Ouagadougou, September 2015

Michèle Audin

I. Symplectic spaces

Euclidean:

$\boldsymbol{E}$ vector space over R, endowed with an inner product
namely a
nondegenerate symmetric bilinear form

$$\begin{aligned} \boldsymbol{E} \times \boldsymbol{E} &\rightarrow \mathrm{R} \\ (\boldsymbol{x}, \boldsymbol{y}) &\mapsto \boldsymbol{x} \cdot \boldsymbol{y} \end{aligned}$$

orthonormal basis $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m)$

Symplectic:

E vector space over R, endowed with a symplectic form

namely a

nondegenerate skewsymmetric bilinear form

$$\begin{aligned} E \times E &\to \mathrm{R} \\ (x, y) &\mapsto \omega(x, y) \end{aligned}$$

symplectic basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$

What is a symplectic basis?

A basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$

such that

$\omega(e_i, e_j) = \omega(f_i, f_j) = 0$ **for all** i, j
$\omega(e_i, f_j) = \delta_{i,j}$ **for all** i, j

Hence $\dim E = 2n$ is even.

Remark:
ω skew-symmetric
$\Rightarrow \omega(x, x) = -\omega(x, x) = 0.$

Hence there exists
no symplectic space of dimension 1.

Proof of the existence of a symplectic basis:
$e_1 \neq 0$
$\exists\, f$ such that $\omega(e_1, f) = a \neq 0$
take $f_1 = f/a$, then $\omega(e_1, f_1) = 1$.

Let F be the orthogonal complement of $\langle e_1, f_1 \rangle$ w.r. to ω.
$\omega|_F$ is nondegenerate.
Conclude by induction on $\dim E$.

$\boldsymbol{E} = \mathrm{R}^n \times \mathrm{R}^n$,
each factor Euclidean,

with the skew-symmetric form
$\omega((p, q), (p', q')) = p \cdot q' - p' \cdot q$

is a symplectic vector space.

$(v_1, \ldots, v_n)$ orthonormal basis of R^n,
let
$e_i = (v_i, 0),\ f_i = (0, v_i)$,

then
$(e_1, \ldots, e_n, f_1, \ldots, f_n)$
is a symplectic basis.
$\omega(e_i, f_j) = v_i \cdot v_j - 0 \cdot 0 = v_i \cdot v_j = \delta_{i,j}$
and so on.

Differential forms.

ω can be understood as a differential 2-form
$\omega = \sum_i dp_i \wedge dq_i$

Note that
$\omega = d\alpha$
for
$\alpha = \sum_i p_i \, dq_i.$
α is the Liouville form.

Subspaces.

F subspace of E.
Denote F° the orthogonal complement,
$F^\circ = \left\{ x \in E \mid \omega(x, y) = 0 \quad \forall\, y \in F \right\}$

Example:
$E^\circ = 0$ (this is to say that
ω is nondegenerate).

Claim:

$$\dim F + \dim F^{\circ} = \dim E$$

Proof:

$$F' = \left\{ \varphi \in E^{\star} \mid \varphi|_F = 0 \right\} \subset E^{\star}.$$

$$\dim F' = \dim E - \dim F$$

F° **is isomorphic to** F'**:**

$$\begin{aligned} F^{\circ} &\to F' \\ x &\mapsto (y \mapsto \omega(x, y)) \end{aligned}$$

Interesting subspaces:

Isotropic: $F \subset F^\circ$
$(\omega|F = 0)$.
Hence $\dim F \leq \dim F^\circ$ and
$\dim F \leq n$.

Co-isotropic: $F \supset F^\circ$.
$\dim F \geq n$.

Lagrangian:
Maximal isotropic
$F = F^\circ$,
$\dim F = n$.

Examples.

- Every dim 1-subspace is isotropic.

- Every hyperplane is co-isotropic.

- If $(e_1, \dots, e_n, f_1, \dots, f_n)$ is a symplectic basis, $\langle e_1, \dots, e_k \rangle$ is isotropic (Lagrangian if $k = n$).

- Conversely, il F is isotropic there exists a basis $(e_1, \dots, e_k)$ (of F) which can be completed into a symplectic basis $(e_1, \dots, e_n, f_1, \dots, f_n)$ of E.

Hence, any isotropic subspace is included in a Lagrangian.

And any co-isotropic subspace contains a Lagrangian.

Example.

Let $\varphi : \mathrm{R}^n \to \mathrm{R}^n$ be a linear map.
Its graph
$L = \left\{(x, \varphi(x)) \mid x \in \mathrm{R}^n\right\} \subset \mathrm{R}^n \times \mathrm{R}^n$
is a Lagrangian subspace if and only if
φ is a symmetric map.

Proof.

$$\omega((x, \varphi(x)), (y, \varphi(y))) = 0 \quad \forall\, x, y$$

if and only if

$$x \cdot \varphi(y) - \varphi(x) \cdot y = 0 \quad \forall\, x, y$$
$$\Leftrightarrow x \cdot \varphi(y) - x \cdot {}^t\varphi(y) = 0 \quad \forall\, x, y.$$

Symplectic reduction:

$L \subset E$ a Lagrangian subspace ($L = L^\circ$).
$F \subset E$ a co-isotropic subspace ($F \supset F^\circ$).
Assume $L + F = E$.

Then the composition
$L \cap F \subset F \to F/F^\circ$
is injective. The space F/F° is symplectic
and the image of $L \cap F$ in F/F°
is a Lagrangian subspace.

Proof:

$\boldsymbol{F/F^\circ}$ **is symplectic.**
and
Kernel of $\boldsymbol{L \cap F \subset F \to F/F^\circ}$ **is**

$$\begin{aligned} \boldsymbol{L \cap F \cap F^\circ} &= \boldsymbol{L \cap F^\circ} \\ &= \boldsymbol{(L^\circ + F)^\circ} \\ &= \boldsymbol{(L + F)^\circ} \\ &= \boldsymbol{E^\circ} \\ &= \mathbf{0.} \end{aligned}$$

Complex setting:
$\mathbf{R}^{2n} = \mathbf{R}^n \times \mathbf{R}^n = \mathbf{C}^n$.

Hermitian product on $\mathbf{C}^n$:

$$\begin{aligned} \langle q + ip, q' + ip' \rangle &= q \cdot q' + p \cdot p' + i(p \cdot q' - q \cdot p') \\ &= \underbrace{q \cdot q' + p \cdot p'}_{\text{real part}} + i\, \underbrace{\omega((p,q),(p',q'))}_{\text{imaginary part}} \\ &= \text{inner product} + i \,\text{symplectic form} \end{aligned}$$

$(u_1, \ldots, u_n)$ a unitary basis of $\mathbf{C}^n$,

Let $f_j = -ie_j$.

Then
$(e_1, \ldots, e_n, f_1, \ldots, f_n)$ is
a symplectic basis.

Proof:

$$\omega(e_i, e_j) = \mathrm{Im}\langle e_i, e_j\rangle = \mathrm{Im}\,\delta_{i,j} = 0$$
$$\omega(f_i, f_j) = \mathrm{Im}\langle ie_i, ie_j\rangle = \mathrm{Im}\langle e_i, e_j\rangle = 0$$
$$\omega(e_i, jf_j) = \mathrm{Im}\langle e_i, -ie_j\rangle = \mathrm{Re}\langle e_i, e_j\rangle = \delta_{i,j}$$

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II. Lagrangian submanifolds, Symplectic manifolds

Lagrangian submanifolds.

$V \subset \mathrm{R}^{2n}$ **is a Lagrangian submanifold if it is a submanifold and** $\forall\, x \in V$, T_xV **is a Lagrangian subspace.**

$f : V \to \mathrm{R}^{2n}$ **is a Lagrangian immersion if it is an immersion and** $\forall\, x \in V$, $T_xf(T_xV)$ **is a Lagrangian subspace.**

So that, in both cases, $\dim V = n$.

In terms of the differential 2-form ω,
$f : V \to \mathrm{R}^{2n}$ is a Lagrangian immersion if

- **$\dim V = n$,**
- **f is an immersion, and**
- **$f^\star\omega = 0$.**

Note that, since $\omega = d\alpha$,
this means that
$f^\star\alpha$ is a closed 1-form on V.

Examples.

Rather rare.

Any (embedded or immersed) curve in R.

Products of Lagrangians:
If $L_1 \subset \mathrm{R}^{2n}$ and $L_2 \subset \mathrm{R}^{2p}$ are Lagrangian submanifolds, then $L_1 \times L_2 \subset \mathrm{R}^{2(n+p)}$ is a Lagrangian submanifold.

Products of circles $\rightarrow$ tori.

Spheres?
In dimension ≥ 2, no Lagrangian submanifold can be a sphere (after a theorem of Gromov).

$$\boldsymbol{S}^n = \left\{ (x_1, \ldots, x_n, y) \in \mathrm{R}^{n+1} \mid \sum x_i^2 + y^2 = 1 \right\}.$$

Lagrangian immersion:

$$\boldsymbol{S}^n \to \mathrm{R}^n \times \mathrm{R}^n$$
$$(x_1, \ldots, x_n, y) \mapsto (x_1, \ldots, x_n, 2yx_1, \ldots, 2yx_n)$$

with a double point

$$(0, \ldots, 0, \pm 1) \mapsto (0, \ldots, 0, 0, \ldots, 0)$$

Graphs.

$F : \mathrm{R}^n \to \mathrm{R}^n$ **a map.**

Its graph

$$V = \left\{(x, F(x)) \mid x \in \mathrm{R}^n\right\}$$

is a Lagrangian submanifold if and only if

$F = \nabla f$ **for a function**

$f : \mathrm{R}^n \to \mathrm{R}.$

Proof.

$T_x V$ is the graph of $(dF)_x$.
It is a Lagrangian subspace if and only if $(dF)_x$ is a symmetric map.

$F = (F_1, \ldots, F_n)$. The matrix of $(dF)_x$ is

$$\left(\frac{\partial F_i}{\partial x_j}\right)_{i,j}.$$

It is symmetric for all $x \in \mathrm{R}^n$ if and only if $\exists\, f$ such that

$$F_i = \left(\frac{\partial f}{\partial x_i}\right) \text{ for all } i.$$

$$\mathrm{R}^{2n} = \mathrm{R}^n \times \mathrm{R}^n = (\mathrm{R}^n)^\star \times \mathrm{R}^n.$$

In $(p, q) \in \mathrm{R}^{2n}$, we consider

p as a linear form on R^n

and q as a vector of R^n.

Replace R^n by a manifold $\boldsymbol{X}$
(of dimension $\boldsymbol{n}$).

Replace $(\mathrm{R}^n)^\star \times \mathrm{R}^n$
by the cotangent bundle
$\boldsymbol{T^\star X}$.
This is an $\boldsymbol{n}$-dimensional manifold.

Locally, X is like R^n

has local coordinates
$(q_1, \ldots, q_n) \in \mathrm{R}^n$.

Then $T^\star X$ has

local coordinates
$(p_1, \ldots, p_n, q_1, \ldots, q_n) \in (\mathrm{R}^n)^\star \times \mathrm{R}^n$.

$(p, q) \in T^\star X$
considered as
$q \in X$
and
$p \in T_q^\star X$,
that is p is a linear form
$p : T_q X \to \mathrm{R}$.

Liouville form
$\alpha = \sum_i p_i \, dq_i$

$q \in X$ **a point,**

$p \in T_q^\star X$ **a linear form.**

$(P, Q) \in T_{(p,q)}(T^\star X)$ **a tangent vector to** $T^\star X$,
$(Q \in T_q X)$,

$\alpha_{(p,q)}(P, Q) = p(Q)$.

Symplectic form

$$\omega = d\alpha = \sum_i dp_i \wedge dq_i$$

ω is a 2-form.

It is closed

$$d\omega = 0$$

(exact: $\omega = d\alpha$)

and nondegenerate.

More generally, a manifold on dimension $2n$ endowed with a closed nondegenerate 2-form is a symplectic manifold.

According to Darboux' theorem, any symplectic manifold is locally isomorphic to
R^{2n} with the symplectic form ω.

Note: no local invariant, implies that all the problems are global.

Different from Riemannian geometry.

Also note.

If W is a compact manifold,
the symplectic form ω
on W cannot be exact.

$d\omega = 0$ but
there is no form α such that $\omega = d\alpha$.

For this course,
back to cotangent bundles

and to Lagrangian submanifolds.

Let $f : X \to \mathrm{R}$ be a function.

Then df can be considered as a section
$df : X \to T^{\star}X$

The graph of df is a
Lagrangian submanifold in $T^{\star}X$.

Notes.

If X is compact, the Lagrangian
"graph of df"
meets the zero section.

η a 1-form on X.
This is a section of $T^{\star}X$,
$\eta : X \to T^{\star}X$.
Its graph is Lagrangian if and only if
η is a closed 1-form.

Generalization.
$f : X \times \mathrm{R}^k \to \mathrm{R}$ **a function.**
The graph of df **is**
a Lagrangian submanifold of
$T^\star(X \times \mathrm{R}^k) = T^\star X \times (\mathrm{R}^k)^\star \times \mathrm{R}^k$.

$T^\star X \times \{0\} \times \mathrm{R}^k$ **is a co-isotropic submanifold**
of $T^\star X \times (\mathrm{R}^k)^\star \times \mathrm{R}^k$.

Consider

$$V = \left\{ (x, a) \in X \times \mathrm{R}^k \mid \frac{\partial f}{\partial a_1} = \cdots = \frac{\partial f}{\partial a_k} = 0 \right\}.$$

Assume f is such that this is a submanifold (then it has dimension n).

$$\begin{aligned} V &\to T^\star X \\ (x, a) &\mapsto (df)_{(x,a)} \end{aligned}$$

is a Lagrangian immersion (by symplectic reduction).

Symplectic reduction:

$L \subset E$ a Lagrangian subspace ($L = L^\circ$).
$F \subset E$ a co-isotropic subspace ($F \supset F^\circ$).
Assume $L + F = E$.

Then the composition
$L \cap F \subset F \to F/F^\circ$
is injective. The space F/F° is symplectic
and the image of $L \cap F$ in F/F°
is a Lagrangian subspace.

Proof:

F/F° **is symplectic.**
and
Kernel of $L \cap F \subset F \to F/F^\circ$ **is**

$$\begin{aligned} L \cap F \cap F^\circ &= L \cap F^\circ \\ &= (L^\circ + F)^\circ \\ &= (L + F)^\circ \\ &= E^\circ \\ &= 0. \end{aligned}$$

Remark: comparison with Viet's notation:

$$
\begin{aligned}
\boldsymbol{q} &\mapsto \boldsymbol{x} \in \mathbf{R}^{n} \\
\boldsymbol{p} &\mapsto \boldsymbol{\xi} \in (\mathbf{R}^{n})^{\star} \\
\boldsymbol{f} &\mapsto \boldsymbol{\varphi} \\
\boldsymbol{a} &\mapsto \boldsymbol{\theta} \\
\boldsymbol{V} &\mapsto \boldsymbol{\Lambda}
\end{aligned}
$$

Example.

$X = \mathrm{R}^n$, $k = 1$,

$$f : \mathrm{R}^n \times \mathrm{R} \to \mathrm{R}$$
$$(x, a) \mapsto a \|x\|^2 + \frac{a^3}{3} - a$$

$$\frac{\partial f}{\partial a} = \|x\|^2 + a^2 - 1.$$

$V = S^n$

and the Lagrangian immersion is the same as above.

Example.

$X = \mathrm{R}^n$, $k = 1$,

For $x = (x_1, \ldots, x_n) \in \mathrm{R}^n$,
$P_x(a) = a^{n+2} + x_1 a^n + \cdots + x_n a$,
polynomial in a.

$$f : \mathrm{R}^n \times \mathrm{R} \to \mathrm{R}$$
$$(x, a) \mapsto a^{n+2} + x_1 a^n + \cdots + x_n a = P_x(a)$$

$$V = \left\{ (x, a) \in \mathrm{R}^n \times \mathrm{R} \mid \frac{\partial f}{\partial a} = 0 \right\}$$
$$= \left\{ (x, a) \in \mathrm{R}^n \times \mathrm{R} \mid P_x'(a) = 0 \right\}$$

(Sub)Example.

For $n = 2$,

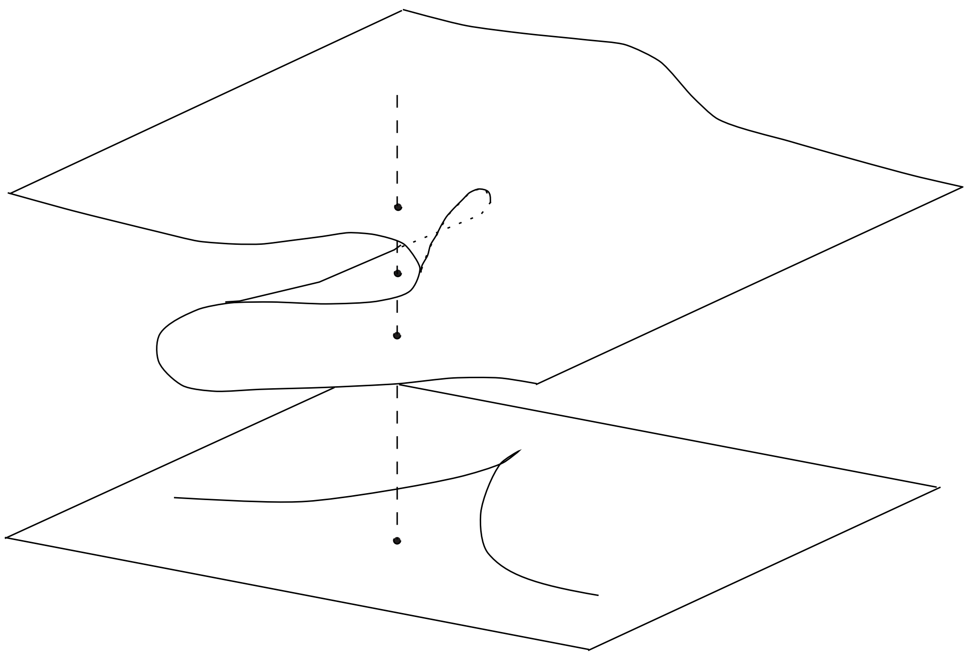
$$V = \left\{(x_1, x_2, a) \in \mathrm{R}^3 \mid 4a^3 + 2x_1 a + x_2 = 0\right\}$$

the Lagrangian immersion

$$V \to T^\star \mathrm{R}^2 = \mathrm{R}^2 \times \mathrm{R}^2$$

is

$$(x_1, x_2, a) \mapsto (x_1, x_2, a^2, a).$$



Conormal bundles.

$Y \subset X$ a submanifold
of dimension k.

$$N^\star Y = \Big\{(p,q) \in T^\star X \mid q \in Y \text{ and } p|_{T_qY} = 0\Big\}.$$

$N^\star Y \subset T^\star X$ is a Lagrangian submanifold.

Proof.
$\alpha|_{N^\star Y} = 0$, and $\dim N^\star Y = k + n - k = n$.

Example.

Let Y be a point $q_0 \in X$.

$T_{q_0}Y = \{0\}.$

$$
\begin{aligned}
N^\star Y &= \Big\{(p,q) \in T^\star X \mid q \in Y \text{ and } p\big|_{T_qY} = 0\Big\} \\
&= \Big\{(p,q_0) \in T^\star X\Big\} \\
&= T^\star_{q_0} X.
\end{aligned}
$$

The fibers in the cotangent bundle are Lagrangian (of course).

This is the Lagrangian with phase the Dirac distribution.

More generally, for an immersion
$f : Y \to X$
the conormal bundle
$N^\star f.$

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III. Lagrangian Grassmannian and the Maslov class

Lagrangian Grassmannian.

This is the space of all Lagrangian subspaces in R^{2n}:

$$\Lambda_n = \left\{ L \subset \mathrm{R}^{2n} \mid L = L^{\circ} \right\}.$$

Example.

$n = 1$.

In R^2, all lines are Lagrangian. Hence Λ_1 is the space of lines in R^2 namely the projective space $\mathrm{P}^1(\mathrm{R})$.

Complex setting:
$\mathbf{R}^{2n} = \mathbf{R}^n \times \mathbf{R}^n = \mathbf{C}^n$.

Hermitian product on $\mathbf{C}^n$:

$$\begin{aligned} \langle q + ip, q' + ip' \rangle &= q \cdot q' + p \cdot p' + i(p \cdot q' - q \cdot p') \\ &= \underbrace{q \cdot q' + p \cdot p'}_{\text{real part}} + i\, \underbrace{\omega((p, q), (p', q'))}_{\text{imaginary part}} \\ &= \text{inner product} + i\, \text{symplectic form} \end{aligned}$$

$(u_1, \ldots, u_n)$ a unitary basis of $\mathbf{C}^n$,

Let $f_j = -ie_j$.

Then
$(e_1, \ldots, e_n, f_1, \ldots, f_n)$ is
a symplectic basis.

Proof:

$$\begin{aligned} \omega(e_i, e_j) &= \mathrm{Im}\langle e_i, e_j\rangle = \mathrm{Im}\,\delta_{i,j} = 0 \\ \omega(f_i, f_j) &= \mathrm{Im}\langle ie_i, ie_j\rangle = \mathrm{Im}\langle e_i, e_j\rangle = 0 \\ \omega(e_i, jf_j) &= \mathrm{Im}\langle e_i, -ie_j\rangle = \mathrm{Re}\langle e_i, e_j\rangle = \delta_{i,j} \end{aligned}$$

Orthogonality and Isotropy.

In $\mathrm{C}^n = \mathrm{R}^{2n}$. Denote $F^\perp$ the Euclidean orthogonal complement of the (real) subspace F.

A real subspace L of C^n is Lagrangian if and only if $L^\perp = iL$.

Proof:

$$\begin{aligned} \omega(Z, Z') = 0 &\Leftrightarrow \mathrm{Im}(Z, Z') = 0 \\ &\Leftrightarrow \mathrm{Re}\langle Z, iZ'\rangle = 0 \\ &\Leftrightarrow Z \cdot (iZ') = 0. \end{aligned}$$

$L \subset \mathbb{C}^n$ a Lagrangian subspace.
$(v_1, \ldots, v_n)$ an orthonormal basis of L.
Then $(v_1, \ldots, v_n)$ is a
complex unitary basis of $\mathbb{C}^n$.

Conversely, if $(v_1, \ldots, v_n)$ is a
complex unitary basis of $\mathbb{C}^n$,
then the real subspace spanned by it
is Lagrangian.

Proof:

$(v_1, \ldots, v_n)$ **an orthonormal basis of** L.

Then
$(v_1, \ldots, v_n, iv_1, \ldots, iv_n)$ **is an orthonormal basis of the real vector space** $\mathbf{C}^n$.

Hence $(v_1, \ldots, v_n)$
is a complex basis of $\mathbf{C}^n$.

Moreover
$\langle v_i, v_j \rangle = v_i \cdot v_j + \omega(v_i, v_j) = \delta_{i,j} - 0.$

Hence this is a unitary basis.

Homogeneous space:
$\Lambda_n = \mathrm{U}(n)/\,\mathrm{O}(n)$

Proof:

L_1, L_2, two Lagrangians in $\mathbb{C}^n$.
Each has an orthonormal basis...
which is a unitary basis of $\mathbb{C}^n$.
Hence there is an element in $\mathrm{U}(n)$
which maps L_1 on L_2:
$\mathrm{U}(n)$ acts transitively on Λ_n.

The stabilizer of $\mathbb{R}^n \subset \mathbb{C}^n$ is $\mathrm{O}(n)$.

Λ_n is a compact and connected manifold of dimension $\dfrac{n(n+1)}{2}$.

Example.
$n = 1$.
$\Lambda_1 = \mathrm{P}^1(\mathrm{R})$ is indeed a compact and connected manifold of dimension 1 (diffeomorphic to a circle).

$\pi_1(\Lambda_n)$ is isomorphic to Z.

Example.
$n = 1$.
Λ_1 is diffeomorphic to a circle, hence its fundamental group is isomorphic to Z.

Proof.

det : U(n) → S^1. If $A \in$ O(n), det $A = \pm 1$ and (det A)$^2 = 1$.

Hence
det^2 : U(n) → S^1 is a well-defined map.

This gives an isomorphism
$\pi_1(\Lambda_n) \to \pi_1(S^1) = \mathbf{Z}$
(because SU(n) is simply connected).

Gauss map.

$V \subset \mathbb{C}^n$ a Lagrangian submanifold (or immersion).

$$\begin{aligned} V &\to \Lambda_n \\ x &\mapsto T_x V \end{aligned}$$

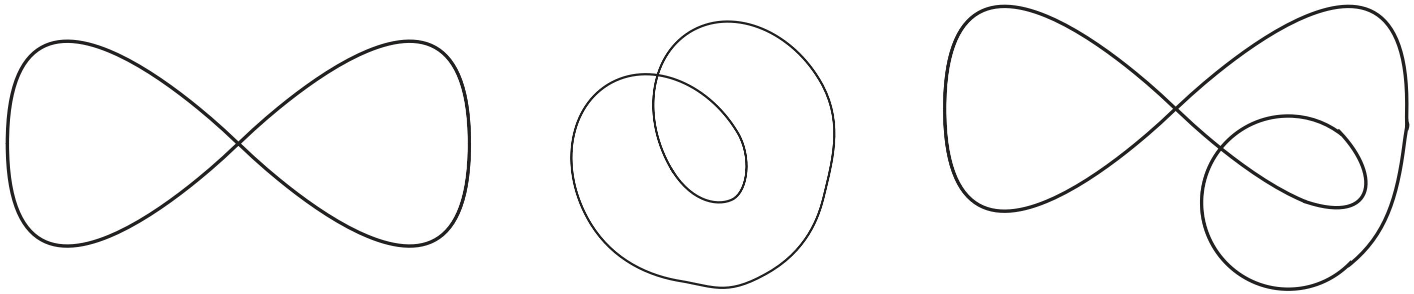
Maslov class.

The Gauss map induces a map

$$\pi_1(V) \to \pi_1(\Lambda_n)$$
$$\pi_1(V) \mapsto \mathbf{Z}$$

To any loop in V is thus associated an integral number.

Examples.
Immersions of the circle in $\mathbb{R}^2$.



Maslov classes?

More abstractly,
this defines an element of $H^1(V)$.

Because a map
$\pi_1(V) \to \mathbf{Z}$
is an element of the dual $H^1(V; \mathbf{Z})$,

This cohomology class is called
the Maslov class of the immersion f.

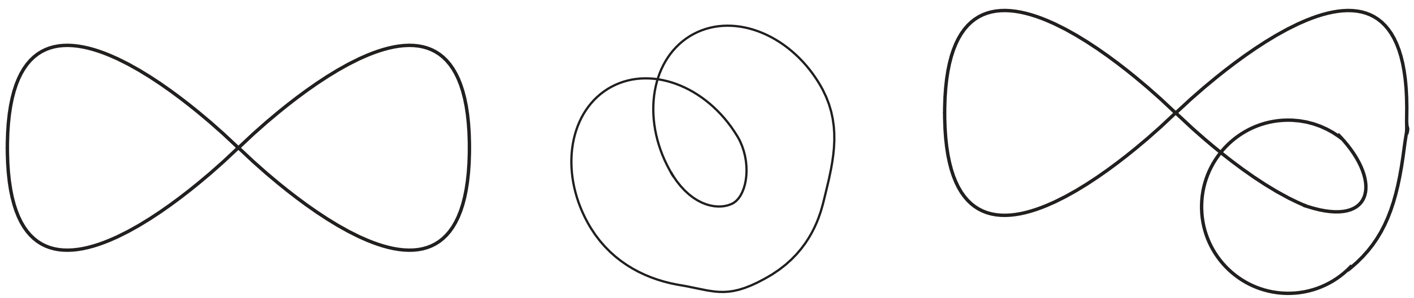
Recall $f^\star\alpha$ is a closed 1-form on V.

This defines a de Rham cohomology class

$$[f^\star\alpha] \in H^1(V;\mathbb{R}),$$

the Liouville class.

Examples.
Immersions of the circle in $\mathbf{R}^2$.



Liouville classes?

(area!)

Exact Lagrangian submanifolds.

$f : V \to T^\star X$
Lagrangian. Then $0 = f^\star\omega = f^\star d\alpha = d(f^\star\alpha)$.

$f^\star\alpha$ is closed. If it is exact, that is
if $\exists\, f : V \to$ R such that
$f^\star\alpha = dF$, then
f is exact.

Then the imge

$$\begin{aligned} V &\to T^\star X \times \mathrm{R} \to X \times \mathrm{R} \\ x &\mapsto (f(x), F(x)) \mapsto (g(x), F(x)) \end{aligned}$$

is called the wave front
of the Lagrangian immersion f.

Notes.
F is only defined
up to the addition of a constant.

Given $(g(x), F(x))$,
f is determined.

Examples. In dimension 1.

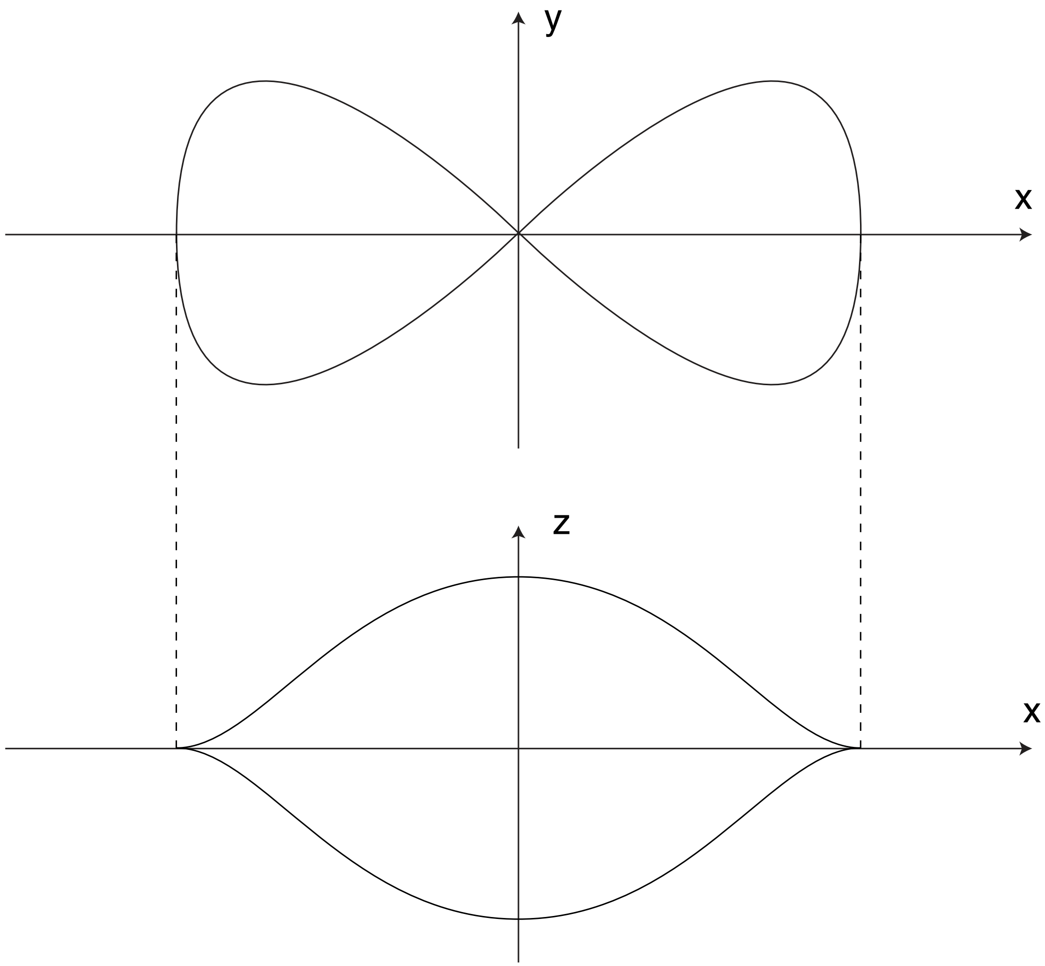
$\dim X = 1$ (coordinate x),

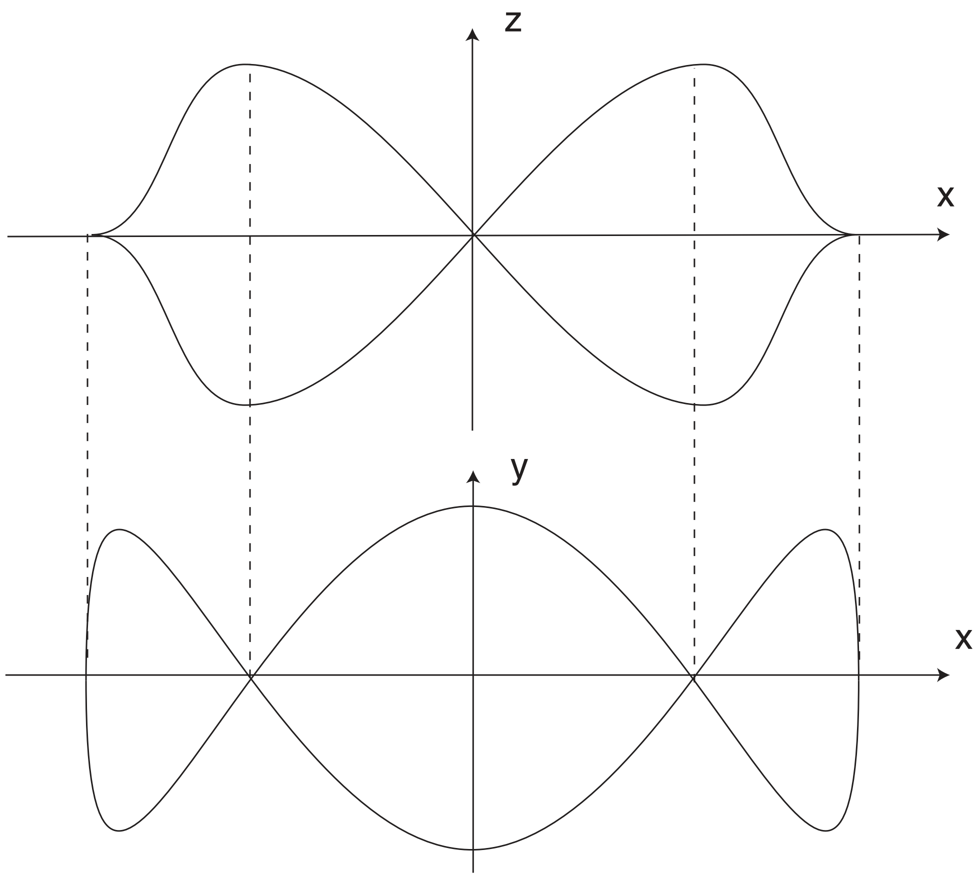
$\dim T^\star X = 2$ (coordinates (x, y)),

$\dim X \times \mathrm{R} = 2$ (coordinates (x, z)).

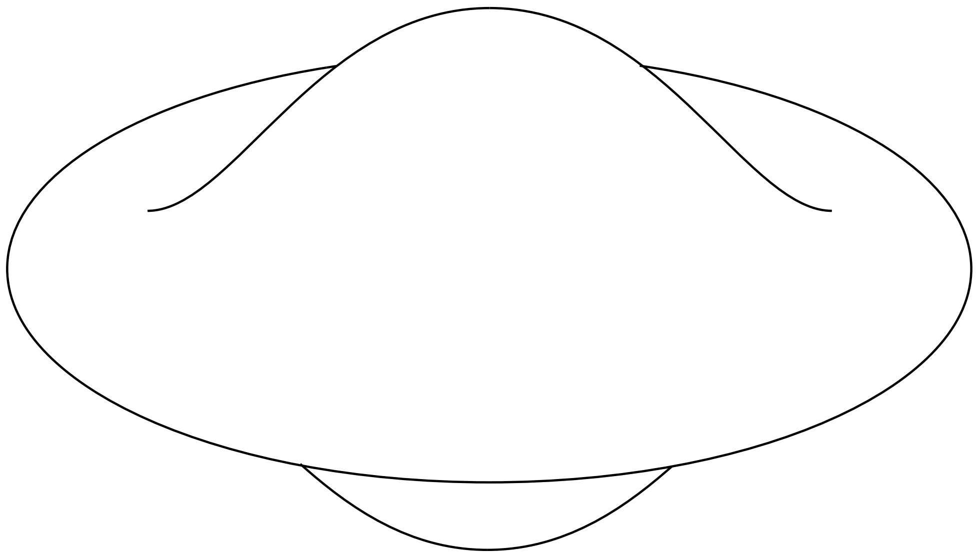
$$t \mapsto (\cos t, \sin 2t)$$

$$y\,dx = -\sin 2t \sin t = -2 \sin^2 t \cos t = -\frac{2}{3}\, d(\sin^3 t).$$


y
x
z
x


z
x
y
x

Examples. In dimension 2.
Note that $\dim \boldsymbol{X} = \mathbf{2} \Rightarrow \dim \boldsymbol{T}^{\star}\boldsymbol{X} = \mathbf{4}$
(hard to draw a Lagrangian submanifold...)
but that $\dim \boldsymbol{X} \times \mathbf{R} = \mathbf{3}$
(easy to draw a wave front).



And if the Lagrangian is not exact?

Note.
There are no exact Lagrangian submanifolds in $\mathbb{R}^{2n}$
(this is a theorem of Gromov).

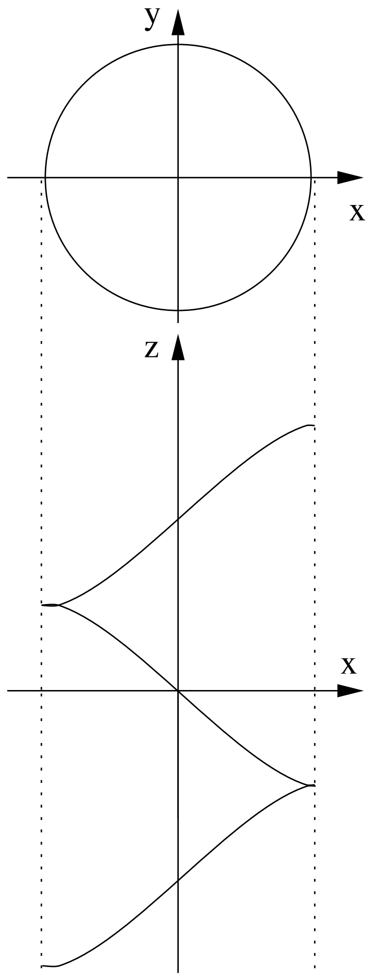
Example.

The circle in R^2.

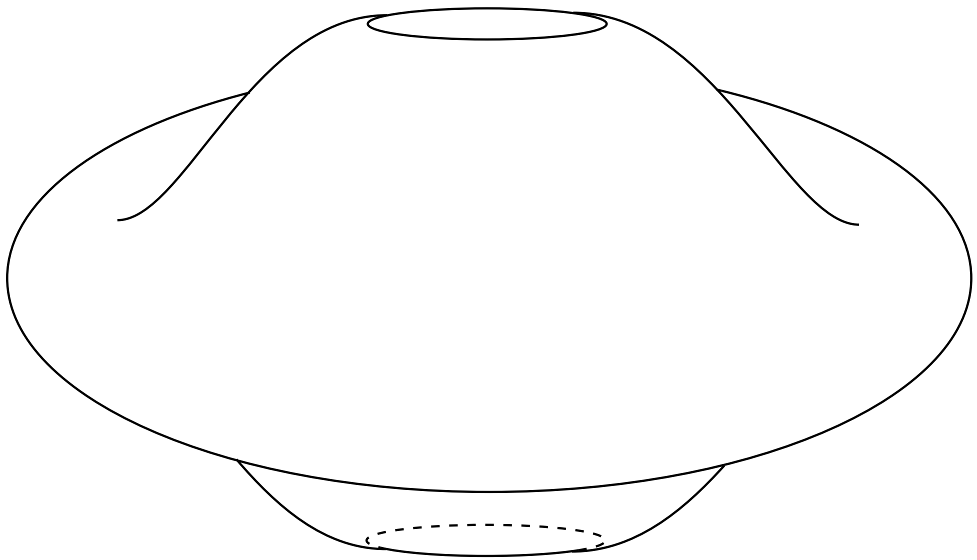
Non exact (why?).

$$t \mapsto (\cos t, \sin t)$$

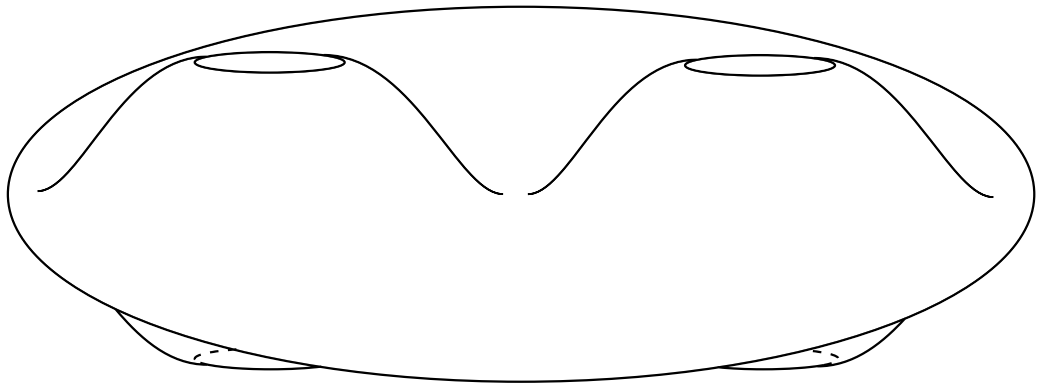
$$y\,dx = -\sin^2 t\,dt = d\Big(\frac{\sin 2t}{4} - \frac{t}{2} + C\Big)$$


y
x
z
x

Other examples of
(non exact) Lagrangian surfaces.



Torus.



Genus 2 surface.

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