Integrability of Hamiltonian Systems

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A satellite moves along its orbit around the Earth. A circular orbit, an apparently quiet revolution, nevertheless accompanied with various motions: the satellite oscillates, rotates, turns upside down.

The *attitude*, this is how physicists call these motions – motions that can obviously not be ignored. Let us pretend that the satellite decides (deplorable attitude) to turn its back to the Earth, the antennae and the cameras now looking at the other side.

This is obviously not what was intended when the satellite was put into orbit. It is thus necessary to know how to describe and control these changes in the attitude. Will the satellite always perform the same motion? Will it wriggle restlessly?

The mechanical system governing the motion of the satellite is a Hamiltonian system. The question I just asked can be made more precise and reformulated as *is this Hamiltonian system integrable?*. A theorem of Morales and Ramis can be applied to answer this question, using differential Galois theory. The aim of this article is to give an idea of the statement of this theorem and the ways to apply it.

Integrable systems

A *Hamiltonian system* is a mechanical system governed by the celebrated Hamilton equations

$$\begin{cases} \dot{q}_1 = \dfrac{\partial H}{\partial p_1}, \dots, \dot{q}_n = \dfrac{\partial H}{\partial p_n}, \\ \dot{p}_1 = -\dfrac{\partial H}{\partial q_1}, \dots, \dot{p}_n = -\dfrac{\partial H}{\partial q_n}. \end{cases}$$

The total energy H of such a system, a function of $q_1, \dots, q_n$ (think of positions) and $p_1, \dots, p_n$ (think of momenta), is constant, and it does not depend on time. It happens that other quantities are conserved as well. They are called *first integrals*.

If there are enough first integrals (and if they commute, in a sense that I will not make more precise here), Liouville showed, in the 19-th century, that the differential system (Hamilton equations) can be solved by quadratures (computing integrals); this is why the system is said to be integrable.

Examples

There are many well known mechanical systems that are integrable.

Let us carry out an entertaining experiment. Rather than putting a satellite in orbit (which is complicated and expensive), we just play with a spinning top. And we observe it spinning, looking carefully at the motion of the end of its axis.

For those of you who do not have a spinning top at home, Figures 1 and 2 show the object and the result of the experiment. The top is considered as a rigid body with a fixed point (the point O at which it meets the horizontal plane) in a constant (vertical) gravitational field. The rigid body has an axis of revolution in this case, so that, in addition to the total energy, the momentum with respect to this axis is a first integral. The end of the axis oscillates between two parallel circles on an (ideal) sphere.

Here is another easy experiment. We fix a ball at an end of a rod, the other end of which is fixed: what we get is a pendulum, usually called a *spherical pendulum*. Here again, the only force present is the gravitation. The momentum with respect to the vertical (direction of the gravitation field) is a first integral. The ball turns, stuck between two parallel circles on a sphere, as shown on Figure 4.

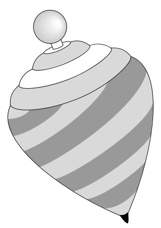

Figure 1: A spinning top

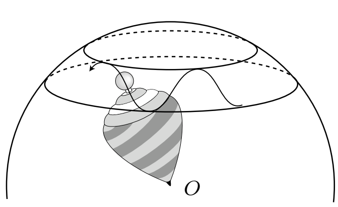

Figure 2: The end of the axis

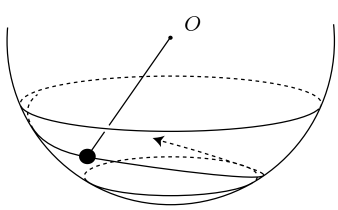

Figure 3: A spherical pendulum

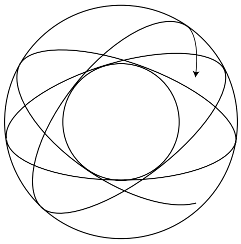

Figure 4: A trajectory of the spherical pendulum (from above)

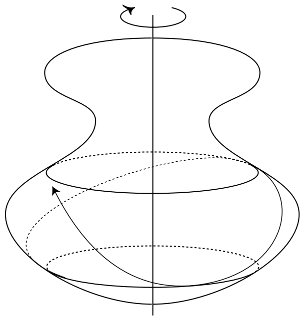

Figure 5: A geodesic on a surface of revolution

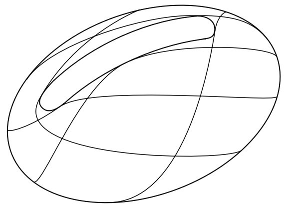

Figure 6: A geodesic on an ellipsoid

You may have noticed the similarities between these two experiments. A similar behaviour (oscillations in a band) can be observed in many other mechanical problems, as, for instance, the motion of a free particle on a surface of revolution or an ellipsoid. A free particle goes the shortest way – this is why the solutions are the geodesics on the surface. Figures 5-6 represent a geodesic on a surface of revolution and of an ellipsoid (respectively). In the case of the surface of revolution, the momentum of the particle with respect to the axis of revolution is a first integral. For a generic ellipsoid there is also a first integral, although this is less obvious (this is due to Jacobi).

A geometrical or dynamical expression of Liouville integrability (as defined above) is the regularity of the solutions. The motion described by an integrable Hamiltonian system is very regular, the trajectories wind on tori (this is part of the Arnold-Liouville theorem), each visiting regularly a neighbourhood of its initial point, and the motion is said to be *quasi-periodic*.

Figure 7 shows a quasi-periodic motion that is drawn in the configuration space (an annulus, on the sheet of paper), or in phase space (in more dimensions, on a torus) that looks very much like the previous ones. The Arnold-Liouville theorem is more precise – it states that these trajectories are *linear* in the sense of the affine structure of the torus, as shown in Figure 8.

Are all Hamiltonian systems integrable?

As we have seen, physics can provide first integrals as the momentum with respect to an axis of revolution (this is what happens for a spinning top, a spherical pendulum, a free particle on a surface of revolution, ...)

There are also many Hamiltonian systems that are *not* integrable. The most famous is the *three-body problem* dealing with three bodies (Sun-Earth-Moon) in gravitational interaction. It is known that the two-body problem (Sun-Earth) is integrable. It was actually to integrate this problem that Lagrange introduced the beginnings of symplectic geometry and Hamiltonian mechanics. Poincaré showed that the three-body problem could not have enough first integrals (*analytic* in positions and momenta).

The method I explain here allows us to prove that this is still true if we accept poles – namely, *meromorphic* first integrals.

Some Hamiltonian systems are suspected to be non-integrable because we have not been able to find enough first integrals and, more seriously, because some experiments or numerical simulations show a chaotic behaviour that seems to be incompatible with the Arnold-Liouville theorem. This is the case for the *Hénon-Heiles* system:

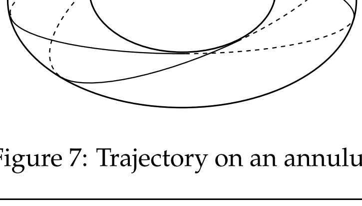

Figure 7: Trajectory on an annulus

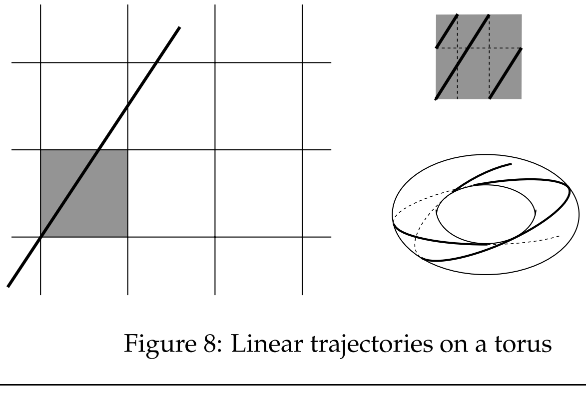

Figure 8: Linear trajectories on a torus

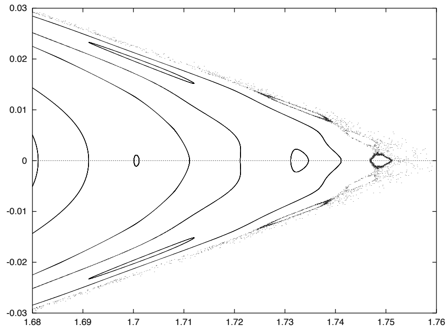

Figure 9: Chaotic behaviour of the Hénon-Heiles system

The Hénon-Heiles system is the Hamiltonian system defined by the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(Aq_1^2 + Bq_2^2) - q_1^2 q_2 - \frac{1}{3}\lambda q_2^3,$$

where A, B and λ are parameters. It is known to be integrable for some values of these parameters. Figure 9 comes from Morales' book and is supposed to illustrate the chaotic behaviour of this system. It shows part of the dynamics (the dynamics of the Poincaré map, to be precise) for the parameters $A = B = 0$ and $\lambda = 3/2$: it seems to indicate that the system is not integrable in this case. Using the method I am going to explain, it is possible to prove that, in general, the Hénon-Heiles system is *not* integrable.

And the attitude? Is the attitude of a satellite an integrable system?

The Galois group, an obstruction to integrability

Following a tradition that goes back, at least, to the *Mécanique céleste* of Poincaré, let us consider the *variational equation*, a linear differential equation that describes the solutions that are *infinitesimally close* to a given solution.

We assume that we know a solution $x(t)$ of the Hamiltonian system. Let $y(t)$ be another solution, that is very close to $x(t)$. Then we can write $y(t) = x(t) + Y(t)$ and, up to order 1, our Hamiltonian system becomes *linear* in Y:

$$\begin{aligned} \dot{Y} &= \dot{y} - \dot{x} \\ &= X(y(t)) - X(x(t)) \\ &= (dX)_{x(t)}(Y(t)), \text{ up to order } 1. \end{aligned}$$

This *linear* differential equation in Y is the *variational equation*.

The Hénon-Heiles example

In the simple case of the Hénon-Heiles system for $A = B = 0$ and $\lambda = 0$, the Hamiltonian is

$$H = \frac{1}{2}(p_1^2 + p_2^2) - q_2 q_1^2.$$

The Hamiltonian system is

$$\begin{cases} \dot{q}_1 = p_1, & \dot{q}_2 = p_2 \\ \dot{p}_1 = 2q_1q_2, & \dot{p}_2 = q_1^2. \end{cases}$$

It is quite easy to find solutions (this is an academic example!). We choose a trajectory that is a straight line:

$$\begin{cases} q_1 = 0, & p_1 = 0, \\ q_2(t) = at - b, & p_2(t) = a \end{cases}$$

for some constants a and b. The variational equation along one of these solutions is the linear differential system

$$\begin{cases} \dot{Q}_1 = P_1, & \dot{Q}_2 = P_2, \\ \dot{P}_1 = 2q_2(t)Q_1, & \dot{P}_2 = 0. \end{cases}$$

We look only at the solutions for which $Q_2 = P_2 = 0$. The linear system $\dot{Q}_1 = P_1$, $\dot{P}_1 = 2q_2(t)Q_1$ is equivalent to the differential equation

$$\ddot{Q} - 2(at - b)Q = 0.$$

This is an Airy equation, the solutions of which, the *Airy functions*, are analytic on the whole complex line, but no solution of which is a rational function or even an algebraic function. For those who like formulas, Airy functions can be written

$$Q(t) = \int_0^\infty \cos(x^3 \pm xt)dx.$$

Differential Galois theory – in about 195 words!

The situation of a linear differential equation with polynomial coefficients, whose solutions are not rational, is reminiscent of the situation of an algebraic equation with rational coefficients and irrational roots.

The base field is the field of meromorphic functions on our trajectory – this is the field $\mathbf{C}(t)$ in the example of Airy. With the linear differential equation is associated a smallest differential field which contains all the solutions of the linear equation (the Picard-Vessiot extension, the differential analogue of the splitting field of an algebraic equation). The differential Galois group is its group of automorphisms. The set of solutions of a linear differential equation is a vector space on which the Galois group acts as a subgroup of the corresponding linear group in the same way as the Galois group of an algebraic equation is a subgroup of the symmetric group, which acts by permutation of the roots. According to a theorem of Kolchin, the differential Galois group is an algebraic group.

In the Airy example, the solutions (and, more precisely, their behaviour at infinity) are intricate enough for the

Galois group to be really huge: it is the entire group $SL(2;\mathbf{C})$.

The Morales-Ramis theorem

The Galois group is the main character of a non-integrability theorem due to Morales and Ramis, in a tradition that goes back to Kowalevskaya, Poincaré, Painlevé and, more recently, Ziglin. This theorem can be considered as an analogue of the theorem on the solvability of equations by radicals. It asserts that *if a Hamiltonian system is integrable, then the Galois group of the variational equation along any trajectory must be almost Abelian* (in the sense that its neutral component is an Abelian group). Notice that, if we neglect finite groups, this is a stronger property than being solvable. Notice also that this theorem, although it gives a very powerful tool for proving non-integrability, is much easier to prove than the algebraic analogue: being an algebraic group, the Galois group can be attacked with infinitesimal tools – the statement is actually that its Lie algebra must be Abelian.

Conclusion

In concrete terms (if I dare say it): you choose a solution, you linearise the system along it, and you compute the Galois group. If it is not (almost) Abelian, your original system was not integrable. Notice that you should have found a solution, that is both simple enough so that you are able to compute the Galois group, and complicated enough so that the latter is not (almost) Abelian.

Applications

Nevertheless, this theorem does have numerous applications; for instance: In the simple case of the Hénon-Heiles example ($A = B = \lambda = 0$), we have seen that the Galois group is $SL(2;\mathbf{C})$. This group is *not* almost Abelian, and hence the system cannot be integrable. However the case $A = B = 0, \lambda = \frac{3}{2}$ illustrated in Figure 9 is still open. A non-symmetric spinning top cannot lead to an integrable system either.

Morales and Ramis have many applications, some of which can be found in Morales' book (cf. reference below).

The satellite

The main difficulty is not to find a particular solution to start with. The Galois group is an algebraic subgroup of the linear group $GL(2n;\mathbf{C})$, where n is the number of degrees of freedom of the system. This number is 2 in most of the examples considered here, which gives a subgroup of $GL(4;\mathbf{C})$. Using the fact that the Hamiltonian is a first integral, this can be reduced to a subgroup of $SL(2;\mathbf{C})$, which allows us to compute it.

However, the attitude of the satellite in a circular orbit is a system with 3 degrees of freedom, so that, after reduction, we still have a subgroup of $SL(4;\mathbf{C})$. This is too big: recall that the group needs only to be *almost Abelian*, so that it is not enough to find two elements that do not commute. With the help of computer algebra (resp. an additional geometric argument), Delphine Boucher (resp. the author of the present paper) were able to prove, in 2003, that the attitude is not integrable.

This does not prevent the satellite SPOT to take its beautiful pictures of the Earth (free advertisement!), since, as the orbits, the attitudes can be corrected.

The end

The first approach to non-integrability goes back to S. Kowalevskaya in 1889. She was investigating the integrability of the rigid body with a fixed point. She asked herself under which assumptions the solutions of the system would all be *meromorphic* functions of the time variable – this is called the *Painlevé test* – that is, their only singularities are poles (no ramification, no logarithm, etc).

She proved that this property is satisfied in only three cases: when the fixed point is the centre of gravity, when the rigid body has an axis of revolution (in these two cases, the system was known to be integrable), and in a new case (now called the *Kowalevskaya top*). She was in fact able to find an additional first integral in this last case too.

The relation between integrability (in the Liouville sense) and the softness of the singularities of the solutions is still not completely clear. The methods of differential algebra used here give precisely such a relation, up to order 1.

Acknowledgments

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A few references

Details, information and references are given in:

- Juan Morales' book, *Differential Galois theory and non-integrability of Hamiltonian systems*, Progress in Math., Birkhäuser, 1999;
- the author's book, *Les systèmes hamiltoniens et leur intégrabilité*, Cours Spécialisés, 8, Société Mathématique de France & EDP Sciences, 2001;
- the references in these two books, and more recent references in other papers of the author are listed on the website: `http:www-irma.u-strasbg.fr/~maudin`.