle = g(\alpha,\beta).$$

Quantum multiplication at a given point *vs* formal series. As we have noticed it above (see § 14), if ξ is a degree-2 class,

$$\langle \alpha \star_\xi \beta, c\rangle = \sum_A \Psi_A(\alpha\otimes\beta\otimes\gamma)\exp\langle \xi, A\rangle,$$

an expression which depends only of the class $[\xi]$ of ξ in $H^2(X;\mathbb{C}/2i\pi\mathbb{Z})$. Recall from § 14 that the small quantum product is given by

$$\langle \alpha \star \beta, c\rangle = \sum_A \Psi_A(\alpha\otimes\beta\otimes\gamma)q^A$$

which is exactly the same thing (write $q^A = \exp\langle \xi, A\rangle$). In other words, to compute $\alpha\star_\xi\beta$ in $H^\star(X;\mathbb{C})$ amounts to the same as to compute $\alpha\star\beta$ in $H^\star(X)\otimes\Lambda$ and to specialise it at the corresponding value of q.

Remarks 16.1.

- One can define in the same mood a variant of the global quantum product: in the formula for $\star_{\xi,q}$, replace q^A by $\exp\langle\omega, A\rangle$ for some fixed class ω, usually the cohomology class of the symplectic form.
- We do not really need that the manifold X be convex: we will use the generalised invariants defined in § 15. See for instance the example of the plane blown-up at a point (Proposition 16.8).

Products. The Gromov-Witten potential of a product $X_1\times X_2$ is not easily expressible in terms of the potentials of the factors (see [KM96]). However, the small quantum cohomology ring behaves very simply with respect to products[7].

[7] This is a good place to mention that there is no obvious functoriality property in this theory (except with respect to isomorphisms).

Proposition 16.2. *The small quantum cohomology ring of a product is isomorphic to the tensor product of the small quantum cohomology rings of the factors:*

$$QH^\star(X_1 \times X_2) \cong QH^\star(X_1) \otimes QH^\star(X_2).$$

Proof. In general, although to give a map $u : C \to X_1 \times X_2$ is equivalent to giving its components (u_1, u_2), because of the marked points and the stability requirement, there is no obvious relation between spaces $\overline{\mathcal{M}}_{0,m}(X_1 \times X_2, A)$ and spaces of stable maps into the factors. However, there is a map

$$\overline{\mathcal{M}}_{0,m}(X_1 \times X_2, A_1 \otimes 1 + 1 \otimes A_2) \longrightarrow \overline{\mathcal{M}}_{0,m}(X_1, A_1) \times \overline{\mathcal{M}}_{0,m}(X_2, A_2)$$

taking $(C, \vec{z}, u = (u_1, u_2))$ to the pair of stable maps obtained by contraction of unstable components in the factors. It is obviously not onto, but its image contains the set of pairs $((C_1, \vec{z}_1, u_1), (C_2, \vec{z}_2, u_2))$ such that $(C_2, \vec{z}_2)$ is isomorphic to $(C_1, \vec{z}_1)$. On this subspace, it is even possible to define an inverse map.

Now, when the number of marked points is 3, our image contains the open set where C_1 and C_2 are irreducible curves, so that there is a birational map

$$\overline{\mathcal{M}}_{0,3}(X_1, A_1) \times \overline{\mathcal{M}}_{0,3}(X_2, A_2) \longrightarrow \overline{\mathcal{M}}_{0,3}(X_1 \times X_2, A_1 \otimes 1 + 1 \otimes A_2).$$

It is defined on the subset of $\overline{\mathcal{M}}_{0,3}(X_1, A_1) \times \overline{\mathcal{M}}_{0,3}(X_2, A_2)$ consisting of pairs of stable maps $((C_1, \vec{z}_1, u_1), (C_2, \vec{z}_2, u_2))$ such that $(C_2, \vec{z}_2)$ is isomorphic to $(C_1, \vec{z}_1)$, which contains the open set where C_1 and C_2 are irreducible curves. It allows to relate fundamental classes and to prove that

$$\Psi_{A_1 \otimes 1 + 1 \otimes A_2}((\alpha_1 \otimes \alpha_2) \otimes (\beta_1 \otimes \beta_2) \otimes (\gamma_1 \otimes \gamma_2)) = \Psi_{A_1}(\alpha_1 \otimes \beta_1 \otimes \gamma_1) \Psi_{A_2}(\alpha_2 \otimes \beta_2 \otimes \gamma_2).$$

Thus

$$\langle (\alpha_1 \otimes \alpha_2) \otimes (\beta_1 \otimes \beta_2), c_1 \otimes c_2 \rangle = \left(\sum \Psi_{A_1}(\alpha_1 \otimes \beta_1 \otimes \gamma_1) q_1^{A_1} \right) \left(\sum \Psi_{A_2}(\alpha_2 \otimes \beta_2 \otimes \gamma_2) q_2^{A_2} \right)$$

and the proposition follows. □

Symplectic reduction. Consider the submanifold

$$B = H^2(X; \mathbb{C}) \xrightarrow{\ j\ } M = \oplus H^{2k}(X; \mathbb{C}).$$

We want to show that, in some cases, the small quantum product defines a Lagrangian subvariety of $T^\star B$. This is achieved using the general symplectic reduction process explained in § 6.

In the specific situation considerd here: to look at $j^\star T^\star M$ is the same thing as to look at the product $\star_\xi$ for a degree-2 class ξ, that is, the small quantum product. The intersection $L \cap j^\star T^\star M$ thus describes the relations in the quantum cohomology ring $QH^\star(X)$.

Let us make now a crucial assumption on the manifold X. We assume that the (classical) cohomology algebra $H^\star(X)$ is generated by $H^2(X)$, its degree-2 part. Denoting by $S^\star(X) = S\left[H^2(X; \mathbb{C})\right]$ the symmetric algebra on $H^2(X; \mathbb{C})$, this means that the natural ring morphism

$$S^\star(X) \longrightarrow H^\star(X)$$

is onto. This is the case, *e.g.* for projective spaces and more generally toric manifolds, complete flag manifolds.

Using a notation already used in § 14 (and that comes from [GK95]), call $(p_1, \ldots, p_k)$ a basis of $H^2(X)$. Assume X to be Fano, so that $\star$ is a product on $H^\star(X; \mathbb{C}) \otimes \Lambda$ (see § 14). The small quantum cohomology ring $QH^\star(X) = (H^\star(X; \mathbb{C}) \otimes \Lambda, \star)$ then consists of polynomials in p and q, with some relations. In other words, we have a surjective homomorphism:

$$S^\star\left(H^2(X; \mathbb{C})\right) \otimes \mathbb{C}[H_2(X; \mathbb{Z})] \longrightarrow QH^\star(X),$$

the kernel of which will be denoted $\mathcal{I}$. The ring in the LHS is nothing other than the ring of regular functions on the cotangent bundle $T^\star\mathcal{B}$ of the torus $\mathcal{B} = H^2(X; \mathbb{C}/2i\pi\mathbb{Z})$. Notice that, with this remarkable notation, the symplectic form on $T^\star\mathcal{B}$ is $\sum dp_i \wedge \dfrac{dq_i}{q_i}$.

As the reduction mod $2i\pi\mathbb{Z}$ is a covering map $B \to \mathcal{B}$ we get:

Corollary 16.3 (Givental & Kim [GK95]). *Assume X is a projective Fano manifold whose cohomology ring is generated by degree-2 classes. Assume that, for some value of q, the quantum product gives $H^\star(X)$ the structure of a semi-simple ring. Then $QH^\star(X)$ is the ring of functions on a Lagrangian subvariety of the cotangent bundle $T^\star\mathcal{B}$ of the torus $\mathcal{B} = H^2(X; \mathbb{C}/2i\pi\mathbb{Z})$.* □

Remark 16.4. In the context of degree-2 classes, Givental and Kim [GK95] have noticed that the homogeneity property with respect to the Euler vector field implies that the Lagrangian is quasi-homogeneous — and that the first Chern class can be considered as a primitive of the Liouville form (see also [Aud98]).

Here are a few examples of small quantum cohomology rings. The manifolds considered are Fano. Moreover, their cohomology ring is generated by degree-2 classes. The first example here is very well known, but difficult to avoid.

The complex projective space. This is of course a convex manifold. Recall that we denote by p the generator of the cohomology of $\mathbb{P}^n$, so that

$$H^\star(\mathbb{P}^n; \mathbb{Z}) = \mathbb{Z}[p]/p^{n+1} \text{ and } c_1 = (n+1)p.$$

We must add an invertible variable q of degree $2(n+1)$. Then

$$QH^\star(\mathbb{P}^n) = \mathbb{C}[p]/p^{n+1} \otimes \mathbb{C}[q, q^{-1}]$$

as a $\mathbb{C}[q, q^{-1}]$-module and the quantum multiplication is $\mathbb{C}[q, q^{-1}]$-linear. As the degree of q is $2(n+1)$, $p^{\star k} = p^k$ for $k \le n$ and we need only to compute $p^{\star(n+1)} = (p^n) \star p$. But

$$\langle p^n \star p, c\rangle = \sum_d \Psi_{dL}(p^n \otimes p \otimes \gamma) q^{dL}.$$

Recall that

$$\Psi_{dL}(p^n \otimes p \otimes \gamma) = \left\langle \mathrm{ev}_3^\star(p^n \otimes p \otimes \gamma), [\overline{\mathcal{M}}_{0,3}(\mathbb{P}^n, dL)]\right\rangle$$

so that it can only be non zero when $\deg\gamma = (n+1)d - 1$. As $0 \le \deg\gamma \le n$, this allows only $d = 1$ and $\deg\gamma = n$. Now we are looking at $\overline{\mathcal{M}}_{0,3}(\mathbb{P}^n, L)$ and at $\Psi_L(p^n \otimes p \otimes p^n)$. But we have already noticed in § 14 that this is the number of lines in $\mathbb{P}^n$ through two general points that meet a general hyperplane somewhere, that is, 1. In conclusion:

Proposition 16.5. *The small quantum cohomology ring $QH^\star(\mathbb{P}^n(\mathbb{C}))$ is isomorphic to the ring*

$$\mathbb{Z}[p, q, q^{-1}]/(p^{n+1} - q).$$

Remark 16.6. In this example the multiplication is obviously generically semi-simple and the Lagrangian is the curve $p^{n+1} = q$ in $\mathbb{C} \times \mathbb{C}^\star$.

The plane blown up at a point. Let us come back to the manifold Y obtained by blowing up a point in the complex projective plane. This is one of the examples investigated in [CM95, RT95, Aud97] for instance. Recall from Examples 12.7 that Y is *not* a convex manifold.

Figure 13 shows Y as the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}(1) \oplus \mathbf{1})$ over $\mathbb{P}^1$. The hachured quadrilateron is Y itself. The triangle represents the plane $\mathbb{P}^2$: the horizontal base is a $\mathbb{P}^1$ not through the blown-up point and the top vertex is the blown-up point. The edge labeled A_1 is the exceptional divisor (the $\mathbb{P}^1$ which replaces the point in Y). The lines in $\mathbb{P}^2$ through the blown-up point become the fibers of the fibration $Y \to \mathbb{P}^1$ so that this fibration is simply the projection from the ghost point to the horizontal line.

All properties of homology and cohomology classes I will write are easily derived from this description. Let A_1 be the class of the exceptional divisor (it was called E in Examples 12.7 and § 15) and A_2 be the class of a line through the blown-up point, or of a fiber (see Figure 13). The two classes A_1 and A_2 form a basis of $H_2(Y; \mathbb{Z})$ with

$$A_1 \cdot A_1 = -1, \quad A_1 \cdot A_2 = 1, \quad A_2 \cdot A_2 = 0.$$

Let (p_1, p_2) be the basis of $H^2(Y;\mathbb{Z})$ dual to (A_1, A_2) (that is, such that $\langle p_i, A_j\rangle = \delta_{i,j}$), so that, denoting again Poincaré duality by D,

$$Dp_1 = A_2, \qquad Dp_2 = A_1 + A_2$$

(as $(Dp_i)\cdot A_j = \delta_{i,j}$). Also,

$$\langle p_i \smile p_j, [Y]\rangle = (Dp_i)\cdot(Dp_j)$$

so that $p_1^2 = 0$, $\langle p_1 \smile p_2, [Y]\rangle = \langle p_2 \smile p_2, [Y]\rangle = 1$. The (classical) cohomology ring is

$$H^\star(Y;\mathbb{Z}) = \mathbb{Z}[p_1, p_2]/\langle p_1^2, p_2^2 - p_1p_2\rangle.$$

The first Chern class is $p_1 + 2p_2$. Let us add two new variables q_1 and q_2 of respective degrees 2 and 4.

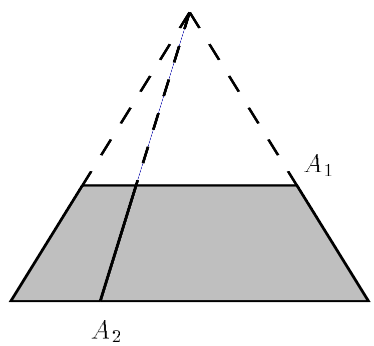


FIGURE 13. The plane blown up at a point

To determine the structure of the quantum cohomology ring, we need only to compute the products $p_i \star p_j$. For this, we need all the invariants $\Psi_A(p_i \otimes p_j \otimes \gamma)$, for all homology classes A containing holomorphic curves and all cohomology classes γ such that $2\langle c_1, A\rangle = \deg\gamma$. This implies in particular that $0 \le \langle c_1, A\rangle \le 2$. Moreover, if a non zero class A can be represented by a rational map $\mathbb{P}^1 \to Y$, it is a multiple of a class A' that can be represented by a somewhere injective rational map. But these must satisfy the adjunction formula, namely

$$A'\cdot A' - \langle c_1, A'\rangle + 2 \ge 0.$$

Writing $A' = m_1' A_1 + m_2' A_2$, this writes

$$(1 - m_1')(m_1' - 2m_2' + 2) \ge 0.$$

From these restrictions, it is seen that the only invariants that we need to compute are Ψ_0, Ψ_{A_1}, Ψ_{A_2} and Ψ_{2A_1} (this is the reason why I have chosen this basis of $H_2(Y;\mathbb{Z})$).

Recall that the class $2A_1$ is the one which prevents Y to be convex (see the examples in section 12). However, a consequence of Proposition 15.2 is:

Lemma 16.7. *The invariant* $\Psi_{2A_1}(p_i \otimes p_j \otimes \gamma)$ *is zero for all* i, j, γ.

Proof. As p_i and p_j have degree 2 and $\langle c_1, 2A_1\rangle = 2$, the invariant $\Psi_{2A_1}(p_i \otimes p_j \otimes \gamma)$ can only be non zero when γ has degree 4. Now the only non zero classes in $H^4(Y)$ have the from $j^\star(a \otimes t)$ for some $a \in H^2(\mathbb{P}^1)$ and we can use Proposition 15.2. □

We are left with the computation of Ψ_{A_1} and Ψ_{A_2}.

- Consider first $\Psi_{A_1}(p_i \otimes p_j \otimes \gamma)$. We know that this is zero unless $\deg\gamma = 2$ (so that all the classes involved have degree 2) and that there is only one curve in the class A_1. Thus

$$\Psi_{A_1}(p_i \otimes p_j \otimes \gamma) = \delta_{i,1}\delta_{j,1}\langle\gamma, A_1\rangle.$$

- Look now at $\Psi_{A_2}(p_i \otimes p_j \otimes \gamma)$. It vanishes unless $\deg \gamma = 4$, so that we must only consider the case where γ is Poincaré dual to a point. Now, there is exactly one curve of the class A_2 through any given point of Y and thus
$$\Psi_{A_2}(p_i \otimes p_j \otimes D[\mathrm{pt}]) = \delta_{i,2}\delta_{j,2}.$$

We can now calculate our monomials $p_i \star p_j$. By definition
$$\langle p_i \star p_j, c\rangle = \langle p_i \smile p_j, c\rangle + \Psi_{A_1}(p_i \otimes p_j \otimes \gamma)q_1 + \Psi_{A_2}(p_i \otimes p_j \otimes \gamma)q_2.$$

- $\langle p_1 \star p_1, c\rangle = (c \cdot A_1)q_1 = \langle DA_1, c\rangle q_1$ and thus
$$p_1 \star p_1 = (p_2 - p_1)q_1,$$
- $\langle p_1 \star p_2, c\rangle = \langle p_1 \smile p_2, c\rangle$, so that
$$p_1 \star p_2 = p_1 \smile p_2,$$
- $\langle p_2 \star p_2, c\rangle = \langle p_2 \smile p_2, c\rangle + \Psi_{A_2}(p_2 \otimes p_2 \otimes \gamma)q_2$ and
$$p_2 \star p_2 = p_2 \smile p_2 + q_2.$$

This gives the desired result:

Proposition 16.8. *The small quantum cohomology ring of the plane blown up at a point is isomorphic to the ring*
$$\mathbb{Z}[p_1, p_2, q_1, q_1^{-1}, q_2, q_2^{-1}]/\langle p_1^2 - (p_2 - p_1)q_1, p_2^2 - p_1p_2 - q_2\rangle.$$
□

Remarks 16.9.
- That the ideal generated by our two relations $f = p_1^2 - (p_2 - p_1)q_1$ and $g = p_2^2 - p_1p_2 - q_2$ defines a Lagrangian subvariety of $T^\star(\mathbb{C}^\star)^2$ is easily checked, as $\{f, g\} = q_1 g$ is in the ideal.
- Proposition 16.8 appears in [CM95] (where the basis used is not symplectic) and in [RT95] (in a different basis[8]).
- There is also a proof of Proposition 16.8 using (a less convenient basis and) symplectic invariants in [Aud97]. There the class $2A_1$ was avoided using spaces of "somewhere injective" holomorphic curves as in [MS94], which gives in this case the same invariants as the "almost" holomorphic curves of [RT95].

Part 4. Appendices

A. Elliptic curves and plane cubics

In this appendix, I try to motivate the introduction of the spaces of stable maps by a description of the difference between (abstract) elliptic curves and plane cubics.

Elliptic curves. An (abstract) elliptic curve is, by definition, a smooth genus-1 complex curve C with a prefered point P on it. To say that the genus is one is to say that the dimension of the complex vector space $H^0(\Omega^1_C)$ of holomorphic 1-forms is one. According to the Abel-Jacobi theorem, there is an isomorphism
$$\begin{array}{ccc} C & \longrightarrow & H^0(\Omega^1_C)^\star/\Lambda \\ Q & \longmapsto & \left(\omega \mapsto \int_P^Q \omega\right) \end{array}$$
where Λ is the period lattice, *i.e.* the isomorphic image of $H_1(C;\mathbb{Z}) \cong \mathbb{Z}^2$ in $H^0(\Omega^1_C)^\star \cong \mathbb{C}$. The marked point P is sent to 0.

The space of isomorphism classes of elliptic curves can then be described as an appropriate quotient of the space of lattices in $\mathbb{C}$.

[8] and with a wrong sign.

Plane cubics. To any such lattice is associated a Weierstrass $\wp$-function. Together with its derivative, it defines an embedding

$$\begin{array}{ccc} \mathbb{C}/\Lambda & \longrightarrow & \mathbb{P}^2(\mathbb{C}) \\ z & \longmapsto & [\wp(z), \wp'(z), 1]. \end{array}$$

Due to the differential equation satisfied by $\wp$, our elliptic curve is a plane cubic, with (affine) equation

$$C' : \qquad y^2 = 4x^3 - g_2 x - g_3$$

where g_2 and g_3 are complex constants defined by the lattice Λ, satisfying $g_2^3 - 27g_3^2 \neq 0$.

Recall that $\wp$ has a double pole at 0, so that the marked point P, alias 0, is sent to $[0,1,0]$, the point at ∞ on the curve C'.

The moduli space of elliptic curves. It turns out that, up to isomorphism, the elliptic curve (C, P) is determined by a single numerical invariant, which can be easily expressed in terms of the constants g_2 and g_3 above:

$$j = \frac{1728 g_2^3}{g_2^3 - 27 g_3^2}.$$

For instance, t being fixed and ε varying, all the curves of equations

$$\Gamma_\varepsilon^t : \qquad y^2 = 4x^3 - \varepsilon^2 x - \varepsilon^3 t$$

should be isomorphic, as they have the same j-invariant (namely $1728/(1-27t^2)$), and they actually are, since $(x, y) \mapsto (\alpha^2 x, \alpha^3 y)$ extends to an isomorphism $\Gamma_1^t \to \Gamma_{\alpha^2}^t$.

Singular cubics. There are of course a lot of different singular plane cubic curves: a conic plus a line, three lines, and so on. Among the cubics having an affine equation of the form

$$y^2 = 4x^3 - g_2 x - g_3$$

there are curves with an ordinary double point ($g_2^3 - 27g_3^2 \neq 0$, g_2 and $g_3 \neq 0$), or with a cusp ($g_2 = g_3 = 0$). Notice that these are rational curves, being parametrised by the set of lines through the singular point.

Compactification of the moduli space. Once we have accepted that the moduli space of elliptic curves is isomorphic[9] to $\mathbb{C}$ (via j), it seems to be easy to compactify: just add a point at infinity, allowing $j = \infty$.

This way, you certainly add the curve with a double point ... but you cannot "add the curve with a cusp point": because of the families Γ_ε^t, all converging to $y^2 = 4x^3$ when ε goes to 0, this curve would be arbitrarly close to *all* elliptic curves.

Notice that $j = \infty$ for curves with a double point, but also that the rational function of g_2 and g_3 has an "indeterminate form" $0/0$ for the cusp curve, and that this is why we were able to get any j:

$$j(\Gamma_\varepsilon^t) = \frac{1728}{1 - 27t^2} \text{ for } \varepsilon \neq 0$$

therefore

$$\lim_{\varepsilon \to 0} j(\Gamma_\varepsilon^t) = j(\Gamma_1^t) = \frac{1728}{1 - 27t^2}.$$

Notice that Theorem 11.4 describes exactly what happens in the family Γ_ε^t when $\varepsilon \to 0$: the genus tends to be concentrated at the singular point. In other words, the map u mentioned in Theorem 11.4 is constant, its graph is horizontal, and there is a bubble whose image is the whole rational cuspidal cubic.

The idea of stable maps is to construct, starting from the space of *maps* of genus-1 curves to the complex projective plane, a space that contains parametrisations of cusp curves taking into account the phenomenon just described.

[9]There should be some structure to give a sense to this word. The whole space $\overline{\mathcal{M}}_{1,1} = \mathbb{C} \cup \{\infty\}$ can be given an algebraic structure. It looks like $\mathbb{P}^1$ but has orbifold-type singularities at points corresponding to elliptic curves which have a non trivial group of automorphisms, namely for $j = 0$, 1728 and ∞.

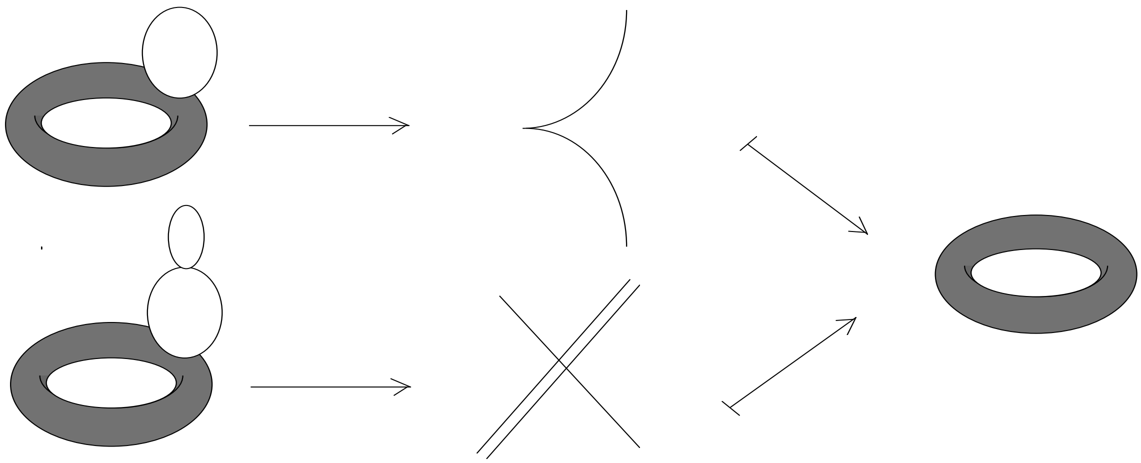


FIGURE 14. Some unstable singular cubics

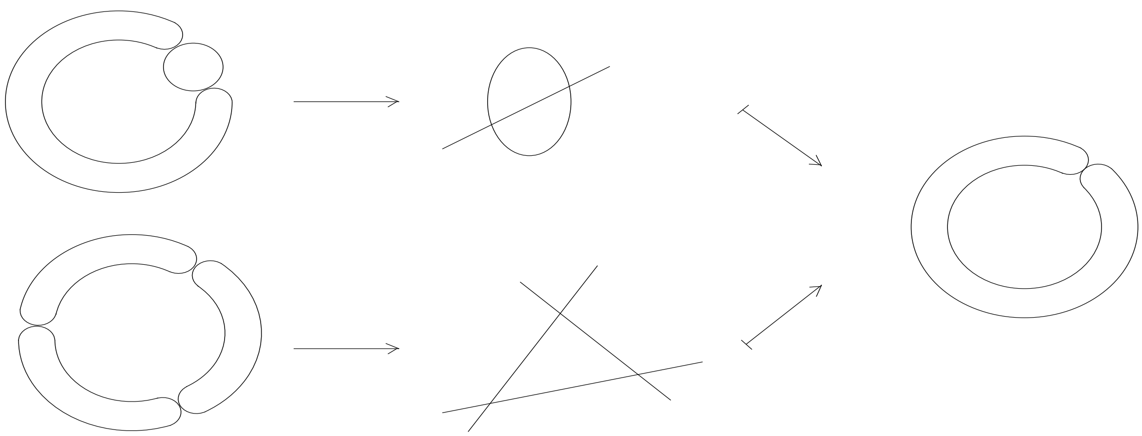


FIGURE 15. Some semi-stable singular cubics

Stable maps. Consider plane cubics parametrised by elliptic curves, that is, consider the contraction mapping

$$\overline{\mathcal{M}}_{1,1}(\mathbb{P}^2, 3L) \longrightarrow \overline{\mathcal{M}}_{1,1}.$$

The figures show some singular plane cubics, (in a mixing of real and complex representations). First a cusp curve and two lines, one being double, parametrised by reducible curves over an arbitrary elliptic curve in $\overline{\mathcal{M}}_{1,1}$ (Figure 14), then various other singular cubics, parametrised by reducible curves over the point at infinity in $\overline{\mathcal{M}}_{1,1}$ (Figure 15).

Remarks A.1. (1) In the space of stable maps $\overline{\mathcal{M}}_{1,1}(\mathbb{P}^2, 3L)$, there are cusp curves over every points of the space $\overline{\mathcal{M}}_{1,1}$ (I have explained above that this is something one should expect). The same property is true for all "unstable cubics", that is, for those who have other singularities than ordinary double points, as the cubics consisting

- of a conic and a (tangent) line,
- or of two lines, one of which is a double line,

- or of three lines through the same point.

(2) In contrast, for "semi-stable" singular cubics (with ordinary double points only), the j-invariant is infinite and all these curves are images of stable maps over the point at infinity in $\overline{\mathcal{M}}_{1,1}$.

(3) There are explicit examples of sequences of smooth cubics converging to singular cubics in Lizan's thesis [Liz96].

I describe further examples of moduli spaces of stable maps, related to algebraic loop spaces and configuration spaces in Appendix B.

B. Algebraic loop spaces and configuration spaces

In this appendix, I give examples of spaces of stable maps from a *fixed* curve to X. For simplicity, I consider only the case of rational curves.

The examples discussed here are not only beautiful examples, they are very useful in Givental's approach to the mirror conjecture [Giv95a, Giv95b, Giv96] for they provide a compactification of the algebraic loop space of a projective manifold.

We start with algebraic loops in X, represented as maps $\mathbb{P}^1 \to X$ ... in other words as *parametrised* rational curves.

We then fix the homology class and consider the space $L_A X$ of maps $\mathbb{P}^1 \to X$ in the class A. This space looks very much like $\mathcal{M}_{0,0}(X, A)$ — except that, in $\mathcal{M}_{0,0}(X, A)$, we have factored out by isomorphisms, thus loosing part of the parametrisation. To keep track of the parametrisation, it if of course sufficient to consider the graph of the maps, that is, to look at the embedding

$$L_A X \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}^1 \times X, L \oplus A).$$

We see that the moduli spaces of stable maps include compactifications of the spaces of algebraic loops.

Example B.1. Take $X = \mathbb{P}^n(\mathbb{C})$ and $A = dL$. We are looking at degree-d algebraic loops in the projective space. An element in $L_d\mathbb{P}^n$ can be represented by $n+1$ degree-d polynomials without common root (modulo multiplication by a scalar). Therefore our space $L_d\mathbb{P}^n$ has an obvious, "naïve", compactification as a projective space $\mathbb{P}^{(n+1)(d-1)-1}$. We thus have two distinct compactifications:

- The naïve one, $L'_A\mathbb{P}^n = \mathbb{P}^{(n+1)(d-1)-1}$ $(A = dL)$, where we are simply allowed to consider polynomials of degree $\leq d$.
- The clever one, $\overline{L}_A\mathbb{P}^n = \overline{\mathcal{M}}_{0,0}(\mathbb{P}^1 \times X, L \oplus A)$, where we allow graphs of degree-$\leq d$ maps $\mathbb{P}^1 \to \mathbb{P}^n$ with vertical components, their total degree being d.

Notice that there exists a map

$$\mu : \overline{L}_A\mathbb{P}^n \longrightarrow L'_A\mathbb{P}^n,$$

used by Givental in [Giv96], that is a morphism and whose set-theoretical definition is the following. Let $u : \Sigma \to \mathbb{P}^1 \times \mathbb{P}^n$ be a bidegree-$(1, d)$ stable map (representing an element of $\overline{L}_A\mathbb{P}^n$). All components of Σ are copies of $\mathbb{P}^1$. One of them, C_0, say, which is sent by u onto the graph of a degree-d' map ($d' \leq d$) from $\mathbb{P}^1$ to $\mathbb{P}^n$. It can be represented by an $(n+1)$-tuple $(f_0, \dots, f_n)$ of degree-d' polynomials. The other components, $C_1, \dots, C_r$ are bubbles, that is, C_i is a rational curve, u has bidegree $(0, d_i)$ on C_i and sends C_i to $z_i \times \mathbb{P}^n$ for some point z_i in $\mathbb{P}^1$ $(d_1 + \cdots + d_r = d - d')$.

Consider a degree-$(d - d')$ polynomial g, that vanishes at each z_i with multiplicity d_i and associate to the stable map u the $(n+1)$-tuple of polynomials $(f_0 g, \dots, f_n g)$, that is an element of $L'_A\mathbb{P}^n$.

Configuration spaces. In this situation, there is an analogue of the contraction morphism considered in § 11, the natural map

$$\overline{\mathcal{M}}_{0,m}\left(\mathbb{P}^1 \times X, L \oplus A\right) \longrightarrow \overline{\mathcal{M}}_{0,m}(\mathbb{P}^1, L).$$

We have already met $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, L)$ and noticed that this was a single point. In the same way, for each degree-1 map

$$u : \Sigma \longrightarrow \mathbb{P}^1,$$

there exists a distinguished component C_0 of Σ. This is the only component on which u is non constant, and where the map u is an isomorphism $C_0 \to \mathbb{P}^1$. Using an isomorphism of Σ, we can assume that $u = \mathrm{Id}$ on C_0, so that we are left with a compactification of the configuration space of m distinct points on $\mathbb{P}^1$ (without quotient).

Such a compactification is considered in greater generality in [FM94] for a general projective variety Y instead of $\mathbb{P}^1$. In the $\mathbb{P}^1$-case, it is easily seen that the compactification as moduli space of stable maps and the Fulton-MacPherson compactification coincide (the description by "screens" in [FM94] actually coincides with that of reducible curves we are using here).

C. The semi-simplicity assumption

One of the reasons why *massive* Frobenius manifolds were interesting for Dubrovin is that they parametrise isomonodromic deformations (see [Dub95, Hit97, Sab98, Man96]).

One cannot expect the quantum cohomology of a projective manifold to be massive in general: if the manifold contains no rational curve (think for instance of a $K3$ surface), all its quantum products are nilpotent. Better candidates for semi-simplicity are the Fano manifolds, as they contain many rational curves. The semi-simplicity of the quantum cohomology of Fano manifolds is conjectured by Tian [Tia95] and Manin [Man96].

Recall (see § 5) that the semi-simplicity of the ring $(T_\xi M, \star_\xi)$ implies the existence of *canonical coordinates* on a neighbourhood of the point ξ in the Frobenius manifold M. In these coordinates, the product is split and the Euler vector field can be written

$$\mathcal{E}(\xi) = \sum_i x_i \frac{\partial}{\partial x_i}.$$

Notice that, by definition, the x_i's are then the eigenvalues of the multiplication by $\mathcal{E}(\xi)$ at ξ. All the data being analytic, this implies:

Proposition C.1. *For a Frobenius manifold, the following properties are equivalent*

(1) *The ring $(T_\xi M, \star_\xi)$ is semi-simple for generic ξ.*
(2) *There exists a point ξ in M such that the ring $(T_\xi M, \star_\xi)$ is semi-simple.*
(3) *There exists a point ξ in M and a vector α in $T_\xi M$ such that all the eigenvalues of the multiplication by α are simple.*
(4) *Multiplication by $\mathcal{E}(\xi)$ at ξ has only simple eigenvalues for generic ξ.* □

Notice also that semi-simplicity of the small quantum cohomology ring (for generic q) would imply semi-simplicity of global quantum cohomology. For instance:

Proposition C.2. *The quantum cohomologies of $\mathbb{P}^n$, $\widetilde{\mathbb{P}^2}$ are massive Frobenius manifolds.*

Proof. It is easily checked that the small quantum cohomology rings computed in § 16 are semi-simple for generic q. The minimal polynomial of multiplication by p in $QH^\star(\mathbb{P}^n)$ is $z^{n+1} - q$, that of multiplication by p_1 in $QH^\star(\widetilde{\mathbb{P}^2})$ is $z^4 + q_1 z^3 + q_1^2 q_2$. Both coincide with the characteristic polynomial and both are split for generic q. □

But there are examples for which the small quantum cohomology ring is not sufficient to check the semi-simplicity.

Proposition C.3. *Let $X^n \subset \mathbb{P}^{n+r}$ be a smooth complete intersection of dimension $n \geq 3$ and of degree $(d_1, \ldots, d_r)$. Assume*

$$\sum_{i=1}^{r} (d_i - 1) \leq \frac{n+1}{2}.$$

There is a value of q for which the small quantum cohomology ring of X is semi-simple if and only if X is a linear subspace or a quadric (that is, $\sum(d_i - 1) = 0$ or 1).

Proof. This is based on Beauville's computation in [Bea95]. According to the Lefchetz theorem, the complex cohomology of X is generated by the class dual to the hyperplane section, $\eta \in H^2(X;\mathbb{C})$ and the primitive cohomology $H_0^n(X;\mathbb{C})$. The first Chern class is $k\eta$, ($k = n+1-\sum(d_i-1)$). According to [Bea95], η satisfies a relation

$$\eta^{n+1} - aq\eta^{n+1-k} = 0$$

for some (non zero) integer a.

In the classical cohomology ring, the minimal polynomial of the cup-product by η, $\eta \smile \cdot$ is z^{n+1}, so that this shows that the minimal polynomial of the quantum multiplication $\eta \star \cdot$ is $z^{n+1} - aqz^{n+1-k}$. This polynomial has a multiple root for all q as soon as

$$n + 1 - k \geq 2, \text{ that is, } \sum(d_i - 1) \geq 2.$$

If this is the case, the endomorphism $\eta\star\cdot$ is *not* diagonalisable ... and this prevents $QH^\star(X)$ to be semi-simple (for all values of q).

When $\sum(d_i-1) = 0$, the complete intersection X is a projective space $\mathbb{P}^n(\mathbb{C})$ and we have already mentioned above that $QH^\star(X)$ was (generically) semi-simple. We are left with the case of quadrics.

If n is odd, the primitive cohomology is zero, so that $z^{n+1} - aqz$ is also the minimal polynomial and all the eigenvalues of $\eta \star \cdot$ are simple for $q \neq 0$.

If n is even, $H_0^n(X)$ is 1-dimensional, generated by the dual to the generator of $H_n(X_{\text{aff}})$ (affine quadric). Still using [Bea95], we know that, for $\alpha \in H_0^n(X)$,

$$\begin{cases} \eta \star \alpha &= 0 & \text{as for } \eta \smile \alpha \\ \alpha \star \alpha &= b(\eta^n - aq) & \text{for some non-zero } b \text{ if } \alpha \neq 0. \end{cases}$$

Thus the endomorphism $\eta\star\cdot$ is diagonalisable (its minimal polynomial is split) but it has 0 as a double eigenvalue. The corresponding eigenspace (kernel) is generated by $\eta^n - aq$ and a generator α of $H_0^n(X)$. However, quantum multiplication by α, restricted to this kernel, has two distinct eigenvalues for $q \neq 0$, since

$$\begin{cases} \alpha \star (\eta^n - aq) &= -aq\alpha \\ \alpha \star \alpha &= b(\eta^n - aq). \end{cases}$$

□

Remarks C.4.
(1) The result for $\mathbb{P}^n$ and quadrics also holds of course for $n = 2$ as well.
(2) The proposition gives *e.g.* the semi-simplicity of the small quantum cohomology ring of $G_2(\mathbb{C}^4)$ (using the Plücker embedding, this is a quadric in $\mathbb{P}^5$). See [Ber97, ST94, Wit93, FP96] for the computation of the small quantum cohomology ring of Grassmannians, [GK95, CF95] for that of flag manifolds.

Thus, there is no hope that the small quantum cohomology ring be semi-simple in general, even for Fano complete intersections. However, for the global quantum cohomology of the same complete intersections, one can prove:

Proposition C.5. *Let X be a smooth complete intersection of odd dimension $n \geq 3$. Let $\eta \in H^2(X;\mathbb{Z})$ be the generator and assume that the degree of X is such that the first Chern class $k\eta$ of X satisfies*

$$\frac{n+1}{2} \leq k \leq n+1.$$

There is a point ξ in $H^\star(X)$ such that all the eigenvalues of $\eta \star_\xi \cdot$ are simple. □

As a consequence (recall we consider only the even part of the cohomology), the quantum cohomology of such an odd dimensional complete intersection is massive. The proof is based on a simple approximation argument that I hope to make more explicit elsewhere. Notice that the massivity of the quantum cohomology for these examples can also be deduced from the computation of [TX96].

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Institut de Recherche Mathématique Avancée, Université Louis Pasteur et CNRS, 7 rue René-Descartes, F–67084 Strasbourg Cedex

E-mail address: Michele.Audin@math.u-strasbg.fr