Symplectic geometry and volumes of moduli spaces in quantum cohomology

Michèle Audin

Institut de Recherche Mathématique Avancée
Université Louis Pasteur et CNRS
7 rue René-Descartes
F–67084 Strasbourg Cedex

maudin@math.u-strasbg.fr

Contents

The aim of this note is to explain, illustrate and prove a few statements of Givental and Kim in [8]. Although the two pages of this paper where the material discussed here comes from do not contain many definitions and proofs, they contain very beautiful ideas which deserve to be explained in detail.

Givental and Kim consider a symplectic manifold X whose complex cohomology algebra is generated by its degree-2 elements. Under a monotonicity assumption, they consider that its quantum cohomology ring is the ring of functions on a subvariety

$$L \subset T^\star H^2 \left(X; \mathbf{C}/2i\pi\mathbf{Z}\right)$$

in the cotangent bundle of the torus $H^2 \left(X; \mathbf{C}/2i\pi\mathbf{Z}\right)$. A very beautiful remark of [8] is that L should be Lagrangian. I will give a precise statement and a proof of this fact. Note that the aim of the original paper [8] was to compute the quantum cohomology ring of complete flag manifolds, in which case the authors prove that L is not just any Lagrangian subvariety but a leaf in a Lagrangian foliation.

Given a weakly monotone[1] symplectic $2n$-manifold (X, ω) and using the Gromov-Witten invariants defined in [13], we will also define a mapping V on $H^2(X; \mathbf{C})$, taking its values in the Novikov ring Λ_ω (a completion of the group algebra $\mathbf{C}\left[H_2(X; \mathbf{Z})\right]$). This mapping can be considered as giving the volumes of the various moduli spaces of holomorphic curves in X.

Under the additional assumption that the complex cohomology algebra of X is generated by its degree-2 classes, we will derive a very simple (although not very effective for computations) description of the quantum cohomology ring of X in terms of V. The Lagrange property of the subvariety L can then be rephrased in terms of the (partial) differential equations satisfied by the function V, leading to a problem which can be of independent interest.

Acknowledgements. — This paper should be considered just as an extension of § 3.3.5 of [2], this meaning that almost all the ideas come from [8]. I thank H. Spielberg for discussions. I wish to thank also McGill University and the Centre de Recherches Mathématiques in Montréal for hospitality and especially J. Hurtubise, L. Jeffrey and E. Markman for useful discussions and informations.

Notation. — I will denote by $H_\star(X; \mathbf{Z})$, $H^\star(X; \mathbf{Z})$ the torsion free part of integral homology and cohomology, and by D the Poincaré duality. Greek characters will usually[2] denote cohomology classes while Latin characters denote homology classes. Corresponding characters in the same formula denote Poincaré dual classes (*e.g.* $x = D\xi$, $a = D\alpha$, $b = D\beta$).

I will also use other coefficients for cohomology. As notation is rather cumbersome, I will sometimes use $H^\star(X)$ for either integral or complex cohomology.

The group ring $\mathbf{Z}[H_2(X; \mathbf{Z})]$ will be denoted Λ. The same notation will be used for the complex group algebra $\mathbf{C}[H_2(X; \mathbf{Z})]$. The Novikov ring (over $\mathbf{Z}$ or over $\mathbf{C}$) associated with the symplectic form ω will be denoted Λ_ω (see [10] or [2] for a definition).

[1] See § 1.1 for the definitions.

[2] There are many exceptions.

1. Some properties of Gromov-Witten invariants

In this §, I will first state quickly the definition of Ruan & Tian [13] of the Gromov-Witten invariants. I will then show that degree-2 cohomology classes play a special role with respect to these invariants.

1.1. Gromov-Witten invariants

The formalism is rather heavy and I do not want to spend much time and place on it. The aim here is just to fix the notation, which I will try to make coherent with that of [2]. I send the reader to [3] and [12] for the properties of holomorphic curves and to the original paper of Ruan & Tian [13] for precisions on Gromov-Witten invariants and proofs.

A symplectic manifold (W, ω) is *monotone* if its first Chern class c_1 is a positive multiple of the cohomology class of the symplectic form ω. It is *weakly monotone* if any homology class A such that $\langle \omega, A\rangle > 0$ and $\langle c_1, A\rangle \geq 3-n$ satisfies $\langle c_1, A\rangle \geq 0$.

This property ensures that, for any generic almost complex structure J calibrated by ω, the first Chern class of X, evaluated on the class of a J-holomorphic sphere, is nonnegative.

On a weakly monotone symplectic manifold (X, ω), Ruan & Tian define Gromov-Witten invariants as elements in bordism groups of pseudo-cycles $\Omega_\star^{ps}\left(X^k \times X^\ell\right)$.

Given a class $A \in H_2(X; \mathbf{Z})$, denote by $\mathcal{M}^A(J, \nu)$ the space of solutions $u : \mathbf{P}^1 \to X$ of the modified Cauchy-Riemann equation $\overline{\partial}_J u = \nu$ which represent the class A. Denote by $\vec{z} = (z_1, \ldots, z_k)$ a k-uple of distinct points in $\mathbf{P}^1$. Suppose now that $k \geq 3$ and $\ell \in \mathbf{N}$ are given, then $\Psi^A_{k,\ell}$ is defined as the class of the evaluation mapping

$$\begin{array}{ccccccc} \mathcal{M}^A(J,\nu) & \times & \underbrace{\mathbf{P}^1 \times \cdots \times \mathbf{P}^1}_{\ell \text{ times}} & \longrightarrow & \underbrace{X \times \cdots \times X}_{k \text{ times}} & \times & \underbrace{X \times \cdots \times X}_{\ell \text{ times}} \\ (u & , & \zeta_1, \ldots, \zeta_\ell) & \longmapsto & (u(z_1), \ldots, u(z_k) & , & u(\zeta_1), \ldots, u(\zeta_\ell)) \end{array}$$

in $\Omega^{ps}_{2n+2\langle c_1, A\rangle + 2\ell}\left(X^k \times X^\ell\right)$ for a generic pair (J, ν) (see [13] or [2]). The intersection number

$$\left(\mathcal{M}^A(J,\nu) \times \left(\mathbf{P}^1\right)^\ell, \mathrm{ev}_{\vec{z},\ell}\right) \cdot (a_1 \otimes \cdots \otimes a_k \otimes b_1 \otimes \cdots \otimes b_\ell)$$

is denoted $\Psi^A_{k,\ell}(a_1 \otimes \cdots \otimes a_k | b_1 \otimes \cdots \otimes b_\ell)$.

If one forgets the assumption $k \geq 3$, one has to take into account the reparametrisation group G_k of $(\mathbf{P}^1, \vec{z})$ (which is trivial if $k \geq 3$), namely

- $PSL(2; \mathbf{C})$ if $k = 0$ (it has real dimension 6),
- the group of affine transformations of $\mathbf{C}$ if $k = 1$ (it has dimension 4),
- the group of dilations of $\mathbf{C}$ if $k = 2$ (it as dimension 2).

The group G_k acts on $\mathcal{M}^A(J,\nu) \times (\mathbf{P}^1)^\ell$ by

$$g \cdot (u, \zeta_1, \ldots, \zeta_\ell) = (u \circ g^{-1}, g\zeta_1, \ldots, g\zeta_\ell)$$

and the evaluation mapping factors through this action, defining

$$\mathrm{ev}_{\vec{z},\ell} : \mathcal{M}^A(J,\nu) \times_{G_k} \left(\mathbf{P}^1\right)^\ell \longrightarrow X^{k+\ell}.$$

If (X,ω) is a weakly monotone symplectic manifold, and if the homology class A is non-zero, so that the G_k-action is free, this evaluation mapping is a pseudo-cycle and defines an element of $\Omega^{ps}_{2n+2\langle c_1,A\rangle+2\ell-\dim G_k}(X^{k+\ell})$ for generic (J,ν) (the proof is the same as for three-or-more-point invariants). As we shall use two-point invariants, let us state a proposition.

1.1.1 Proposition. — *Let (X,ω) be a weakly monotone symplectic manifold of dimension $2n$ and let A be a non-zero element of $H_2(X;\mathbf{Z})$. Let G be the subgroup of all elements of $PSL_2(\mathbf{C})$ that fix 0 and ∞, that is, the group $\mathbf{C}^\star$ of dilations of $\mathbf{C}$. Then, for generic (J,ν), the evaluation map*

$$\begin{array}{ccc} \mathcal{M}^A(J,\nu)/G & \longrightarrow & X \times X \\ [u] & \longmapsto & (u(0), u(\infty)) \end{array}$$

defines a pseudo-cycle

$$\varphi_2^A : H_\star(X \times X) \longrightarrow \mathbf{Z}$$

of dimension $2n + 2\langle c_1, A\rangle - 2$. □

1.2. Degree-2 classes in quantum cohomology

It is in the nature of the Gromov-Witten invariants that codimension-2 cycles play a special role. Here are a few elementary properties that are supposed to illustrate this assertion.

1.2.1 Proposition. — *Let ξ be a degree-2 cohomology class. Then, for all $a \in H_\star(X^k)$, $b \in H_\star(X^\ell)$,*

$$\Psi^A_{k,\ell+1}(a|b \otimes x) = \langle \xi, A\rangle \Psi^A_{k,\ell}(a|b).$$

Proof. — One wants to check that the diagram

$$\begin{array}{ccc} H_d(X^k \times X^\ell) & \longrightarrow & H_{d+2(n-1)}(X^k \times X^{\ell+1}) \\ \Psi^A_{k,\ell} \Big\downarrow & & \Big\downarrow \Psi^A_{k,\ell+1} \\ \mathbf{Z} & \xrightarrow{(x \cdot A)} & \mathbf{Z} \end{array}$$

(in which the top arrow is $y \mapsto y \otimes x$) is commutative, which is more or less obvious from the definition of the Gromov-Witten invariants:

$$\left(\mathcal{M}^A(J,\nu) \times \left(\mathbf{P}^1\right)^{\ell+1}, \mathrm{ev}_{\vec{z},\ell+1}\right) \cdot (y \otimes x) = \left(\left(\mathcal{M}^A(J,\nu), \mathrm{ev}_{\vec{z},\ell}\right) \cdot y\right) \left(u_\star[\mathbf{P}^1]\right).$$

□

1.2.2 Corollary. — *For any $\xi \in H^2(X;\mathbf{Z})$, any $A \in H_2(X;\mathbf{Z})$ and any $a \in H_\star(X^3)$,*

$$\sum_\ell \frac{1}{\ell!} \Psi^A_{3,\ell}(a|x \otimes \cdots \otimes x) = \Psi^A_{3,0}(a) \exp\langle \xi, A\rangle.$$

As an easy induction gives

$$\Psi^A_{k,\ell}(a|x\otimes\cdots\otimes x) = (\langle\xi, A\rangle)^\ell\,\Psi^A_{k,0}(a).$$

□

With respect to two-point invariants, it is also straightforward to check:

1.2.3 PROPOSITION. — *Let* (X,ω) *be a weakly monotone symplectic manifold of dimension* $2n$. *Let* A *be a non-zero element of* $H_2(X;\mathbf{Z})$. *For any* a, $b\in H_\star(X;\mathbf{Z})$,

$$\Psi^A_{3,0}(a\otimes b\otimes x) = \langle\xi, A\rangle\varphi^A_2(a\otimes b).$$

□

Quantum multiplication at a given point vs formal series. — Corollary 1.2.2 allows us to define a multiplication law on the cohomology of X:

1.2.4 PROPOSITION. — *Let* (X,ω) *be a monotone symplectic manifold. For any degree-*2 *cohomology class* ξ, *the formula*

$$\langle\alpha\star_\xi\beta, c\rangle = \sum_A \Psi^A_{3,0}(a\otimes b\otimes c)\exp\langle\xi, A\rangle$$

defines a ring structure $\star_\xi$ *on* $H^\star(X;\mathbf{C})$.

Proof. — The associativity of the product $\star_\xi$ follows from a simple version of the composition rules of [13]. □

What we have got here is, *for any* $\xi\in H^2(X;\mathbf{C})$, a ring structure on the vector space $H^\star(X;\mathbf{C})$, depending on ξ; notice however that it depends only on the class of ξ in the torus $H^2(X;\mathbf{C}/2i\pi\mathbf{Z})$.

Another point of view on the quantum product (on a monotone symplectic manifold) consists in considering a product *not* depending on any cohomology class but defined on the extended module

$$QH^\star(X) = H^\star(X)\otimes\Lambda.$$

If $A\in H_2(X;\mathbf{Z})$, the corresponding element of Λ will be denoted by q^A. The quantum product is defined by

$$\langle\alpha\star\beta, c\rangle := \sum_A \Psi^A_{3,0}(a\otimes b\otimes c)q^A.$$

In other words (write $q^A = \exp\langle\xi, A\rangle$), to compute $\alpha\star_\xi\beta$ in $H^\star(X;\mathbf{C})$ amounts to the same as to compute $\alpha\star\beta$ in $QH^\star(X)$ and to specialize it at the corresponding value of q.

Monotonicity assumptions. — The formal viewpoint allows us to consider more general symplectic manifolds, namely *weakly monotone* symplectic manifolds: it allows to replace Λ by the Novikov ring Λ_ω defined by the symplectic form. However, there are non-monotone weakly monotone symplectic manifolds for which the sum

$$\sum_A \Psi^A_{3,0}(a \otimes b \otimes c) q^A$$

is finite and defines a Laurent polynomial in q, and the product $\star_\xi$ is well-defined. Let us say that such a symplectic manifold is *almost monotone.*

1.3. The canonical 1-form on $H^2(X; \mathbf{C}/2i\pi\mathbf{Z})$

Let us assume now that (X, ω) is an almost monotone symplectic manifold. Associated with the product $\star_\xi$, there is a canonical 1-form Ω on the torus $B = H^2(X; \mathbf{C}/2i\pi\mathbf{Z})$ with values in $\operatorname{End}(H^\star(X; \mathbf{C}))$, defined by

$$\Omega_{[\xi]}(\beta) \cdot \alpha = \alpha \star_\xi \beta$$

in which formula $[\xi] \in H^2(X; \mathbf{C}/2i\pi\mathbf{Z})$ is the class of an element $\xi \in H^2(X; \mathbf{C})$, β is an element of $H^2(X; \mathbf{C}) = T_{[\xi]} H^2(X; \mathbf{C}/2i\pi\mathbf{Z})$, so that $\Omega_{[\xi]}(\beta)$ is an endomorphism of $H^\star(X; \mathbf{C})$. By definition, $\Omega_{[\xi]}(\beta)$ is just multiplication by β at the point ξ. A consequence of the simple remark above (Proposition 1.2.3) is the following:

1.3.1 Proposition. — *The canonical* 1*-form* Ω *is closed.*

Proof. — One just computes the differential $d\Omega$. As the manifold $H^2(X; \mathbf{C}/2i\pi\mathbf{Z})$ is a torus, one can use *constant* vector fields $\beta, \gamma \in H^2(X; \mathbf{C})$, so that $[\beta, \gamma] = 0$ and

$$(d\Omega)_{[\xi]}(\beta, \gamma) = \gamma \cdot \left(\Omega_{[\xi]}(\beta)\right) - \beta \cdot \left(\Omega_{[\xi]}(\gamma)\right).$$

Now,

$$\begin{aligned}
\langle \gamma \cdot \left(\Omega_{[\xi]}(\beta) \cdot \alpha\right), y \rangle &= \gamma \cdot \langle \Omega_{[\xi]}(\beta) \cdot \alpha, y \rangle \\
&= \gamma \cdot \langle \alpha \star_\xi \beta, y \rangle \\
&= \gamma \cdot \left(\sum_A \Psi^A_{3,0}(a \otimes b \otimes y) \exp\langle \xi, A \rangle \right) \\
&= \sum_{A \neq 0} \Psi^A_{3,0}(a \otimes b \otimes y) \langle \gamma, A \rangle \exp\langle \xi, A \rangle \\
\text{(the derivative of the } A = 0\text{-term vanishes)} \quad &= \sum_{A \neq 0} \varphi^A_2(a \otimes y) \langle \beta, A \rangle \langle \gamma, A \rangle \exp\langle \xi, A \rangle
\end{aligned}$$

which is symmetric in β and γ, so that

$$\langle (d\Omega)_{[\xi]}(\beta, \gamma) \cdot \alpha, y \rangle = 0$$

for all $\alpha \in H^\star(X; \mathbf{Z})$ and for all $y \in H_\star(X; \mathbf{Z})$ and thus $d\Omega = 0$. □

Remark. — It is also easy to find (locally) a primitive of Ω. Precisely, let

$$S : H^2(X;\mathbf{C}) \longrightarrow \mathrm{End}\,(H^\star(X;\mathbf{C}))$$

be the mapping defined by

$$\langle S(\xi)\cdot\alpha, c\rangle = (a\cdot c\cdot x) + \sum_{A\neq 0}\varphi_2^A(a\otimes c)\exp\langle\xi, A\rangle.$$

Denoting by

$$\pi : H^2(X;\mathbf{C}) \longrightarrow H^2\left(X;\mathbf{C}/2i\pi\mathbf{Z}\right)$$

the universal covering map ($\pi(\xi) = [\xi]$), one easily checks that $\pi^\star\Omega = dS$.

Flat connections. — The expression $d + t\Omega$ defines a connection ∇_t on the trivial vector bundle with fiber $H^\star(X;\mathbf{C})$ over $H^2\left(X;\mathbf{C}/2i\pi\mathbf{Z}\right)$.

1.3.2 Proposition. — *The connection ∇_t is flat for all values of t.*

Proof. — The assertion actually means two things: firstly, that Ω is closed—that we have already proved—, secondly, that $\Omega\wedge\Omega = 0$. Now,

$$\Omega_{[\xi]}(\beta)\cdot\left(\Omega_{[\xi]}(\gamma)\cdot\alpha\right) = (\alpha\star_\xi\gamma)\star_\xi\beta,$$

so that

$$\begin{aligned}(\Omega\wedge\Omega)_{[\xi]}(\beta,\gamma)\cdot\alpha &= \frac{1}{2}\left[\Omega_{[\xi]}(\beta),\Omega_{[\xi]}(\gamma)\right]\cdot\alpha \\ &= (\alpha\star_\xi\gamma)\star_\xi\beta - (\alpha\star_\xi\beta)\star_\xi\gamma.\end{aligned}$$

This is thus a consequence of both the associativity and the commutativity of the product $\star_\xi$ that $\Omega\wedge\Omega = 0$. □

Remarks.

1. Here we are very close to the notion of a Frobenius manifold investigated by Dubrovin [6] (see also [9]). In some cases (including monotone symplectic manifolds), one is able to define a quantum product $\star_\xi$ for any $\xi\in H^\star(X)$ (recall that our $\star_\xi$ was defined only for degree-2 classes ξ), and a 1-form $\widetilde{\Omega}$ on $H^\star(X)$ with values in $\mathrm{End}\,H\star(X)$ by

   $$\widetilde{\Omega}_\xi(\beta)\cdot\alpha = \alpha\star_\xi\beta$$

   (now α and β are cohomology classes of any degree). There is a function

   $$F : H^\star(X)\longrightarrow\mathbf{C},$$

   the Gromov-Witten potential, defined by

   $$F(\xi) = \sum_A\sum_{\ell\geq 0}\frac{1}{(\ell+3)!}\Psi_{3,\ell}^A(x\otimes x\otimes x|x\otimes\cdots\otimes x)$$

and whose second derivative, viewed as a map

$$\widetilde{S}: H^\star(X) \longrightarrow \mathrm{Sym}\,(H^\star(X)) \subset \mathrm{End}\, H^\star(X)$$

is a primitive of $\widetilde{\Omega}$. The quantum product $\star_\xi$ being associative and commutative, the 1-form $\widetilde{\Omega}$ satisfies $\widetilde{\Omega} \wedge \widetilde{\Omega} = 0$ as above. But $\widetilde{\Omega}$ is nothing other than the third derivative of F, so that the function F has to be a solution of a system of order 3 partial differential equations, called the WDVV equation (see [6, 9]).

Note that the map $\widetilde{S}$, when restricted to $H^2(X;\mathbf{C}) \subset H^\star(X;\mathbf{C})$ is also a primitive of our 1-form $\pi^\star\Omega$.

2. Let $E \to B$ be the trivial vector bundle $B \times H^\star(X;\mathbf{C})$, so that Ω can be considered as a morphism

$$TB \longrightarrow \mathrm{End}\, E$$

or as a morphism

$$E \longrightarrow E \otimes T^\star B.$$

Since $\Omega \wedge \Omega = 0$, the pair (E, Ω) is a Higgs pair in the sense of [14].

2. Symplectic geometry in quantum cohomology

In this section, I will work with Laurent polynomials in q, so that I will assume that (X, ω) is almost monotone. Let us make a crucial assumption: we assume that the cohomology algebra $H^\star(X)$ is generated by $H^2(X)$, its degree-2 part. Denoting by $S^\star(X) = S\left[H^2(X;\mathbf{C})\right]$ the symmetric algebra on $H^2(X;\mathbf{C})$, this means that the natural ring morphism

$$S^\star(X) \longrightarrow H^\star(X)$$

is onto. We thus have a surjective homomorphism:

$$S^\star\left(H^2(X;\mathbf{C})\right) \otimes \mathbf{C}\left[H_2(X;\mathbf{Z})\right] \longrightarrow QH^\star(X),$$

the kernel of which will be denoted by $\mathcal{I}$. The left hand ring is nothing other that the ring of regular functions on the cotangent bundle $T^\star H^2\left(X;\mathbf{C}/2i\pi\mathbf{Z}\right)$ of the torus $B = H^2\left(X;\mathbf{C}/2i\pi\mathbf{Z}\right)$. As any cotangent bundle, this one carries a canonical Poisson structure. The aim of this § is to prove that the ideal $\mathcal{I}$ is stable under Poisson bracket: the Poisson bracket of two relations in $QH^\star(X)$ is again a relation. Unfortunately I cannot prove this without an additional assumption (the precise statement is 2.2.2 below).

2.1. The cotangent bundle and its structures

On $T^\star B$ we have

- the Liouville 1-form λ, the pairing

$$\lambda_{([\xi],a)}(\widetilde{\beta}) = \langle \beta, a \rangle$$

where $a \in H_2(X;\mathbf{C})$, $\widetilde{\beta} \in T_{([\xi],a)}\left(T^\star B\right)$ has projection β on $T_{[\xi]}B = H^2(X;\mathbf{C})$,

- a symplectic form $d\lambda$, which, being non-degenerate, associates a vector field X_F to each function,
- the associated Poisson bracket

$$\{F, G\} = d\lambda(X_F, X_G).$$

The symplectic structure, in coordinates. — Using a basis $(p_1, \dots, p_r)$ of $H^2(X; \mathbf{Z})$, the dual basis $(A_1, \dots, A_r)$ of $H_2(X; \mathbf{Z})$ and its multiplicative counterpart, the coordinates $(q_1, \dots, q_r)$ on $H^2(X; \mathbf{C}/2i\pi\mathbf{Z})$, our function ring is just $\mathbf{C}[p_1, \dots, p_r, q_1, q_1^{-1}, \dots, q_r, q_r^{-1}]$. As the notation (which also come from [8]) is very well chosen, the Liouville form is

$$\lambda = \sum_{i=1}^{r} p_i \wedge \frac{dq_i}{q_i},$$

the symplectic form is

$$d\lambda = \sum_{i=1}^{r} dp_i \wedge \frac{dq_i}{q_i},$$

and the Poisson bracket is simply

$$\{F, G\} = \sum_{i=1}^{r} q_i \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i} \frac{\partial F}{\partial q_i} \right).$$

2.2. The characteristic subvariety

The ideal $\mathcal{I}$ of relations in quantum cohomology describes a subvariety L of $T^\star B$. As $QH^\star(X)$ is defined as the free Λ-module $H^\star(X) \otimes \Lambda$, the geometric counterpart of the exact sequence

$$0 \to \mathcal{I} \longrightarrow S^\star(X) \otimes \Lambda \longrightarrow QH^\star(X) \to 0,$$

that is, the composition

$$L \subset T^\star B \longrightarrow B$$

is a degree-N morphism, where $N = \dim_{\mathbf{C}} H^\star(X; \mathbf{C})$. Givental and Kim noticed that L should be a *Lagrangian* (totally isotropic, maximal) subvariety of the symplectic manifold $T^\star B$. The idea is to use the fact that the canonical 1-form Ω defined by the quantum product is closed (Proposition 1.3.1) to deduce that the Liouville form, when restricted to L, is closed as well, so that L is isotropic.

The canonical 1-form Ω of § 1.3 is a matrix valued 1-form while the Liouville form is a scalar valued 1-form. I will now make a technical assumption that will insure that $j^\star\lambda$, the form induced on L by the Liouville form, is in some sense an *eigenvalue* of Ω so that the closedness of Ω implies that of $j^\star\lambda$.

Assumption. — *There is at least one point $\xi_0 \in H^2(X; \mathbf{C})$ such that the algebra $(H^\star(X), \star_{\xi_0})$ is semi-simple.*

Comments.

- This means that, for any ξ in a neigbourhood of ξ_0, there exists a basis $(w_1(\xi),\ldots,w_N(\xi))$ of $H^\star(X;\mathbf{C})$ such that
$$w_i(\xi)\star w_j(\xi)=\mu_i(\xi)\delta_{i,j}w_i(\xi)$$
for some non-zero $\mu_i(\xi)$: the algebra $(H^\star(X),\star_\xi)$ splits as a sum $\bigoplus_{i=1}^N \mathbf{C}\cdot w_i$.

- Notice that this would be a consequence of the fact that there exists an element α in $H^\star(X;\mathbf{C})$ such that the endomorphism $\alpha\star_{\xi_0}\cdot$ has distinct eigenvalues. Another stronger assumption would be that $p_i\star_{\xi_0}\cdot$ is diagonalisable for all the generators p_i's. In both cases a basis $(w_1,\ldots,w_N)$ of eigenvectors will work.

- If there is one value of ξ_0 satisfying the assumption, the same will hold for almost all choices of ξ_0.

- The assumption implies that L is reduced ($\mathcal{I}=\sqrt{\mathcal{I}}$).

Geometric consequences. — With this assumption, on a neighbourhood of $[\xi_0]$, $\mathcal{I}$ is generated by the relations
$$\begin{cases} w_i(\xi)^2-\mu_i(\xi)w_i(\xi) \\ w_i(\xi)w_j(\xi) \text{ for } i\neq j.\end{cases}$$
A point $([\xi],a)$ of $T^\star B$ is in L if and only if
$$\begin{cases} \langle w_i(\xi)^2-\mu_i(\xi)w_i(\xi),a\rangle=0 \\ \langle w_i(\xi)w_j(\xi),a\rangle=0 \text{ for } i\neq j.\end{cases}$$

The N branches of L correspond to the N vectors $(w_1,\ldots,w_N)$. In coordinates (p,q), this can be expressed as follows: the w_i's are eigenvectors of the multiplication by any element, in particular
$$p_i\star_\xi w_j=p_i^j(q)w_j$$
and the jth branch is described by the local (analytic) equations:
$$p_1=p_1^j(q),\ldots,p_r=p_r^j(q).$$

Remark. — Associated with any Higgs pair, there is a *spectral cover* (see [5]), that is, a subvariety of $T^\star B$ describing the eigenvalues of the matrix Ω. We have just checked that, under our assumptions, the spectral cover of (E,Ω) is the variety L whose ring of functions is the quantum cohomology of X.

The Liouville form as an eigenvalue. — Choose one of the branches, that is, an eigenvector $w=w_j$. Let f_w be the inclusion of the corresponding local branch of L. The next remark is:

2.2.1 Proposition. — *The Liouville form is an eigenvalue of the canonical form, i.e.*
$$\Omega\cdot w=(f_w^\star\lambda)\,w.$$

Proof. — This is just the definition. In coordinates (p, q) for instance, this is simply

$$\sum_{i=1}^{r} p_i \star_\xi w \frac{dq_i}{q_i} = \left(\sum_{i=1}^{r} p_i(q) \frac{dq_i}{q_i}\right) w.$$

□

Let us now turn to the main result:

2.2.2 Proposition. — *Under the semi-simplicity assumption above, L is a Lagrangian subvariety of $T^\star B$.*

Proof. — This means that for any branch (that is, any w), $f_w^\star\lambda$ is a closed 1-form. The only thing to do is to differentiate the relation of Proposition 2.2.1 above:

$$(d\Omega) \cdot w + \Omega \cdot dw = (d(f_w^\star\lambda))w + (f_w^\star\lambda)dw.$$

The canonical 1-form is closed (Proposition 1.3.1) and we just need an evaluation of dw. Recall that $w = w_j$. As we are in a vector space (trivial bundle), dw_j can be written in the given basis

$$dw_j = \sum a_j^i w_i$$

for some 1-forms a_j^i. Recall now the relations between the w_j's and differentiate that giving $w_j \star w_j$:

$$2w_j \star dw_j = (d\mu_j)w_j + \mu_j dw_j.$$

Using that $w_j \star w_i = 0$ for $i \neq j$, we get

$$2a_j^j w_j = d\mu_j w_j + \mu_j \sum a_j^i w_i$$

thus $a_j^i = 0$ for $i \neq j$ and $dw_j = a_j^j w_j$. With this relation ($dw = aw$), the relation computing $d(f_w^\star\lambda)$ becomes

$$(d(f_w^\star\lambda))\, w = a\,(\Omega \cdot w - (f_w^\star\lambda)w) = 0,$$

and thus $f_w^\star\lambda$ is indeed a closed 1-form. □

2.2.3 Corollary. — *Under the assumptions above, $\mathcal{I}$ is a Poisson ideal.*

Since any Lagrangian subvariety is co-isotropic and thus involutive. □

Remark. — In [7] (see also [16]), the Lagrangian subvariety L is described as a characteristic subvariety using a module over the ring $\mathcal{D}$ of differential operators on the torus B which would be the S^1-equivariant Floer cohomology of a covering of the loop space LX, if this was defined.

2.3. Homogeneity property and the first Chern class

Recall now that $QH^\star(X)$ is a graded ring, the grading being defined by that of $H^\star(X)$ and the first Chern class on Λ: if $A \in H_2(X;\mathbf{Z})$, $\deg q^A = 2\langle c_1, A\rangle$. As a consequence, the ideal $\mathcal{I}$ is quasi-homogeneous. Let E be the vector field on $T^\star B$ which is the infinitesimal generator of the $\mathbf{C}^\star$-action

$$e^z \cdot ([\xi], a) = ([\xi + zc_1], e^z a).$$

Write the Cartan formula $\mathcal{L}_E = di_E + i_E d$ and apply it to the Liouville form. Obviously $\mathcal{L}_E \lambda = \lambda$ and $i_E\lambda = c_1$, so that, if $j : L \subset T^\star B$ is the inclusion,

$$j^\star\lambda = \mathcal{L}_E(j^\star\lambda) = di_E j^\star\lambda + i_E dj^\star\lambda.$$

As $j^\star\lambda$ is closed, we get

$$j^\star\lambda = d\left(c_1|_L\right) :$$

the first Chern class is a primitive of the Liouville form on L.

2.4. Integrable systems

The main result (or conjecture[3]) of [8] is that, in the case where X is the manifold F_{r+1} of complete flags in $\mathbf{C}^{r+1}$, the ideal $\mathcal{I}$ is generated by r *Poisson commuting* polynomials $f_1, \dots, f_r$ in $p_1, \dots, p_r, q_1, \dots, q_r$ ($r = \operatorname{rk} H^2(F_{r+1}) = \dim B$ in this case), and more precisely first integrals of a (non-periodic) Toda lattice. This means that the Lagrangian subvariety L is, in this case, a leaf in a Lagrangian foliation: for any $a \in \mathbf{C}^r$, $f_1 = a_1, \dots, f_r = a_r$ is a Lagrangian subvariety and L corresponds to $a = 0$.

A natural question would be to understand whether this property could hold in the more general situation we are investigating in this paper. The aim of this § is to give simple examples which show that this is not the case.

Non-complete intersections. — Notice that, if this were the case, the ideal $\mathcal{I}$ would be generated by r polynomials. However, it is easy to find an example that shows:

2.4.1 Proposition. — *There exists a Fano manifold X with $h^{2,0} = 0$, whose cohomology ring is generated by degree-2 classes, and whose quantum cohomology ring $QH^\star(X)$ is the ring of regular functions on a Lagrangian subvariety $L \subset T^\star(H^2(X;\mathbf{C}/2i\pi\mathbf{Z}))$ which is not a complete intersection.*

Let us look for a simply connected complex surface (so that $H^\star(X)$ will automatically be generated by $H^2(X)$), with $h^{2,0} = 0$ and for which we know that $\mathcal{I}$ has good chances to have too many generators. The ordinary cohomology will be some quotient

$$\mathbf{C}[p_1, \dots, p_r]/I,$$

[3] This result has been proved by Ciocan-Fontanine [4].

and the ideal I will be such that $\dim H^4(X) = 1$. Now in $\mathbf{C}[p_1, \ldots, p_r]$, there are $\frac{1}{2}r(r+1)$ monomials p_ip_j, so that the ideal I will have $\frac{1}{2}r(r+1) - 1$ generators. We are thus looking for an example where $\frac{1}{2}r(r+1) - 1 > r$, that is, $r \geq 3$.

As X is Fano, the p_i's can be chosen as classes of Kähler forms and no negative powers of the corresponding q_i's are needed to construct the quantum cohomology ring. Suppose $\mathcal{I}$ can be generated by r polynomials $f_1(p,q), \ldots, f_r(p,q)$. Then I will be generated by $f_1(p,0), \ldots, f_r(p,0)$ and thus will also be generated by r elements.

Let X be the simplest example one can imagine, that is, $\mathbf{P}^2$ blown up at two points (or $\mathbf{P}^1 \times \mathbf{P}^1$ blown up at one point, this is the same manifold), so that the ordinary cohomology ring of X is

$$H^\star(X; \mathbf{C}) = \mathbf{C}[p_1, p_2, p_3]/(p_1^2, p_2^2, p_3^2 + p_1p_2, p_1p_3, p_2p_3).$$

This is an easy exercise to show that the ideal I cannot be generated by three elements. □

Complete intersections. — There are even examples where L is a complete intersection but *not* a level of an algebraic integrable system, in the sense that the ideal $\mathcal{I}$ cannot be generated by commuting polynomials. The simplest possible example is that of $\widetilde{\mathbf{P}}^2$, the plane $\mathbf{P}^2$ blown up at one point.

Recall[4] that $QH^\star(\widetilde{P^2}) \cong \mathbf{Z}[p_1, p_2, q_1, q_2]/\langle p_1^2 - (p_1 - p_2)q_1, p_2^2 - p_1p_2 - q_2\rangle$ with $\deg p_1 = \deg p_2 = \deg q_1 = 2$, $\deg q_2 = 4$, so that $\mathcal{I}$ is generated by two degree-4 elements

$$f = p_1^2 - (p_1 - p_2)q_1, \quad g = p_2^2 - p_1p_2 - q_2.$$

These generators satisfy

$$\{f, g\} = q_1 g,$$

so that $\{f, g\}$ is indeed in $\mathcal{I}$. Now any set f_1, g_1 of generators of $\mathcal{I}$ will consist of degree-4 elements, so that

$$f_1 = af + bg, \quad g_1 = cf + dg$$

for some *constants* a, b, c and d such that $ad - bc \neq 0$. Hence

$$\{f_1, g_1\} = (ad - bc)\{f, g\} = (ad - bc)q_1 g \neq 0.$$

The characteristic Lagrangian subvariety of the quantum cohomology of $\widetilde{P}^2$ is not the zero level of two commuting polynomials and thus is not a level of an integrable system.

3. The volume function

In this section, (X, ω) denotes a weakly monotone symplectic manifold.

[4] see *e.g.*[2], with a different basis here to avoid negative powers of q_1, q_2.

3.1. Definition and first properties of V

The "volume function" is another variant of the Gromov-Witten potential, the mapping V which associates, with any $\xi \in H^2(X;\mathbf{C})$, the formal series

$$V_\xi = \sum_{A\in H_2(X;\mathbf{Z})} \frac{\Psi^A_{n+\langle c_1,A\rangle,0}(x\otimes\cdots\otimes x)}{(n+\langle c_1,A\rangle)!} q^A.$$

As ξ is a degree-2 class, notice that

$$\Psi^A_{N,0}(x\otimes\cdots\otimes x) = 0 \text{ if } N \neq n+\langle c_1,A\rangle,$$

that is,

$$V_\xi = \sum_{\substack{A\in H_2(X;\mathbf{Z})\\ N\in\mathbf{Z}}} \frac{\Psi^A_{N,0}(x\otimes\cdots\otimes x)}{N!} q^A.$$

By semipositivity, $\langle c_1, A\rangle \geq 0$ for all classes A that contain rational J-holomorphic curves, so that $n+\langle c_1,A\rangle \geq n$ is rather large.

As a consequence of Gromov's compactness theorem, the set of classes A that contain J-holomorphic curves and such that $\langle \omega, A\rangle \leq K$ is finite for all K. In particular, the set of classes A such that the coefficient of q^A is non-zero and $\langle \omega, A\rangle \leq K$ is finite for all K. This is to say that V_ξ belongs to the complex Novikov ring associated with ω: the volume function is a mapping

$$V : H^2(X;\mathbf{C}) \longrightarrow \Lambda_\omega.$$

Remark. — One usually needs the Novikov ring to define the quantum product for weakly monotone symplectic manifolds that are not monotone. Here it appears in the monotone case as well: there are infinitely many terms in the q-expansion of V even in the simple (and monotone) case of $\mathbf{P}^1$ (see below).

Homogeneity of V. — The complex Novikov ring Λ_ω is endowed with a $\mathbf{C}^\star$-action defined by the first Chern class c_1:

$$\mu \cdot \left(\sum_A \lambda_A q^A\right) = \sum_A \lambda_A \mu^{\langle c_1,A\rangle} q^A.$$

3.1.1 Proposition. — *The volume function is (quasi-)homogeneous of degree n. More precisely*

$$\mu\cdot(V_{\mu^{-1}\xi}) = \mu^n V_\xi. \quad \square$$

Expansion of V_ξ. — Choose a basis $(p_1,\ldots,p_r)$ of $H^2(X;\mathbf{Z})$, write $\xi = \sum_{i=1}^r z_i p_i$ ($z_i \in \mathbf{C}$), and expand

$$V_\xi = \sum_{\substack{j=(j_1,\ldots,j_r)\\ A\in H_2(X;\mathbf{Z})}} \frac{\Psi^A_{n+\langle c_1,A\rangle,0}\left(H_1^{\otimes j_1}\otimes\cdots\otimes H_r^{\otimes j_r}\right)}{j_1!\ldots j_r!} z^j q^A$$

(where of course $H_i = Dp_i \in H_{2n-2}(X; \mathbf{Z})$). Obviously, for any multi-exponent j,

$$\partial_z^j V_\xi|_{\xi=0} = \sum_A \Psi^A_{n+\langle c_1, A\rangle,0} \left(H_1^{\otimes j_1} \otimes \cdots \otimes H_r^{\otimes j_r}\right) q^A$$

and it turns out, as a consequence of the composition rule of [13], that this is exactly the "quantum intersection index"

$$\langle p_1^{j_1} \star \cdots \star p_r^{j_r}, [X]\rangle.$$

Volume function for $\mathbf{P}^n$. — Recall that $QH^\star(\mathbf{P}^n) = \mathbf{Z}[p,q]/p^{n+1} - q$, so that

$$\langle p^k, [\mathbf{P}^n]\rangle = \begin{cases} q^d & \text{if } k = (n+1)d + n \\ 0 & \text{otherwise.} \end{cases}$$

As this is (up to a factorial) the coefficient of z^k in V_z, we get

$$V_z = \sum_{d\geq 0} \frac{z^{(n+1)d+n}}{[(n+1)d+n]!} q^d.$$

3.2. Why is V called a volume function?

Consider first the "constant term" (the $A = 0$ term) in V_ξ. This is

$$\frac{1}{n!}\Psi^0_{n,0}(x \otimes \cdots \otimes x) = \frac{1}{n!}\langle \xi^n, [X]\rangle.$$

If ξ was the cohomology class of a symplectic form on X, this would be the symplectic volume of X. Notice that X is the moduli space of constant holomorphic curves in X, that is, $\mathcal{M}^0(J,\nu)$.

More generally, the coefficient of q^A is $\Psi^A_{N,0}(x \otimes \cdots \otimes x)/N!$ where $2N = \dim \mathcal{M}^A(J,\nu)$ and $\Psi^A_{N,0}(x \otimes \cdots \otimes x)$ is defined as the intersection number of the pseudo-cycle

$$\begin{array}{ccc} \mathcal{M}^A(J,\nu) & \xrightarrow{\mathrm{ev}_{\vec{z}}} & X \times \cdots \times X \\ u & \longmapsto & (u(z_1), \ldots, u(z_N)) \end{array}$$

with the class $x \otimes \cdots \otimes x$ dual to $\xi \otimes \cdots \otimes \xi$, so that it is tempting to consider that $\Psi^A_{N,0}(x \otimes \cdots \otimes x)/N!$ is the "symplectic volume of the moduli space $\mathcal{M}^A(J,\nu)$"... which it would be if $\mathrm{ev}^\star_{z_1}\xi$ was the class of a symplectic form on $\mathcal{M}^A(J,\nu)$ and if $(\mathrm{ev}_{\vec{z}})^\star(\xi \otimes \cdots \otimes \xi) = \left(\mathrm{ev}^\star_{z_1}\xi\right)^N$ could be integrated on $\mathcal{M}^A(J,\nu)$.

The case of $\mathbf{P}^1$. — In the case of $\mathbf{P}^1$, the volume function is

$$V_z = \sum_{d\geq 0} \frac{z^{2d+1}q^d}{(2d+1)!} (= q^{-1/2}\sinh(q^{1/2}z)).$$

We may interpret this result as the fact that the space of degree-d rational maps $u : \mathbf{P}^1 \to \mathbf{P}^1$ has volume $z^{2d+1}/(2d+1)!$ for the "volume form" defined by $z\omega$. This is rather satisfactory as this space has a naïve compactification as $\mathbf{P}^{2d+1}$ (using the coefficients in the rational functions)... although not really sufficient to use this expected volume property as a definition of V.

3.3. Differential equations satisfied by the volume function

Let us now make the same assumption we made in § 2, namely that the cohomology algebra $H^\star(X)$ is generated by $H^2(X)$, its degree-2 part. As the quantum product is defined as a ring structure on the free Λ_ω-module $QH^\star(X) = H^\star(X) \otimes \Lambda_\omega$, this automatically implies that the natural ring morphism

$$S^\star(X) \otimes \Lambda_\omega \longrightarrow QH^\star(X)$$

is surjective. Let us call again its kernel $\mathcal{I}$ so that $QH^\star(X) = S^\star(X) \otimes \Lambda_\omega/\mathcal{I}$.

The ring $S^\star(X) \otimes \Lambda_\omega$ acts on the space of mappings $H^2(X; \mathbf{C}) \to \Lambda_\omega$: any element β of $H^2(X)$ can be considered as a (constant) vector field on $H^2(X)$ and thus as a derivation acting on functions.

Using coordinates, that is, a basis $(p_1, \dots, p_r)$ of $H^2(X; \mathbf{Z})$, any element of $S^\star(X)$ is a polynomial in p_i's. A mapping $F : H^2(X; \mathbf{Z}) \to \Lambda_\omega$ is a series $F(z_1, \dots, z_r, q)$ and

$$(R \cdot F)(z_1, \dots, z_r, q) = R(\partial_{z_1}, \dots, \partial_{z_r}, q) \cdot F(z_1, \dots, z_r, q).$$

The next result is a very simple characterisation of $\mathcal{I}$ in terms of the volume function: this is the ideal of all the differential equations it satisfies.

3.3.1 Proposition. — *Assume the natural map*

$$S^\star(X) \longrightarrow H^\star(X)$$

is onto. Then the kernel $\mathcal{I}$ of

$$S^\star(X) \otimes \Lambda_\omega \longrightarrow QH^\star(X)$$

is the ideal of all differential equations satisfied by the volume function:

$$\mathcal{I} = \{R \in S^\star(X) \otimes \Lambda_\omega \mid R \cdot V = 0\}.$$

In other words, the relation $R(p, q) = 0$ holds in the quantum cohomology ring $QH^\star(X)$ *if and only if* $R(\partial_z, q) \cdot V(z, q) = 0$.

Example. — The volume function of $\mathbf{P}^n$ satisfies the differential equation

$$\frac{\partial^{n+1}}{\partial z^{n+1}} V_z = q V_z$$

and any other differential equation it satisfies is a consequence of it.

Proof. — Let us work with coordinates. First of all, as V is defined as a power series, $R \cdot V = 0$ if and only if

$$p^j \cdot (R \cdot V)|_{z=0} = 0$$

for all $p^j = p_1^{j_1} \dots p_r^{j_r} \in S^\star(X)$. This last condition is equivalent to

$$(p^j R) \cdot V|_{z=0} = 0 \qquad \forall p^j \in S^\star(X).$$

But the derivatives of V at 0 describe the quantum intersection indices as noticed above (§ 3.1), so that $R \cdot V = 0$ if and only if $\langle p^j R(p), [X] \rangle = 0$ for all the multi-exponents j. Let now h_j be the homology class $D(p_j)$. As

$$\langle p^j R(p), [X] \rangle = \langle R(p), h_j \rangle,$$

what we have proved so far is that $R \cdot V = 0$ if and only if $\langle R(p), h_j \rangle = 0$ for all j. As the monomials p^j generate $H^\star(X)$, we deduce that $R \cdot V$ vanishes if and only if $\langle R(p), b \rangle = 0$ for every homology class b, a fact that ends the proof of the proposition. □

Although this result does not seem to be very effective as a way of determining $\mathcal{I}$, it is a beautiful description of this ideal in terms of a single function V. Notice that Proposition 2.2.2 can be rephrased as:

3.3.2 COROLLARY. — *Suppose (X, ω) is an almost monotone symplectic manifold which satisfies the semi-simplicity assumption and such that the natural map $S^\star(X) \to H^\star(X)$ is onto. Then the ideal of differential equations satisfied by the volume function V is stable under Poisson bracket.* □

It might be interesting to understand which functions have the property that the ideal of differential equations they satisfy is Poisson.

Bibliography

[1] M. AUDIN, *Symplectic and almost complex manifolds*, in [3].

[2] M. AUDIN, *Cohomologie quantique*, in *Séminaire Bourbaki 1995-96*, Astérisque (to appear).

[3] M. AUDIN, J. LAFONTAINE, EDS, *Holomorphic curves in symplectic geometry*, Progress in Math. 117, Birkhäuser, 1994.

[4] I. CIOCAN-FONTANINE, *Quantum cohomology of flag varieties*, preprint (1995).

[5] R. DONAGI, E. MARKMAN, *Spectral covers, algebraically completely integrable Hamiltonian systems and moduli of bundles*, Lecture Notes in Math. **1620** (1995).

[6] B. DUBROVIN, *Geometry of 2D topological field theories*, Lecture Notes in Math. **1620** (1995).

[7] A. GIVENTAL, *Homological geometry I. Projective hypersurfaces*, preprint (1994).

[8] A. GIVENTAL, B. KIM, *Quantum cohomolgy of flag manifolds and Toda lattices*, Commun. Math. Phys. **168** (1995), 609–641.

[9] N. HITCHIN, *Lectures on Frobenius manifolds*, Séminaire de Mathématiques Supérieures, Montréal (1995).

[10] H. HOFER, D. SALAMON, *Floer homology and Novikov rings*, in *the Floer memorial volume*, Progress in Math. 133, Birkhäuser, 1995.

[11] D. McDuff, *Positivity of intersections*, in [3].

[12] D. McDuff, D. Salamon, *J-holomorphic curves and quantum cohomology*, Amer. Math. Soc., 1994, 2nd printing, 1995.

[13] Y. Ruan, G. Tian, *A mathematical theory of quantum cohomology* (announcement), Math. Res. Letters **1** (1994), 269–278, and *A mathematical theory of quantum cohomology*, Journal of Diff. Geometry, to appear.

[14] C. Simpson, *Higgs bundles and local systems*, Publ. Inst. Hautes Etudes Sci. **75** (1992), 5–95.

[15] G. Tian, *Quantum cohomology and its associativity*, preprint (1995).

[16] C. Voisin, *Symétrie miroir*, Panoramas et Synthèses, Société Mathématique de France, to appear.